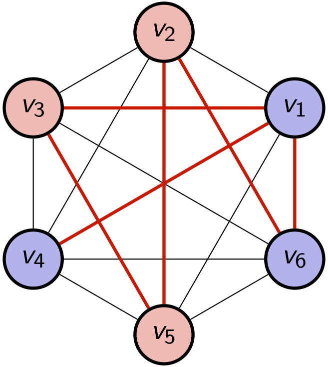
LECTURE 3

The voter model on dense dynamic random graphs

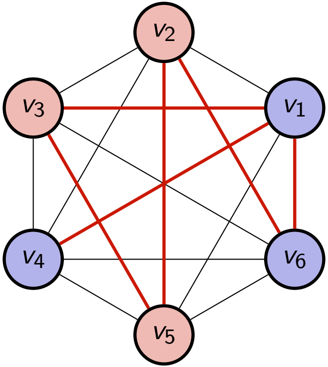
Part II

Model 2: Definition



At time $t = 0$ we start with the complete graph and the active edges are chosen independently with probability p_0.

Model 2: Definition



Each vertex x has a rate–β Poisson clock, when this rings, vertex x chooses a uniform neighbour (through active edges) and adopts its opinion.

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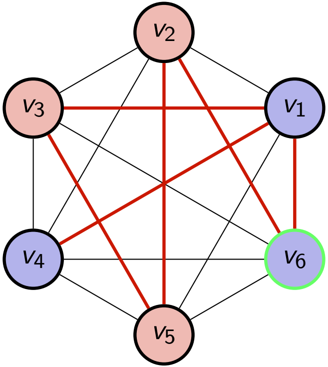

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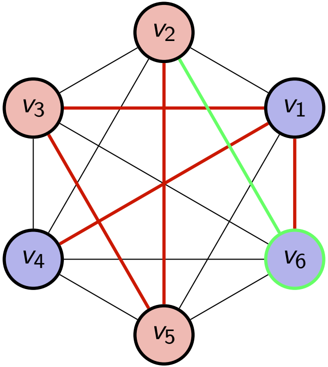

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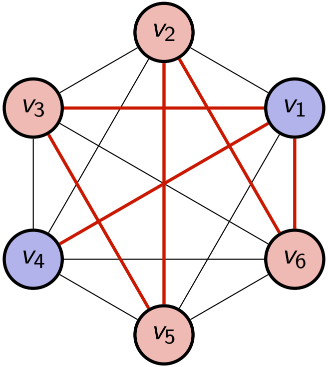

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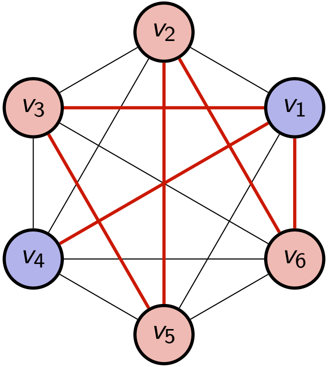

Each edge has a rate-1 *Poisson clock, when this rings,*
the edge ij *is active with probability* $(\pi_i + \pi_j)/2$.

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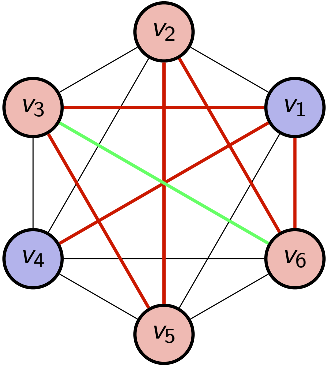

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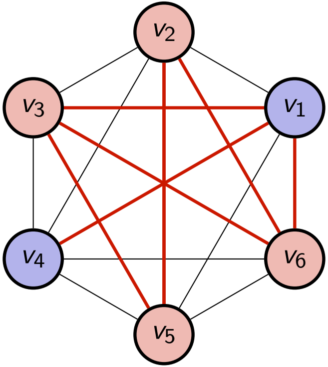

Each edge has a rate-1 Poisson clock, when this rings, the edge ij is active with probability $(\pi_i + \pi_j)/2$.

- This model is co-evolutionary, in that the opinions of the adjacent vertices affect the probability that an edge is active (through π_+ and π_-), and the state of adjacent edges affect the opinions of the vertices (because the vertices copy the opinions of their neighbours)

- When the number of vertices is finite the process is almost surely absorbed in a configuration in which all the vertices have the same opinion. Because of the update mechanisms of the connections, the edge dynamics never stops

- The framework used for Model 1 cannot be applied: $(x_i(t), y_i(t))_{i\in[n]}$ are now dependent. What to do?

Strategy

1. Define a process, referred to as mimicking process $(G_n^*(t))_{t\in[0,T]}$, with a one-way feedback interaction between vertex and edge dynamics

2. Characterise the limiting path F for the mimicking process

3. Show that the original model and the mimicking process are close "in some sense" as $n \to \infty$

Definition of the mimicking process

- The edge dynamics is the same as the co-evolutionary model
- The vertex dynamics in the mimicking process is defined as follows: each vertex is assigned an independent rate-β Poisson clock, and when the clock rings the vertex chooses opinion $+$ with probability $\alpha(t; u, \vec{v})$, where u is the type of the selected vertex and $\vec{v} = \vec{v}(t, u) = (f_+(t, u), f_-(t, u))^\top$.
- The expression of $\alpha(t; u, \vec{v})$ will be specified later to make the mimicking process asymptotically equivalent to the original model as $n \to \infty$

The infinitesimal generator of the mimicking process is

$$\begin{aligned}(\mathcal{L}_t f)(x,y) &= \beta((1-\alpha(t;y,\vec{v}))\mathbf{1}\{x=+\}+\alpha(t;y,\vec{v})\mathbf{1}\{x=-\})[f(x',y)-f(x,y)] \\ &\quad +b(x,y)\frac{\partial}{\partial y}f(x,y),\end{aligned}$$

where x' is the opinion of the selected vertex after switching its opinion x at time t and $b(x,\cdot)$ is the drift term when starting from $x \in \{-,+\}$ and has the same shape as for Model 1

$$\begin{aligned}\frac{\partial}{\partial t}f_+(t,u)+(1-u)\frac{\partial}{\partial u}f_+(t,u) &= \beta f_-(t,u)\,\alpha(t;u,\vec{v})-(\beta(1-\alpha(t;u,\vec{v}))-1)f_+(t,u), \\ \frac{\partial}{\partial t}f_-(t,u)-u\frac{\partial}{\partial u}f_-(t,u) &= \beta f_+(t,u)\,(1-\alpha(t;u,\vec{v}))-(\beta\alpha(t;u,\vec{v})-1)f_-(t,u).\end{aligned}$$

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\frac{\partial}{\partial t} f_+(t,u) + (1-u)\frac{\partial}{\partial u} f_+(t,u) &= \beta f_-(t,u)\,\alpha(t;u,\vec{v}) - (\beta(1-\alpha(t;u,\vec{v})) - 1) f_+(t,u), \\
\frac{\partial}{\partial t} f_-(t,u) - u\frac{\partial}{\partial u} f_-(t,u) &= \beta f_+(t,u)\,(1-\alpha(t;u,\vec{v})) - (\beta\alpha(t;u,\vec{v}) - 1) f_-(t,u).
\end{aligned}
$$

Theorem (FLLN for the densities of opinions) $(F_n(t;\cdot))_{t\in[0,T]}$ converges weakly to $(F(t;\cdot))_{t\in[0,T]}$ as $n \to \infty$ on $D((\mathcal{M}, d_L), [0,T])$.

Model 2: Main results

Define

$$p_+ = \lim_{t\to\infty} \lim_{n\to\infty} \frac{N_+(t)}{n}$$

as the limiting proportion of vertices with opinion $+$

Theorem (Limiting type density)

- If $\pi_+ = \pi_-$, then

$$\lim_{t\to\infty} \begin{pmatrix} f_+(t,u) \\ f_-(t,u) \end{pmatrix} = \begin{pmatrix} u \\ 1-u \end{pmatrix} f_{\beta p_+, \beta(1-p_+)}(u), \quad u \in [0,1].$$

- If $\pi_+ \neq \pi_-$, then only consensus is admissible.

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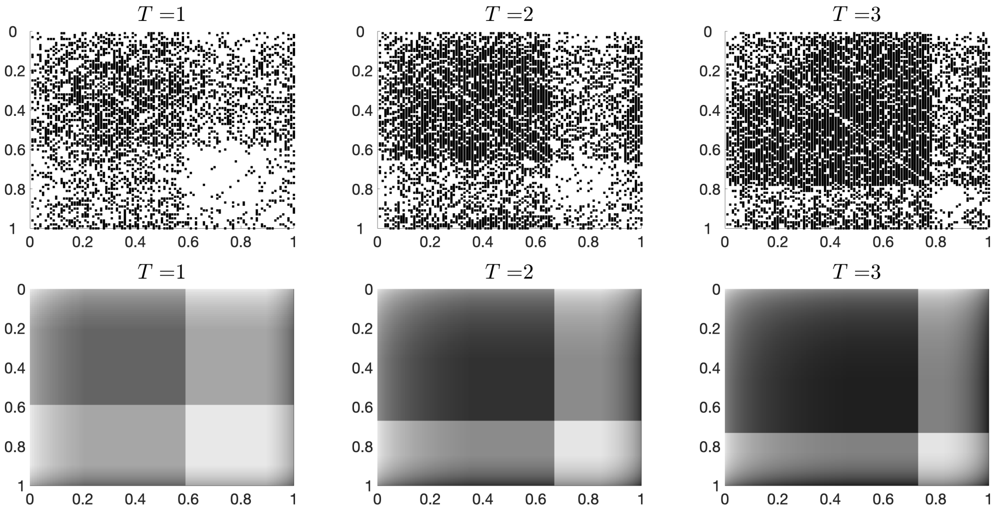
- If $\pi_+ \neq \pi_-$, then only consensus is admissible.

Theorem (FLLN for the dynamic random graph) $(h^{G_n(t)}(\cdot,\cdot))_{t\in[0,T]}$ converges weakly to $(g^{[F]}(t;\cdot,\cdot))_{t\in[0,T]}$ as $n \to \infty$ on $D((W, d_\square), [0, T])$, where $g^{[F]}(t;x,y) = H(t; \bar{F}(t;x), \bar{F}(t;y))$ is defined as in Model 1

Model 2: Discussion

- In the case $\pi_+ = \pi_-$:
  - the theorem suggests that the model moves towards one of the fixed points described by the Beta-distributions quickly, i.e., in a finite time that does not increase with *n*.
  - There should be a diffusion on the manifold by these fixed-point solutions for the process defined on a different timescale.
- In the case $\pi_+ \neq \pi_-$, the model moves toward consensus quickly with the proportion of vertices holding the minority opinion falling below any $\varepsilon > 0$ in a finite time that does not increase with *n*.

Model 2: Numerical simulations with $\pi_+ = 0.9$ and $\pi_- = 0.1$



The mimicking process is close to the original model

- In our case, the vertex dynamics in the mimicking process is defined as follows: each vertex is assigned an independent rate-β Poisson clock, and when the clock rings the vertex chooses opinion $+$ with probability

$$\alpha(t; u, \cdot) = \frac{\displaystyle\int_0^1 \mathrm{d}y \; f_+(t, y) H(t; y, u)}{\displaystyle\int_0^1 \mathrm{d}y \; [f_+(t, y) + f_-(t, y)] H(t; y, u)}$$

- If $h^{G_n^*} \implies g^{[F]}$, then $\alpha(t; u, \cdot)$ can be interpreted as the number of vertices having opinion $+$ connected to a vertex with type in $[u - du, u + du]$ divided the total number of vertices connected to a vertex with type in $[u - du, u + du]$

- The probability that the vertex takes opinion $+$ when its clock rings is then asymptotically equivalent to the probability that it takes opinion $+$ by copying one of its neighbors

Why does such a mimicking process exist?

- Construction of the mimicking process, i.e., a process that approximates the co-evolutionary process in such a way that discrepancies during the time interval $[0, T]$ have a sufficiently small probability.

- The existence of such a mimicking process is guaranteed by the density of the graph, meaning that each vertex is adjacent to order n active edges, so that, as $n \to \infty$, a law of large numbers holds at each vertex.

Discussion

1. If $\pi_- \neq \pi_+$, then the model moves toward consensus in a finite time that does not increase with n. Since consensus is absorbing, this (effectively) ends the dynamics of opinions. This suggests that, in the case $\pi_+ > \pi_-$ (resp. $\pi_+ < \pi_-$), the model reaches consensus on opinion $+$ (resp. opinion $-$)

2. If $\pi_+ = \pi_-$, the model moves towards one of the fixed points described by a continuum of fixed point solutions in a finite time that does not increase with n. However, unless this fixed point is consensus, the dynamics of the opinions does not end

3. We conjecture that if $\pi_+ = \pi_-$, then as $n \to \infty$ the process $(h^{G_n}(nt))_{t \in [0,T]}$ converges to a diffusion on the manifold defined by these fixed point solutions

4. Model 2 does not lead to polarisation

Nonlinear version of Model 2

- Same dynamics of the second model, but now an edge connecting $+$ and $-$ is active with probability $[(\pi_+ + \pi_-)/2]^q$, $q > 0$

- q represents group interactions among individuals with discordant opinions

- q tunes the convergence of the process to consensus or polarisation
    - $q = 1$: we recover the linear Model 2
    - $0 < q < 1$: the probability of having connections between $+$ and $-$ is higher
    - $q > 1$: the probability of having connections between $+$ and $-$ is lower

When $q > 1$, the expression for $p_{ij}(t)$ no longer simplifies to an equation that can be written only in terms of the types of i and j. Consequently, for those q we are unable to define H as required, and this issue appears to be fundamental in the sense that it cannot be solved by changing the definition of the vertex type

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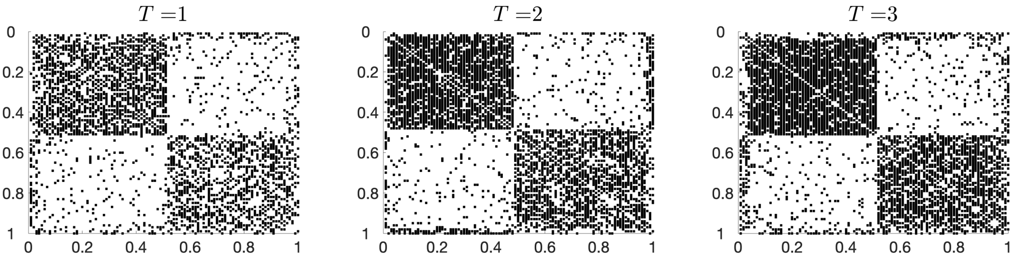

Model 3: Definition

- While the second model is a co-evolutionary network, it does not capture the key feature of a voter model on a dynamic network: vertices holding different opinions are less likely to be connected

- Consider $2q$ copies of Model 2 evolving in parallel: q green and q red graphs

- The green and red graphs have different parameters (π_+^g, π_-^g and π_+^r, π_-^r)

- The opinions of the vertices are the same in all the copies, but the states of the edges are different

- The $2q$ graphs are conditionally independent of each other, given the opinions of the vertices. The resulting graph is a function of the q copies

- Vertex dynamics: Each vertex is assigned an independent rate-β Poisson clock. Each time the clock rings, the vertex selects one of its neighbours in the resulting graph uniformly at random and copies the opinion of that vertex.

- Edge dynamics:
  - ▶ In the green and red graphs, each edge is assigned an independent rate-1 Poisson clock. Each time the clock rings in the k-graph, with $k \in \{\text{green}, \text{red}\}$, the edge ij is active with probability $(\pi_i^k + \pi_j^k)/2$
  - ▶ The resulting graph is a function of the green and red copies:
    - edge ij is active in the resulting graph if edge ij is active in exactly all the copies of the k-graph for one $k \in \{\text{green}, \text{red}\}$
    - edge ij is inactive in the resulting graph in all the other cases

An active edge ij in all the green graphs can be interpreted as an individual (either i or j) sharing a positive thought about opinion $+$ with the other individual, and an active edge ij in all the red graphs can be interpreted as an individual (either i or j) sharing a positive thought about opinion $-$ with the other individual

- If i and j share positive thoughts about only one opinion, then they are friends and are therefore connected in the resulting graph

- If they share positive thoughts about neither opinion (they do not contact each other) or both opinions (they argue), then they are not connected in the resulting graph

As an illustration, suppose $\pi^r_- = \pi^g_+ = 1$ and $\pi^r_+ = \pi^g_- = 0$

- If vertices i and j have both held opinion $+$ for an infinitely long period of time, then the probability that edge ij is active is 1 (the same applies if they have held opinion $-$)

- If vertex i has held opinion $+$ and vertex j opinion $-$ for an infinitely long period of time, then the probability that edge ij is active is $2 \cdot 0.5^q$

Consequently, vertices that share the same opinion (generally) have a higher probability of being connected

As a function of the parameter q, we make it less likely that individuals holding different opinion are connected. As a result, we obtain different levels of polarisation depending on the value of q. In the limit $q \to \infty$, we expect to obtain strong polarisation

Definition of the mimicking process

- The edge dynamics is the same as the co-evolutionary Model 3
- The vertex dynamics in the mimicking process is defined as follows: each vertex is assigned an independent rate-β Poisson clock, and when the clock rings the vertex chooses opinion $+$ with probability $\widetilde{\alpha}(t; u, \vec{v})$, where u is the type of the selected vertex and $\vec{v} = \vec{v}(t, u) = (f_+(t, u), f_-(t, u))^\top$.
- The expression of $\widetilde{\alpha}(t; u, \vec{v})$ will be specified later to make the mimicking process asymptotically equivalent to the original model as $n \to \infty$

The infinitesimal generator of the mimicking process is

$$\begin{aligned}(\mathcal{L}_t f)(x,y) &= \beta((1-\widetilde{\alpha}(t;y,\vec{v}))\mathbf{1}\{x=+\}+\widetilde{\alpha}(t;y,\vec{v})\mathbf{1}\{x=-\})[f(x',y)-f(x,y)] \\ &\quad + b(x,y)\frac{\partial}{\partial y}f(x,y),\end{aligned}$$

where x' is the opinion of the selected vertex after switching its opinion x at time t and $b(x,\cdot)$ is the drift term when starting from $x \in \{-,+\}$ and has the same shape as for Model 1 and Model 2

$$\begin{aligned}\frac{\partial}{\partial t}f_+(t,u)+(1-u)\frac{\partial}{\partial u}f_+(t,u) &= \beta f_-(t,u)\,\widetilde{\alpha}(t;u,\vec{v})-(\beta(1-\widetilde{\alpha}(t;u,\vec{v}))-1)f_+(t,u), \\ \frac{\partial}{\partial t}f_-(t,u)-u\frac{\partial}{\partial u}f_-(t,u) &= \beta f_+(t,u)\,(1-\widetilde{\alpha}(t;u,\vec{v}))-(\beta\widetilde{\alpha}(t;u,\vec{v})-1)f_-(t,u).\end{aligned}$$

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$$\frac{\partial}{\partial t}f_-(t,u) - u\frac{\partial}{\partial u}f_-(t,u) = \beta f_+(t,u)\,(1-\widetilde{\alpha}(t;u,\vec{v})) - (\beta\widetilde{\alpha}(t;u,\vec{v})-1)f_-(t,u).$$

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The probability that an edge between vertices of type u and v at time t is present in a green and red graph is

$$H_g(t;u,v) = \mathrm{e}^{-t}p_0 + \tfrac{1}{2}\left[\pi_+^g u + \pi_-^g(1-\mathrm{e}^{-t}-u) + \pi_+^g v + \pi_-^g(1-\mathrm{e}^{-t}-v)\right]$$
$$H_r(t;u,v) = \mathrm{e}^{-t}p_0 + \tfrac{1}{2}\left[\pi_+^r u + \pi_-^r(1-\mathrm{e}^{-t}-u) + \pi_+^r v + \pi_-^r(1-\mathrm{e}^{-t}-v)\right]$$

Due to the conditional independence between green and red graphs with respect to the type of the vertices, the probability that there is an edge in the resulting graph between vertices of type u and v at time t is

$$H_R(t;u,v) = H_g(t;u,v)^q\,(1-H_r(t;u,v))^q + H_r(t;u,v)^q\,(1-H_g(t;u,v))^q$$

Define the limiting graphon as $g^{[F]}(t;x,y) = H_R(t,\bar{F}(t;x),\bar{F}(t;y))$

Model 3: Main results

Theorem (FLLN for the dynamic random graph) $(h^{G_n(t)}(\cdot,\cdot))_{t\in[0,T]}$ converges weakly to $(g^{[F]}(t;\cdot,\cdot))_{t\in[0,T]}$ as $n \to \infty$ on $D((\mathcal{W}, d_\square), [0, T])$.

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Theorem (Limiting type density)

- Let p_+ be the limiting proportion of vertices with opinion $+$. If $\pi_+^g = \pi_-^g$ and $\pi_+^r = \pi_-^r$, then

$$\lim_{t\to\infty} \begin{pmatrix} f_+(t,u) \\ f_-(t,u) \end{pmatrix} = \begin{pmatrix} u \\ 1-u \end{pmatrix} f_{\beta p_+, \beta(1-p_+)}(u), \quad u \in [0,1].$$

- If both (A) $\pi_+^g \neq \pi_-^g$ or $\pi_+^r \neq \pi_-^r$ and (B) $\pi_+^g \neq \pi_-^r$ or $\pi_+^r \neq \pi_-^g$, then only consensus is admissible.

Model 3: Conjecture (Limiting type density)

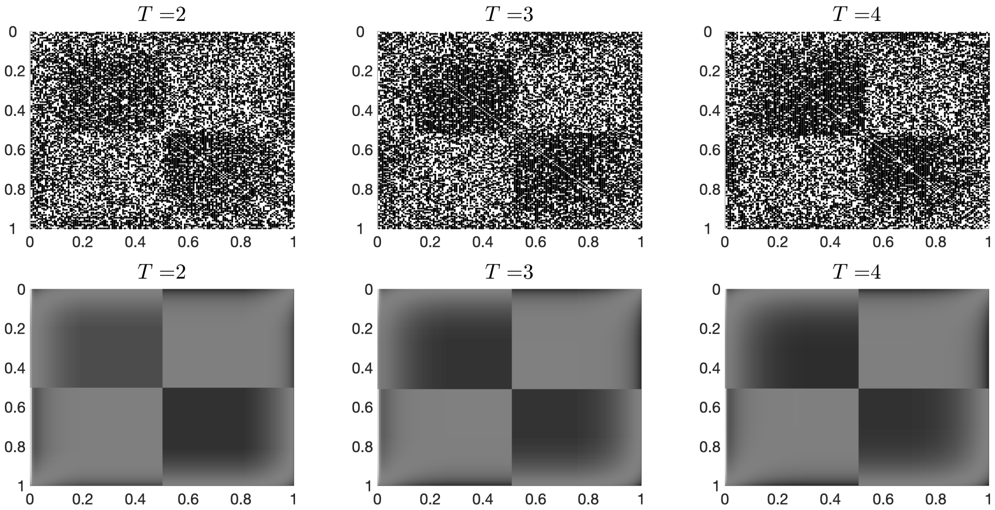
Our limiting result provides insight into the structure of the evolution for large t, even though it remains a major challenge to give a complete picture of the limiting behaviour, as shown by the following conjecture

Conjecture

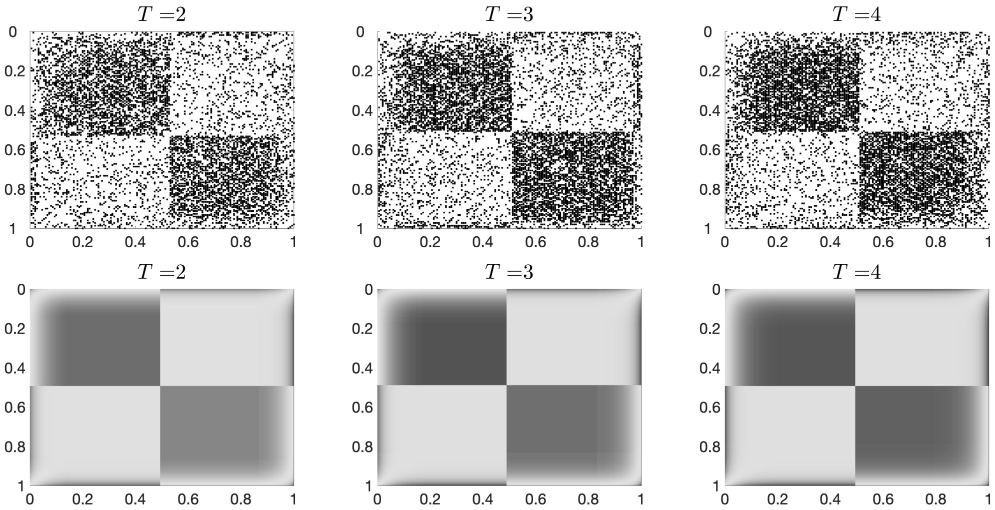
(i) If $\pi_+^g = \pi_-^r$ and $\pi_+^r = \pi_-^g$, then for any $q \in \mathbb{N}$ there exist unique limiting densities f_+ and f_- corresponding to consensus, or there are different levels of polarisation depending on the value of q, and the densities of edges connecting vertices having different opinions is a non-zero monotone decreasing function in q.

(ii) If $\pi_+^g = \pi_-^r$ and $\pi_+^r = \pi_-^g$, then in the limit as $q \to \infty$ only consensus occurs, or strong polarisation is admissible, and the densities of edges connecting vertices having different opinions is zero.

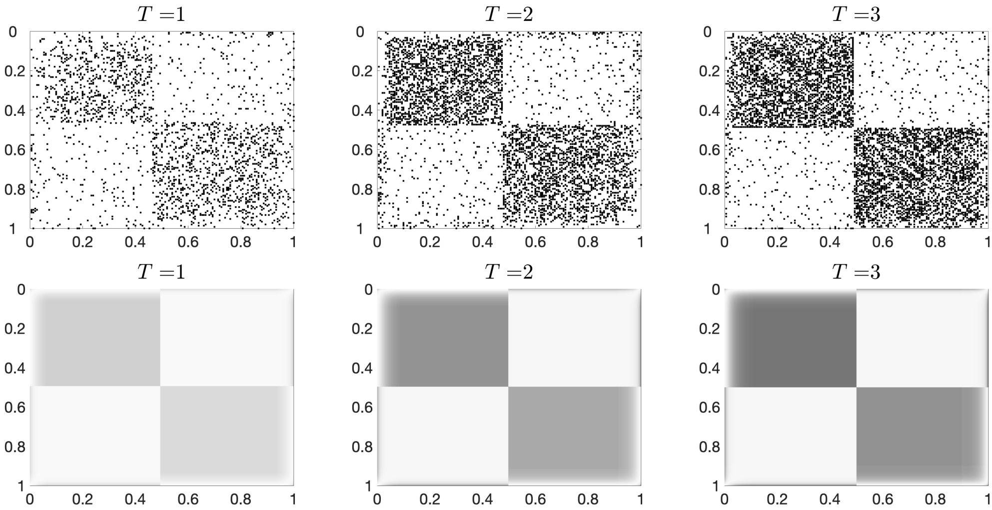
Model 3: Numerical simulations for $q = 1$



Model 3: Numerical simulations for $q = 2$



Model 3: Numerical simulations for $q = 3$



The mimicking process is close to the original process

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$$\widetilde{\alpha}(t;u,\cdot) = \frac{\int_0^1 \mathrm{d}y \; f_+(t,y) H_R(t;y,u)}{\int_0^1 \mathrm{d}y \; [f_+(t,y) + f_-(t,y)] H_R(t;y,u)}$$

- If $h^{G_n^*} \implies g^{[F]}$, then $\widetilde{\alpha}(t;u,\cdot)$ can be interpreted as the number of vertices having opinion $+$ connected to a vertex with type in $[u-du, u+du]$ divided the total number of vertices connected to a vertex with type in $[u-du, u+du]$

- The probability that the vertex takes opinion $+$ when its clock rings is then asymptotically equivalent to the probability that it takes opinion $+$ by copying one of its neighbors

Discussion

1. We proved functional laws of large numbers for three examples of the voter model on a dynamic random graph, both with one-way and two-way feedback. These functional laws of large numbers were used to visualise the evolution of the process

2. The approach seems robust and can possibly be applied more in general

3. For the nonlinear version of Model 2 it is unclear how to write down a functional law of large numbers. It remains to be determined whether functional laws of large numbers can be established in such cases, and if so, what mathematical framework is required

4. Understand what happens in the Potts model and for opinions in $[0, 1]$

5. Find other co–evolutionary models in which polarisation can be hoped for

6. Study of the fluctuations of type CLT for the subgraph counts for one–way and two–way feedback model (joint with N. Kriukov)

7. Parameter estimation for one–way and two–way feedback model (joint with J. Wang)