LECTURE 2

The voter model on dense dynamic random graphs

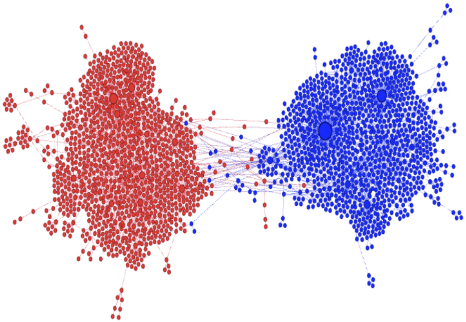
Part I

Motivation

- The understanding of processes on dynamic networks is still in its infancy [Holme, Newman, 2006], [Durrett et al, 2012], [Basu, Sly, 2017]

- Double dynamics are prevalent in most real–world networks

- Strong evidence that polarisation can only occur in a co-evolutionary network



Holme-Newman model

Consider a graph with N vertices and M edges. Each vertex x has an opinion $\xi(x)$ from a set of G possible opinions. Assume that $\gamma_N = N/G$ is bounded as N gets large

At each step of the process, a vertex x is picked at random

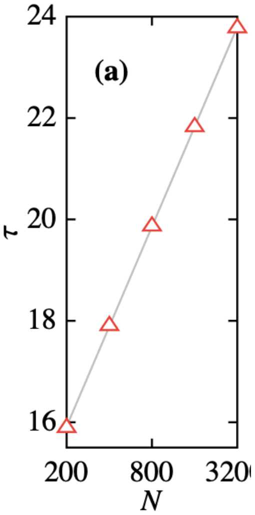
- If its degree is zero, nothing happens
- If its degree is strictly positive
  - (i) with probability α an edge attached to vertex x is selected and the other end of that edge is moved to a vertex chosen at random from those with opinion $\xi(x)$
  - (ii) with probability $1 - \alpha$ a random neighbor y of x is selected and we set $\xi(x) = \xi(y)$

This process continues until there are no longer any edges connecting individuals with different opinions

Holme-Newman model

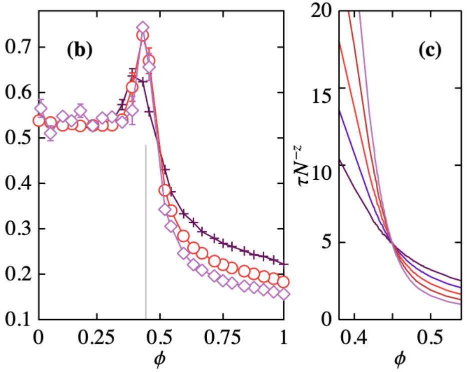
- When $\alpha = 0$, the system reduces to the voter model on a static graph. The average convergence time is then of order N (Sood, Redner 2005)

- When $\alpha = 1$, only rewiring steps occur, so once all of the M edges have been touched, the graph has been disconnected into G components, each consisting of individuals who share the same opinion. This requires approximately $M \log M$ updates. The average convergence time is then $M \log M / N \sim \log N$ for fixed average degree $2M/N$



Holme-Newman model

Using simulation and finite size scaling, Holme and Newman showed that there is a critical value α_c so that for $\alpha > \alpha_c$ all of the opinions have a small number of followers at the end of the process, whereas for $\alpha < \alpha_c$ "a giant community of like-minded individuals forms". When the average degree $2M/N = 4$ and $\gamma_N \to 10$, this transition occurs at $\alpha_c \approx 0.46$



Durrett et al

The "rewire-to-same" model differs from that of Holme and Newman in two ways:

(i) each vertex can have only opinion 0 or 1

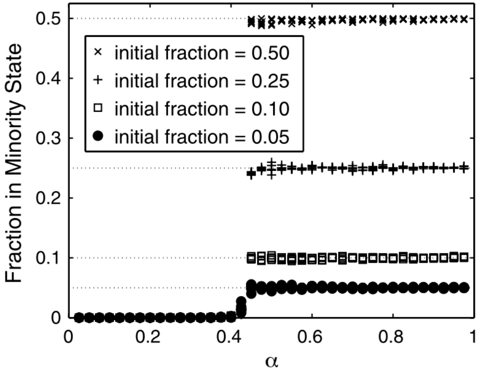
(ii) at each step, a discordant edge (x, y) is picked

- with probability α x breaks its connection to y and makes a new connection to a voter chosen at random from those that share its opinion
- with probability $1 - \alpha$ the voter at x adopts the opinion of the voter at y

The process continues until there are no edges connecting voters that disagree

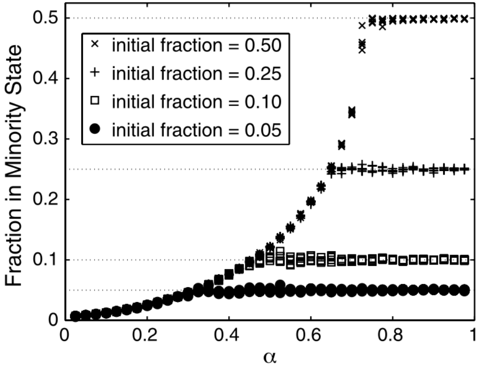
Durrett et al

This model has a phase transition similar to that of Holme and Newman. In particular, the final fraction of voters with the minority opinion undergoes a discontinuous transition at a value α_c that does not depend on the initial density



Durrett et al

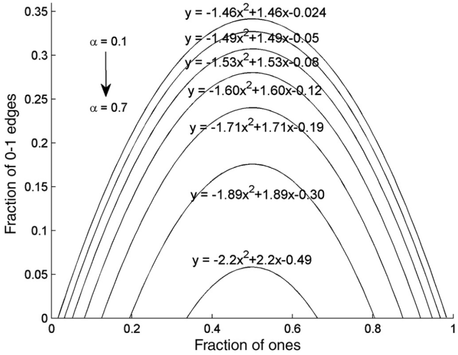
The "rewire-to-random" variant of the model differs from the "rewire-to-same" model in only one way: x makes its new connection to a voter chosen at random from all of the vertices in the graph. This single difference leads to fundamentally different model outcomes



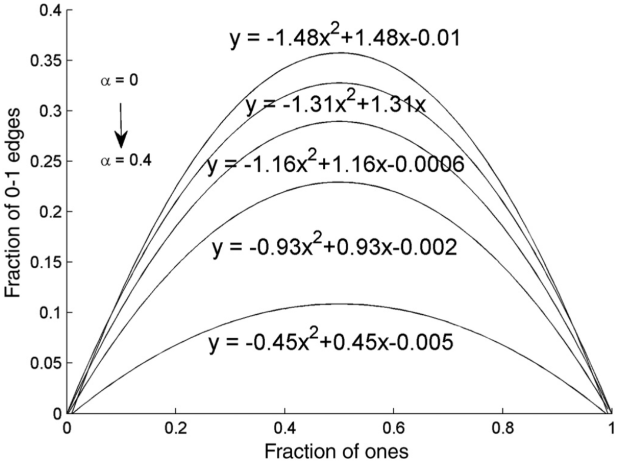
Conjecture 1 In the rewire-to-random model, if $\alpha < \alpha_c(1/2)$ and $\nu(\alpha) < u \leq 1/2$, then starting from product measure with density u, at time tN the evolving voter model looks locally like $\nu_{\alpha,\theta(t)}$, where the density changes according to

$$\mathrm{d}\theta_t = \sqrt{(1-\alpha)[c_\alpha \theta_t(1-\theta_t) - b_\alpha]}\mathrm{d}W_t$$

until θ_t reaches $\nu(\alpha)$ or $1-\nu(\alpha)$, where $(W_t)_{t \geq 0}$ is a standard Brownian motion



Conjecture 2 In the rewire-to-same model, the behavior is as described in Conjecture 1 but now $b_\alpha = 0$, so α_c is independent of the initial density u, and for $\alpha < \alpha_c$, $\rho \approx 0$



Basu, Sly

Basu and Sly considered the dense version of the model considered by Durrett et al with the renormalization with $1 - \alpha = \beta/n$

In order to state the main theorems, we need some notations and definitions

1. τ denotes the first time to reach an absorbing state

2. For $\varepsilon < 1/2$, let $\tau_*(\varepsilon)$ be the alertfirst time that the fraction of the minority opinion reaches ε

3. $N_*(t)$ is the number of vertices holding the minority opinion at time t

Main theorems

Theorem 1 Let $0 < \varepsilon' \le 1/2$ be given. For both variants of the model, there exist $0 < \beta_0 < \beta_*(\varepsilon') < \infty$ such that each of the following hold.

(i) For all $\beta < \beta_0$ and any $\eta > 0$, $\{\tau < 10n^2, N_*(\tau) \ge (1/2 - \eta)n\}$ holds w.h.p.

(ii) For all $\beta > \beta_*(\varepsilon')$, $\tau_*(\varepsilon') \le \tau$ w.h.p. and

$$\lim_{c \downarrow 0} \liminf_{n} \mathbb{P}(\tau > cn^3) = 1$$

Theorem 2 Let $\beta > 0$ fixed. For the rewire-to-random model there exists $\varepsilon_* = \varepsilon_*(\beta) > 0$ such that $\tau < \tau_*(\varepsilon_*)$ with high probability

Open problems

- In the rewire-to-same model, is there a positive fraction of both opinions present when the process reaches an absorbing state?
- Is there any value β_0 such that for $\beta < \beta_0$ we have behaviour as in Theorem 1(i) and for $\beta > \beta_0$, we have behaviour as in Theorem 1(ii)?

Target

The focus is on the voter model on dense dynamic random graphs. Our goal is to understand and describe the occurrence of consensus versus polarisation over long periods of time

Our main results are functional laws of large numbers for the densities of the two opinions and for the dynamic random graphs, and a characterisation of the limiting densities in terms of Beta-distributions

Baldassarri, Braunsteins, den Hollander, Mandjes 2024

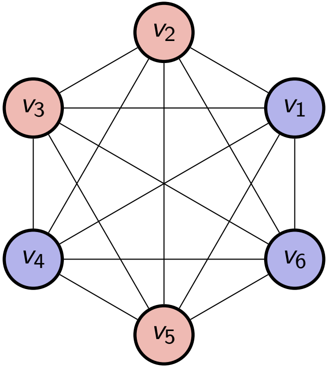
Models of increasing complexity

- Model 1: vertex dynamics not affected by edges, but not the reverse.

- Model 2: vertex dynamics affected by edges and viceversa.

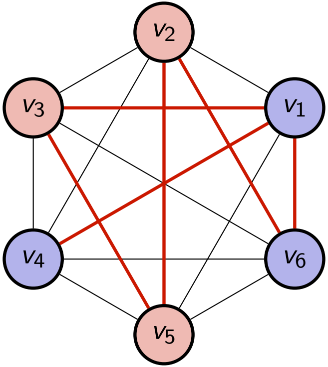
- Model 3: co-evolutionary network such that vertices holding different opinions are less likely to be connected.

Model 1: Definition



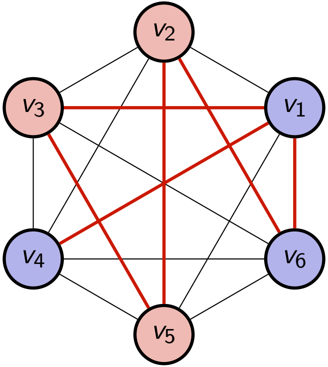
We start at time $t = 0$ with a complete graph.

Model 1: Definition



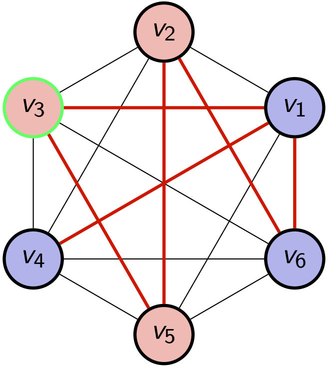
At time $t = 0$ the active edges are chosen independently with probability p_0.

Model 1: Definition



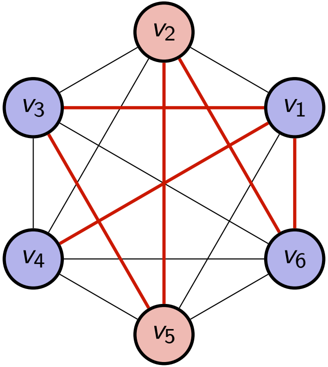
If a vertex holds opinion $+1/-1$*, it switches its opinion at rate* γ_{+-}/γ_{-+}.

Model 1: Definition



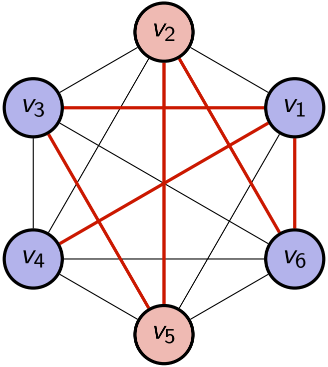
If a vertex holds opinion $\color{red}{+1}/\color{blue}{-1}$*, it switches its opinion at rate* $\color{red}{\gamma_{+-}}/\color{blue}{\gamma_{-+}}$.

Model 1: Definition



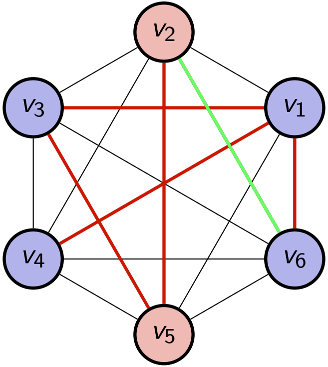
If a vertex holds opinion $+1$/-1, it switches its opinion at rate γ_{+-}/γ_{-+}.

Model 1: Definition



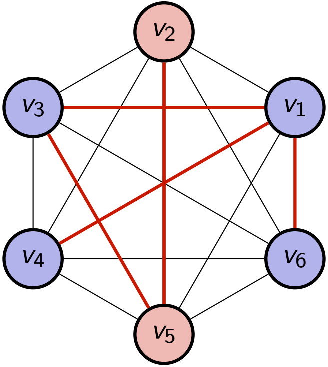
Each edge has a rate−1 Poisson clock, when this rings, the edge ij is active with probability $(\pi_i + \pi_j)/2$.

Model 1: Definition



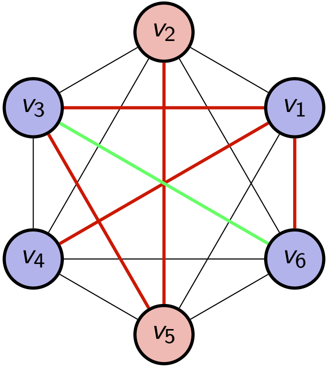
Each edge has a rate-1 *Poisson clock, when this rings, the edge* ij *is active with probability* $(\pi_i + \pi_j)/2$.

Model 1: Definition



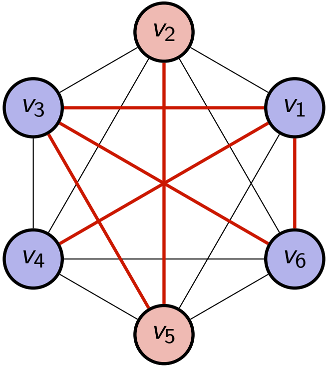
Each edge has a rate–1 Poisson clock, when this rings, the edge ij is active with probability $(\pi_i + \pi_j)/2$.

Model 1: Definition



Each edge has a rate-1 Poisson clock, when this rings, the edge ij is active with probability $(\pi_i + \pi_j)/2$.

Model 1: Definition



Each edge has a rate-1 *Poisson clock, when this rings,*
the edge ij *is active with probability* $(\pi_i + \pi_j)/2$.

Model 1: Key quantities

- $x_i(t)$ is the opinion of vertex i at time t.

- Keeping track of the opinions of vertices only is not enough to capture the structure of the resulting graph.

- $y_i(t) = \int_0^t ds\ e^{-s} \mathbf{1}\{x_i(t-s) = +1\}$ is the type of vertex i at time t.

- $X_i(t) = (x_i(t), y_i(t))$ is the generalised type of vertex i at time t.

The probability to have an active edge at time t between two vertices can be expressed in terms of their types only

$$
\begin{aligned}
p_{ij}(t) &= \mathrm{e}^{-t} p_0 + \int_0^t \mathrm{d}s\, \mathrm{e}^{-s} \tfrac{1}{2} \left[\pi_{x_i(t-s)} + \pi_{x_j(t-s)}\right] \\
&= \mathrm{e}^{-t} p_0 + \int_0^t \mathrm{d}s\, \mathrm{e}^{-s} \tfrac{1}{2} \big[\pi_+ \mathbf{1}\{x_i(t-s) = +\} + \pi_-(1 - \mathbf{1}\{x_i(t-s) = +\}) \\
&\qquad\qquad + \pi_+ \mathbf{1}\{x_j(t-s) = +\} + \pi_-(1 - \mathbf{1}\{x_j(t-s) = +\})\big] \\
&= \mathrm{e}^{-t} p_0 + \tfrac{1}{2} \big[\pi_+ y_i(t) + \pi_-(1 - \mathrm{e}^{-t} - y_i(t)) + \pi_+ y_j(t) + \pi_-(1 - \mathrm{e}^{-t} - y_j(t))\big]
\end{aligned}
$$

$$
H(t; u, v) = \mathrm{e}^{-t} p_0 + \tfrac{1}{2} \left[\pi_+ u + \pi_-(1 - \mathrm{e}^{-t} - u) + \pi_+ v + \pi_-(1 - \mathrm{e}^{-t} - v)\right]
$$

- $F_n(t;+,u) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\{X_i(t) \in (+,[0,u])\}$ is the empirical type process

- $F(t;+,u) = \int_0^u f_+(t,s)\,\mathrm{ds}$, where $f_+(t,s)\,\mathrm{ds} = \mathbb{P}(X_i(t) \in (+,\mathrm{ds}))$

The limiting path F, as n grows large, can be determined by using the Kolmogorov forward equations of the Markov process starting from its infinitesimal generator

$$(\mathcal{L}f)(x,y) = (\gamma_{+-}\mathbf{1}\{x=+\} + \gamma_{-+}\mathbf{1}\{x=-\})[f(x',y) - f(x,y)] + b(x,y)\frac{\partial}{\partial y}f(x,y),$$

where x' is the opinion of the selected vertex after switching its initial opinion x and $b(x,\cdot)$ is the drift term when starting from $x \in \{-,+\}$, with

$$b(+,y) = 1-y, \qquad b(-,y) = -y$$

Model 1: Main results

$$\frac{\partial}{\partial t} f_+(t,u) - (1-u)\frac{\partial}{\partial u} f_+(t,u) = -\gamma_{+-} f_+(t,u) + \gamma_{-+} f_-(t,u),$$
$$\frac{\partial}{\partial t} f_-(t,u) + u\frac{\partial}{\partial u} f_-(t,u) = -\gamma_{-+} f_-(t,u) + \gamma_{+-} f_+(t,u)$$

Theorem (FLLN for the densities of opinions) $(F_n(t;\cdot))_{t\in[0,T]}$ converges weakly to $(F(t;\cdot))_{t\in[0,T]}$ as $n \to \infty$ on $D((\mathcal{M}, d_L), [0,T])$.

Model 1: Main results

$$\frac{\partial}{\partial t} f_+(t,u) - (1-u)\frac{\partial}{\partial u} f_+(t,u) = -\gamma_{+-} f_+(t,u) + \gamma_{-+} f_-(t,u),$$
$$\frac{\partial}{\partial t} f_-(t,u) + u\frac{\partial}{\partial u} f_-(t,u) = -\gamma_{-+} f_-(t,u) + \gamma_{+-} f_+(t,u)$$

Theorem (FLLN for the densities of opinions) $(F_n(t;\cdot))_{t\in[0,T]}$ converges weakly to $(F(t;\cdot))_{t\in[0,T]}$ as $n\to\infty$ on $D((\mathcal{M},d_L),[0,T])$.

Theorem (Limiting type density)

(i) As $n\to\infty$ followed by $t\to\infty$, almost surely the proportion of $+$ converges to $\gamma_{-+}/(\gamma_{+-}+\gamma_{-+})$ and the proportion of $-$ to $\gamma_{+-}/(\gamma_{+-}+\gamma_{-+})$

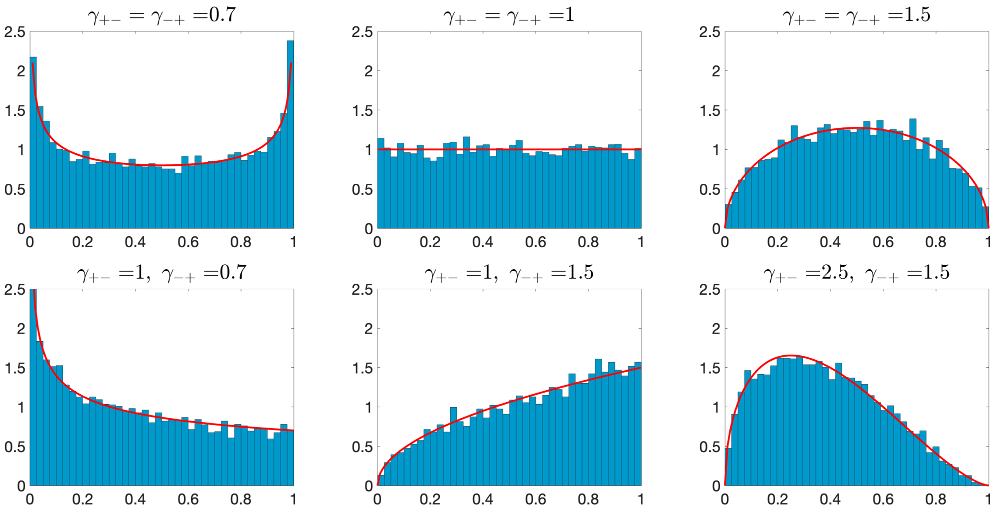
(ii)
$$\lim_{t\to\infty}\begin{pmatrix} f_+(t,u) \\ f_-(t,u)\end{pmatrix} = \begin{pmatrix} u \\ 1-u \end{pmatrix} f_{\gamma_{-+},\gamma_{+-}}(u), \quad u\in[0,1].$$

Remark

The type is close to zero if the vertex has held opinion $-$ for an extended period of time, close to one if the vertex has held opinion $+$ for an extended period of time, and close to 0.5 if the vertex has been regularly flipping between opinions $+$ and $-$. As a consequence, assuming that $\gamma_{+-} = \gamma_{-+} = \gamma$,

- if γ is large, then we expect that most vertices have been regularly flipping between opinions $+$ and $-$, and therefore have type close to 0.5
- if γ is small, then we expect that most vertices have held either opinion $+$ or $-$ for an extended period of time, and therefore have type close to 0 or 1

Model 1: Numerical simulations for the type density



Graphons

- Natural way to embed the adjacency matrix of a graph into the set of graphons

$$W = \{h : [0,1]^2 \to [0,1] : \ h(x,y) = h(y,x) \, \forall \, x,y\}.$$

- Define the empirical graphon associated to the graph G as

$$h^G(x,y) := \begin{cases} 1 & \text{if } \lceil nx \rceil \sim \lceil ny \rceil, \\ 0 & \text{if } \lceil nx \rceil \nsim \lceil ny \rceil. \end{cases}$$

Graphons

- Natural way to embed the adjacency matrix of a graph into the set of graphons

$$W = \{h : [0,1]^2 \to [0,1] : \; h(x,y) = h(y,x) \; \forall \, x, y\}.$$

- Define the empirical graphon associated to the graph G as

$$h^G(x,y) := \begin{cases} 1 & \text{if } \lceil nx \rceil \sim \lceil ny \rceil, \\ 0 & \text{if } \lceil nx \rceil \nsim \lceil ny \rceil. \end{cases}$$

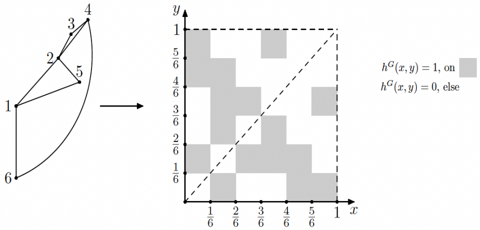
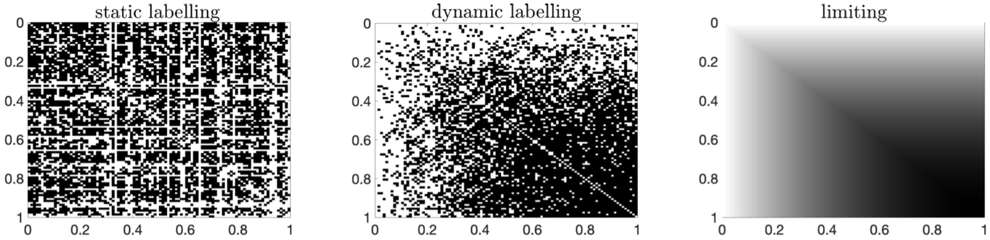


Figure taken by F. den Hollander et al (2018).

Dynamic labelling

The labels of the vertices are updated dynamically so that, at any time t, they are ordered lexicographically: all opinion $+$ vertices have a lower label than opinion $-$ vertices, and between vertices with the same opinion the vertices are ordered by increasing type



Model 1: Main results

Define

$$\bar{F}(t;y) = \begin{cases} \inf\{s \in [0,1]\colon\ F(t;+,s) \geq y\}, & \text{if } 0 \leq y \leq F(t;+,1), \\ \inf\{s \in [0,1]\colon\ F(t;+,1) + F(t;-,s) \geq y\}, & \text{otherwise,} \end{cases}$$

and the graphon

$$g^{[F]}(t;x,y) = H(t;\bar{F}(t;x),\bar{F}(t;y)),$$

where $H(t;u,v)$ is the probability that there is an active edge connecting two vertices with type u and v at time t

Model 1: Main results

Define

$$\bar{F}(t;y) = \begin{cases} \inf\{s \in [0,1]\colon\ F(t;+,s) \geq y\}, & \text{if } 0 \leq y \leq F(t;+,1), \\ \inf\{s \in [0,1]\colon\ F(t;+,1) + F(t;-,s) \geq y\}, & \text{otherwise,} \end{cases}$$

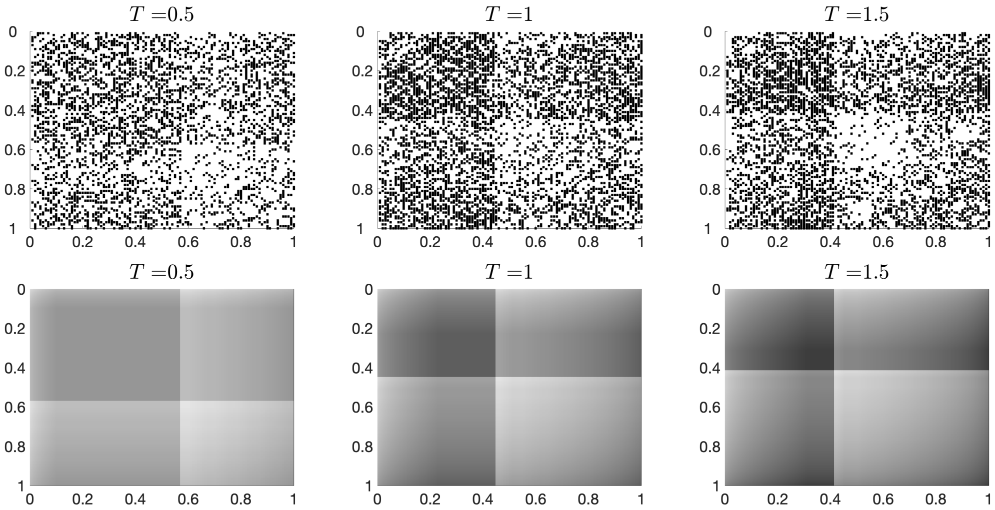
and the graphon

$$g^{[F]}(t;x,y) = H(t;\bar{F}(t;x),\bar{F}(t;y)),$$

where $H(t;u,v)$ is the probability that there is an active edge connecting two vertices with type u and v at time t

Theorem (FLLN for the dynamic random graph) $(h^{G_n(t)}(\cdot,\cdot))_{t\in[0,T]}$ converges weakly to $(g^{[F]}(t;\cdot,\cdot))_{t\in[0,T]}$ as $n \to \infty$ on $D((W,d_\square),[0,T])$.

Model 1: Numerical simulations for the graphons



Discussion

1. The above theorems fully accomplish our goal of characterising the functional law of large numbers in the space of graphons for this first model. In particular, we deduce that this model leads to coexistence only, meaning that both opinions survive, but there appear many edges connecting different opinions in the resulting graph

2. Although there is only a one-way feedback between vertex and edge dynamics, there appears to be no explicit expression for the densities f_+ and f_-, while the limiting proportion of vertices having opinion $+$ and $-$, as well the limiting densities f_+ and f_-, are fully characterised

3. As we will see in the next lecture, this is still possible for the second model, even though it gives rise to a richer phenomenology, while for the third model only weaker results are obtained