How opinions and infections evolve on dynamic random graphs

Simone Baldassarri

Gran Sasso Science Institute

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GRAN SASSO SCIENCE INSTITUTE

SCHOOL OF ADVANCED STUDIES

Scuola Universitaria Superiore

Outline

- Lecture 1. Background and motivation for opinion dynamics and infection models on static graphs
- Lecture 2. Voter model on dense dynamic random graphs: Part I
- Lecture 3. Voter model on dense dynamic random graphs: Part II
- Lecture 4. SIR epidemic model on dense dynamic random graphs

Goals

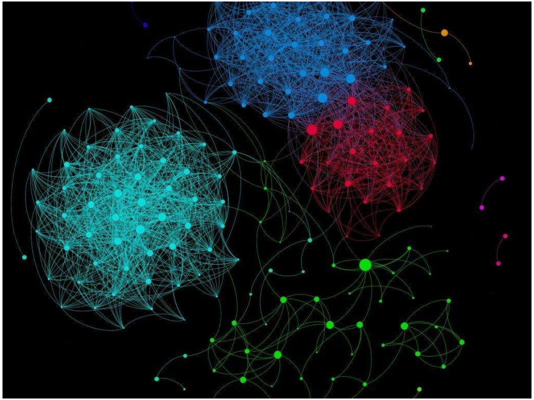
- Describe the evolution of dynamic networks together with dynamic processes running on them
- Highlight the challenging when dealing with co-evolution
- Outline some open questions and possible directions for future work

LECTURE 1

Opinion dynamics and infection models on static graphs

Interacting systems on networks: A big picture

Many complex systems can be seen as dynamical processes evolving on networks

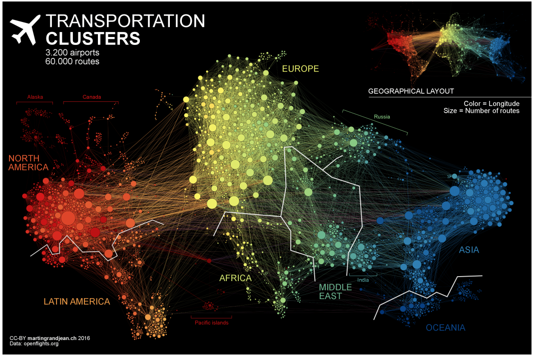


What is a network?

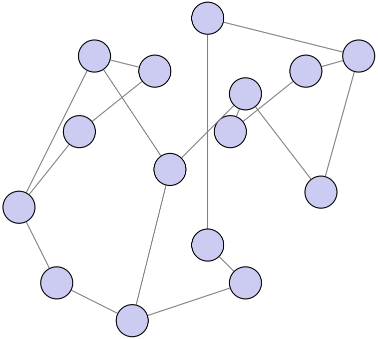
- A network is a set of nodes (people, computers, genes, neurons) connected by edges (interactions, friendships, communication, infection pathways, influence)
- Key features
  - Node degree: number of connections per node
  - Network density: fraction of possible edges that exist
  - Communities: groups of nodes more connected among themselves
  - Dynamics on networks: how node states evolve along edges (opinions, infections, activity…)

Examples of real-world networks

- Social networks: friendships, online interactions
- Biological networks: spread of an infection, protein interactions, neurons
- Technological networks: internet, communication systems, power grids
- Transportation networks: air traffic, road systems

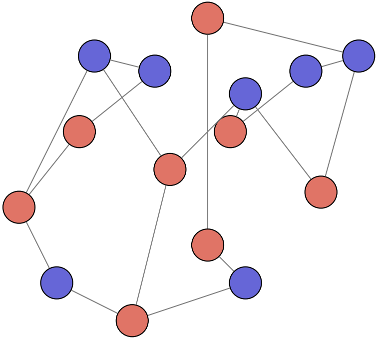


Dynamics on networks



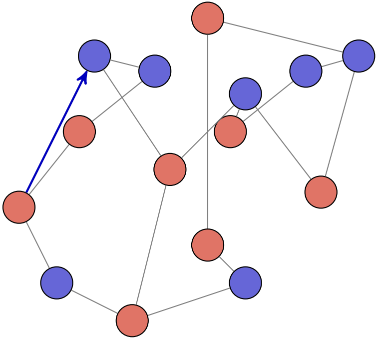
nodes = agents, edges = interactions

Dynamics on networks



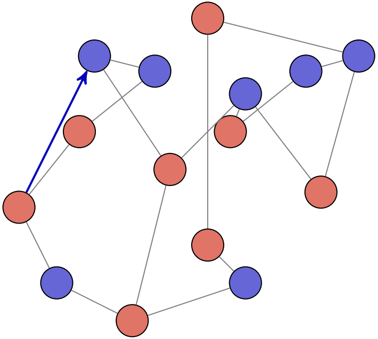
red/blue = states of agents

Dynamics on networks



nodes copy neighbors along arrows

Dynamics on networks



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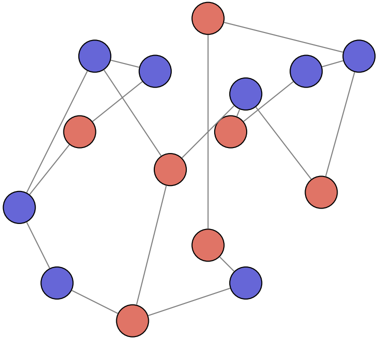


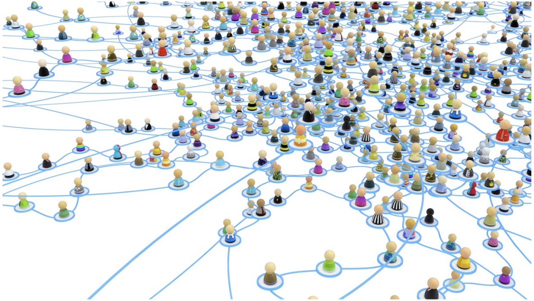
network after one step of the dynamics

Why study dynamic processes on networks?

- These processes are inherently dynamic
  - states change over time
  - small local interactions can lead to large-scale effects
- Key questions
  - What are the possible long-term behaviors?
  - How fast does the system reaches equilibrium (if any)?
  - How can local interactions produce global behavior?

The voter model

Given a connected graph $G = (V, E)$, each vertex is associated with an opinion 0 or 1. The voter model is a Markov process $(\xi_t)_{t\geq 0}$ on the state space $\{0,1\}^V$, where each vertex at rate 1 selects uniformly at random one of its neighbours and adopts its opinion



Write [0] and [1] to denote the configurations $\eta \equiv 0$ and $\eta \equiv 1$, respectively. These configurations are both traps for the dynamics: if all sites have the same opinion, then no change of opinion occurs

Denote by $\xi_t(i)$ the opinion of vertex i at time t. We analyse the evolution of the fraction of discordant edges

$$\mathcal{D}_t = \frac{D_t}{M}, \qquad D_t = |\{(i,j) \in E : \xi_t(i) \neq \xi_t(j)\}|,$$

where $M = |E|$. This is an interesting quantity because it monitors the size of the boundary between the two opinions

The consensus time is defined as

$$\tau_{cons} = \inf\{t \geq 0 : \xi_t(i) = \xi_t(j) \, \forall \, i, j \in V\}$$

For finite graphs, we know that $\tau_{cons} < \infty$ with probability 1 (either at [0] or [1])

The interest lies then in the determining the relevant time scales on which consensus is reached, and how it is reached.

The voter model on the complete graph

The number of 1-opinions at time t, given by

$$O_t = \sum_{i \in V} \xi_t(i),$$

performs a continuous-time nearest-neighbour random walk on the set $\{0, 1, \ldots, n\}$ ($n = |V|$) with transition rates

$$\begin{aligned} m \to m+1 &\quad \text{at rate } m(n-m)\frac{1}{n-1} \\ m \to m-1 &\quad \text{at rate } (n-m)m\frac{1}{n-1} \end{aligned}$$

This is the same as the Moran model from population genetics

Set $\mathcal{O}_t = \frac{1}{n} O_t$ the fraction of 1-opinions at time t. Then the process $(\mathcal{O}_{sn})_{s\geq 0}$ converges in law as $n \to \infty$ to the Fisher-Wright diffusion $(\theta_s)_{s\geq 0}$ on $[0, 1]$ given by

$$\mathrm{d}\theta_s = \sqrt{2\theta_s(1-\theta_s)}\mathrm{d}W_s,$$

where $(W_s)_{s\geq 0}$ is a standard Brownian motion

The number of discordant edges is

$$D_t = \frac{O_t(n-O_t)}{2}$$

and therefore the proportion of discordant edges is

$$\mathcal{D}_t = \frac{O_t(n-O_t)}{n(n-1)} = \frac{n}{n(n-1)}\mathcal{O}_t(1-\mathcal{O}_t)$$

It follows that the process $(\mathcal{D}_{sn})_{s\geq 0}$ converges in law as $n\to\infty$ to the process

$$(\theta_s(1-\theta_s))_{s\geq 0}$$

In the mean field setting of the complete graph, the fraction of discordant edges is the product of the fractions of the two opinions.

The latter property on non-complete graphs, in particular, on random graphs

The voter model on random regular graphs

Consider the regular random graph $G_{d,n} = (V, E)$ of degree $d \geq 3$, consisting of

$$|V| = n \text{ vertices}$$
$$|E| = m = \frac{dn}{2} \text{ edges}$$

Recall that $\mathcal{O}_t$ denotes the fraction of 1-opinions at time t. Then Chen, Choi, Cox 2016 showed that the process $(\mathcal{O}_{sn})_{s\geq 0}$ converges in law as $n \to \infty$ to the Fisher-Wright diffusion $(\theta_s)_{s\geq 0}$ given by

$$\mathrm{d}\theta_s = \sqrt{2\alpha_d \theta_s(1-\theta_s)}\,\mathrm{d}W_s,$$

where $(W_s)_{s\geq 0}$ is a standard Brownian motion, and

$$\alpha_d = \frac{d-2}{d-1}$$

The voter model on directed sparse random graphs

Avena, Capannoli, Hazra, Quattropani 2023, Capannoli 2024

Under mild conditions on the sequence of in- and out-degrees, the same scaling applies and an explicit formula can be derived for the diffusion constant $\alpha_{d^{in},d^{out}}$

For instance, if $d^{in} = d^{out}$ (Eulerian graph), then

$$\alpha_{d^{in},d^{out}} = \left(\frac{m_2}{m_1^2} - 1 + \sqrt{1 - \frac{1}{m_1}} \right)^{-1},$$

where m_1, m_2 are the first and second moment of the limit of the empirical degree distribution

The contact process

Given a connected graph $G = (V, E)$, each vertex is associated with a state 0 (healthy) or 1 (infected). The contact process is a Markov process $(\xi_t)_{t \geq 0}$ on the state space $\{0, 1\}^V$.

Each infected vertex becomes healthy at rate 1, independently of the states of the other vertices, while each healthy vertex becomes infected at rate λ times the number of infected neighbours, with $\lambda \in (0, \infty)$ the infection rate

The configuration at time t is $\xi_t = \{\xi_t(i) : i \in V\}$, with $\xi_t(i)$ the state at time t of vertex i

Note that [0] is a trap for the dynamics: if all sites are healthy, then no infection will ever occur. The focus will be on understanding how the extinction time

$$\tau_{[0]} = \inf\{t \geq 0 : \xi_t(i) = 0 \, \forall \, i \in V\}$$

behaves as $|V| = n \to \infty$, depending on the value of λ and the properties of the graph. In particular, we will look at the average extinction time starting from the configuration where every vertex is infected $\mathbb{E}_{[1]}[\tau_{[0]}]$

The contact process is much harder than the voter model because it does not have a nice dual. In fact, it is self-dual

The contact process on the complete graph

The number of infections at time t, given by

$$I_t = \sum_{i \in V} \xi_t(i),$$

performs a continuous-time nearest-neighbour random walk on the set $\{0, 1, \ldots, n\}$ with transition rates

$$\begin{aligned} m &\to m+1 \quad \text{at rate } \lambda m(n-m) \\ m &\to m-1 \quad \text{at rate } m \end{aligned}$$

Write $\mathcal{I}_t = \frac{1}{n} I_t$ for the fraction of infections at time t. This process is continuous-time nearest-neighbour random walk on the set $\left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}$ with transition rates

$$\begin{aligned} x &\to x + n^{-1} \quad \text{at rate } \lambda x(1-x)n^2 \\ x &\to x - n^{-1} \quad \text{at rate } xn \end{aligned}$$

This process has a strong drift upward, which becomes zero for $x = 1 - \frac{1}{\lambda n}$, i.e., very close to full infection when $\lambda n \gg 1$

Let $\tau_{[0]}$ be the extinction time. Then

$$\log \mathbb{E}_{[1]}[\tau_{[0]}] = n(1 + \log(\lambda n)) + o(n), \quad n \to \infty$$

The contact process on the complete graph is supercritical for all $\lambda > 0$ as $n \to \infty$

For any $\lambda \in (0, \infty)$,

$$\lim_{n\to\infty} \mathbb{P}_{[1]}\left(\frac{\tau_{[0]}}{\mathbb{E}_{[1]}[\tau_{[0]}]} > t\right) = \mathrm{e}^{-t} \quad \forall\, t > 0 \qquad \text{metastability}$$

In the mean-field setting of the complete graph, the fraction of infected vertices performs a random walk.

The latter property fails on non-complete graphs, in particular, on random graphs: $(\mathcal{I}_t)_{t\geq 0}$ loses the Markov property

- Chatterjee, Durrett 2009, Mountford, Mourrat, Valesin, Yao 2016: the contact process on the configuration model with power law degree distribution is supercritical regardless of the value of λ
- Can, Schapira 2015, Mourrat, Valesin, Yao 2013, Linker, Mitsche, Schapira, Valesin 2020: the case $\tau \in (1, 2]$ is included
- Cator, Don 2021: contact process on configuration model with i.i.d. degrees

Limitations of static networks

So far, the underlying graph is fixed in time. This assumption is often unrealistic:

- Social contacts change over time
- Individuals move and form new connections
- Interaction patterns are inherently dynamic

Two coupled dynamics

In many systems, we have two interacting processes:

- State dynamics:
  - opinions, infections, activity
- Network dynamics:
  - edges appear and disappear
  - connections are resampled

These two levels influence each other:

- Who you meet affects your state
- Your state may affect whom you meet

This leads to a co-evolutionary feedback between vertex and network dynamics