Elements of Mathematical Foundations of Quantum Mechanics

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Last update Nov 14, 2024

1. Preliminaries

1.1 Brief Summary on Measures

Definition 1.1 (σ-algebra)**.** Given a set X, we denote its power set[1] by $P(X)$ and $\Sigma \subseteq P(X)$ is called a *σ-algebra* if

- $X \in \Sigma$;
- given $A \in \Sigma$, then $X \setminus A \in \Sigma$;
- given $\{A_n\}_{n\in\mathbb{N}} \subset X$, then $\bigcup_{n\in\mathbb{N}} A_n \in \Sigma$.

Remark 1.1. *From the definition of a σ-algebra we also have*

- $\emptyset \in \Sigma$;
- *given* $\{A_n\}_{n\in\mathbb{N}} \subset X$, *then by* De Morgan's law $X \setminus \bigcup_{n\in\mathbb{N}} (X \setminus A_n) = \bigcap_{n\in\mathbb{N}} A_n \in \Sigma$.

The smallest possible σ-algebra on X is $\{X, \emptyset\}$, while the largest is $P(X)$.

Definition 1.2 (Borel σ-algebra)**.** Given (X, τ) a topological space, the *Borel σ-algebra* of X, denoted by $\mathfrak{B}(X)$, is the smallest σ-algebra containing the topology (*i.e.* the open sets)

$$\mathfrak{B}(X) = \bigcap_{\substack{\mathscr{F}(X)\ \sigma\text{-algebra on } X:\\ \mathscr{F}(X) \supseteq \tau}} \mathscr{F}(X).$$

Definition 1.3 (Measure)**.** Given a set X and a σ-algebra Σ on it, a map $\mu \colon \Sigma \longrightarrow [0, +\infty]$ is said a (unsigned) *measure* if

- $\exists\, E \in \Sigma : \quad \mu(E) < +\infty$;
- given $\{E_n\}_{n\in\mathbb{N}} \subset \Sigma$ pairwise disjoint, one has

$$\mu(E) = \sum_{n\in\mathbb{N}} \mu(E_n), \qquad \text{with } E = \bigcup_{n\in\mathbb{N}} E_n \in \Sigma. \qquad \longleftarrow \quad \sigma\text{-additivity}$$

(X, Σ, μ) is said a *measure space* and the elements of Σ are called measurable sets.
In case $\Sigma = \mathfrak{B}(X)$, μ is said a *Borel measure*.
A measure space is said *complete* if for any $N \in \Sigma$ s.t. $\mu(N) = 0$ one has $\{E \subset X \mid E \subset N\} \subset \Sigma$.
In case $\mu(X) = 1$, μ is called a *probability measure* and (X, Σ, μ) a *probability space*.

Remark 1.2. *From the definition of measure we deduce*

[1] The power set of a set is the set of all its subsets, including itself and $\emptyset$.

- $\mu(\emptyset) = 0$;
- $E_1, E_2 \in \Sigma$ *with* $E_1 \subseteq E_2$ *implies* $\mu(E_1) \leq \mu(E_2)$ $\longleftarrow$ *monotonicity*
- *given* $E_1, E_2 \in \Sigma$ *one has* $\mu(E_1 \cap E_2) + \mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$;
- *given* $\{E_n\}_{n\in\mathbb{N}}, \{F_n\}_{n\in\mathbb{N}} \subset \Sigma$ *with* $E_n \subseteq E_{n+1}$ *and* $F_n \supseteq F_{n+1}$, *one has*

$$\mu\Big(\bigcup_{n\in\mathbb{N}} E_n\Big) = \lim_{n\to\infty} \mu(E_n), \qquad \mu\Big(\bigcap_{n\in\mathbb{N}} F_n\Big) = \lim_{n\to\infty} \mu(F_n).$$

Definition 1.4. Given a measure space (X, Σ, μ), the measure μ is said

- *finite*, if $\mu(X) < +\infty$;
- *σ-finite*, if X can be covered with at most countably many sets in Σ with finite measure;
- if X is a Hausdorff topological space[2], μ is said *locally finite*, if for any $p \in X$ there exists $G \in \Sigma$ open such that $p \in G$ and $\mu(G) < +\infty$.

Definition 1.5 (Regular Measure)**.** Given the measure space $(X, \mathfrak{B}(X), \mu)$, the Borel measure μ is said *inner regular* (or tight[3]) if

$$\mu(E) = \sup_{K\subseteq E} \big\{\mu(K) \,\big|\, K \in \Sigma \text{ and } K \text{ compact}\big\}, \qquad \forall\, E \in \Sigma.$$

It is *outer regular* if

$$\mu(E) = \inf_{G\supseteq E} \big\{\mu(G) \,\big|\, G \in \Sigma \text{ and } G \text{ open}\big\}, \qquad \forall\, E \in \Sigma.$$

The measure μ is **regular** if it is both inner and outer regular.

Definition 1.6 (Radon Measure)**.** Let $(X, \mathfrak{B}(X), \mu)$ be a measure space with X Hausdorff topological space. The Borel measure μ is said **Radon** if it is locally finite and tight.

Proposition 1.1. *Let X be a proper metric space (*i.e.* any finite closed ball is compact) and μ a σ-finite Borel measure on $\mathfrak{B}(X)$. Then, μ is regular.*

Definition 1.7. Given a set X with a σ-algebra $\Sigma \subseteq P(X)$ on it, we provide two distinct measures $\mu, \nu \colon \Sigma \longrightarrow [0, +\infty]$.

- μ and ν are (mutually) **singular** ($\mu \perp \nu$), if there exists $N \in \Sigma$ s.t. $\mu(N) = \nu(X \setminus N) = 0$;
- ν is **absolutely continuous** with respect to μ ($\nu \ll \mu$), if $\mu(A) = 0$ implies $\nu(A) = 0$.

Proposition 1.2. *Suppose μ and ν are tight Borel measures. Then, $\nu \ll \mu$ if and only if $\mu(N) = 0$ implies $\nu(N) = 0$ for every compact set N.*

[2]Namely, given two distinct points in the space, they have two respective neighbourhoods which are disjoint.

[3]Some authors distinguish the inner regularity from tightness by requiring measurable sets to be arbitrary close (in measure) to some closed set (inner regularity) or compact set (tightness).

Definition 1.8 (Measurability)**.** Given a measure space (X, Σ_X, μ) and a topological space Y, a function $f: X \longrightarrow Y$ is said *μ-measurable* if for any $A \in \mathfrak{B}(Y)$ one has

$$f^{-1}(A) := \{x \in X \mid f(x) \in A\} \in \Sigma_X.$$

In case $\Sigma_X = \mathfrak{B}(X)$, a μ-measurable function f is called a **Borel function**.

Definition 1.9 (Continuous function)**.** A function $f : X \longrightarrow Y$ between X and Y topological spaces is said **continuous** at $x \in X$ if for any open set $G \subset Y$ such that $f(x) \in G$ one has $f^{-1}(G)$ is a open set in X.

Remark 1.3. *A continuous function* (i.e. *continuous at every point) is also a Borel function, but not the converse. Consider for instance the* Dirichlet function *(which assigns* 1 *to rational numbers and* 0 *to irrational numbers). For any open $G \subseteq \mathbb{R}$ one has*

$$f^{-1}(G) = \begin{cases} \emptyset, & \text{if } \{0,1\} \cap G = \emptyset; \\ \mathbb{Q}, & \text{if } 1 \in G \wedge 0 \notin G; \\ \mathbb{R} \smallsetminus \mathbb{Q} & \text{if } 0 \in G \wedge 1 \notin G; \\ \mathbb{R} & \text{if } \{0,1\} \subset G. \end{cases}$$

In all cases, the preimage is a Borel set ($\mathbb{Q}$ is a countable union of points, i.e. *closed sets). Hence we exhibited a non-continuous Borel function.*

Definition 1.10 (Support)**.** Given X topological space and the measure space $(X, \mathfrak{B}(X), \mu)$, consider a μ-measurable function $f: X \longrightarrow \mathbb{C}$. One defines its *(essential) support* as the closed set

$$\operatorname{supp} f := X \smallsetminus \bigcup_{G \in \mathcal{N}_f} G, \qquad \mathcal{N}_f := \left\{G \subseteq X \mid G \text{ open}, f = 0 \ \mu-a.e.^4 \text{ in } G\right\}.$$

Proposition 1.3. *Let (X, Σ_X, μ) be a measure space and Y, Z two topological spaces. Suppose $f: X \longrightarrow Y$ is a μ-measurable function and $g: Y \longrightarrow Z$ a Borel function. Then, $g \circ f: X \longrightarrow Z$ is μ-measurable as well.*

Remark 1.4. *In particular, a continuous function composed with a measurable function f is measurable,* e.g. $|f|$, *whereas the composition of Borel functions is again a Borel function.*

Proposition 1.4. *Sums and products of two complex-valued measurable functions are measurable.*

Proposition 1.5. *Suppose $f_n: X \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ is a sequence of measurable functions[5]. Then*

$$\inf_{n \in \mathbb{N}} f_n(x), \qquad \sup_{n \in \mathbb{N}} f_n(x), \qquad \liminf_{n \in \mathbb{N}} f_n(x), \qquad \limsup_{n \in \mathbb{N}} f_n(x)$$

are measurable as well.

[4] A property holds $\mu - a.e.$ (almost everywhere) if it is valid for all points in $X \smallsetminus N$ with $\mu(N) = 0$.
[5] The standard topology of $\mathbb{R} \cup \{\pm\infty\}$ is generated by the base $\{[-\infty, a), (a, b), (b, +\infty] \mid a, b \in \mathbb{R}\}$.

Remark 1.5. *This last result implies that if f and g are measurable, so are* $\max(f,g), \min(f,g)$.

Definition 1.11 (Simple Functions)**.** In a measure space (X, Σ, μ), we say that a measurable function[6] $s\colon X \longrightarrow \mathbb{C}$ is *simple* if its image is finite, namely if there exist a set of disjoint measurable sets $\{A_k\}_{k=1}^n \subset \Sigma$ and values $\{\alpha_k\}_{k=1}^n \subset \mathbb{C}$ for some $n \in \mathbb{N}$ such that

$$s(x) = \sum_{k=1}^n \alpha_k \mathbb{1}_{A_k}(x), \quad \forall x \in X \quad \Longrightarrow \quad s(X) = \{\alpha_k\}_{k=1}^n, \quad s^{-1}(\{p\}) = \begin{cases} A_k, & \text{if } p = \alpha_k; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Definition 1.12 (Lebesgue Integral - Simple functions)**.** Given a measure space (X, Σ, μ) and a non-negative simple function $s\colon X \longrightarrow [0, +\infty]$ we define[7] for all $A \in \Sigma$

$$\int_A d\mu(x)\ s(x) := \sum_{k=1}^n \alpha_k\, \mu(A_k \cap A), \qquad \text{if } s : x \longmapsto \sum_{k=1}^n \alpha_k \mathbb{1}_{A_k}(x).$$

Proposition 1.6. *The integral for non-negative simple functions fulfils*

i) $\displaystyle\int_A d\mu(x)\ s(x) = \int_X d\mu(x)\ \mathbb{1}_A(x)\, s(x), \qquad A \in \Sigma;$

ii) $\displaystyle\int_{\bigcup_{n\in\mathbb{N}} A_n} d\mu(x)\ s(x) = \sum_{n\in\mathbb{N}} \int_{A_n} d\mu(x)\ s(x), \qquad \{A_n\}_{n\in\mathbb{N}} \subset \Sigma;$

iii) $\displaystyle\int_A d\mu(x)\ \left[\alpha\, s_1(x) + \beta\, s_2(x)\right] = \alpha \int_A d\mu(x)\ s_1(x) + \beta \int_A d\mu(x)\ s_2(x), \qquad \alpha, \beta \geq 0,\ A \in \Sigma;$

iv) $A, B \in \Sigma\ :\ A \subseteq B \quad \Longrightarrow \quad \displaystyle\int_A d\mu(x)\ s(x) \leq \int_B d\mu(x)\ s(x);$

v) $s_1 \leq s_2 \quad \Longrightarrow \quad \displaystyle\int_A d\mu(x)\ s_1(x) \leq \int_A d\mu(x)\ s_2(x), \qquad A \in \Sigma.$

Definition 1.13 (Lebesgue Integral - Non-negative Measurable functions)**.** Given a measure space (X, Σ, μ) and a non-negative measurable function $f\colon X \longrightarrow [0, +\infty]$ we define

$$\int_A d\mu(x)\ f(x) := \sup_{\substack{s \text{ simple functions}:\\ 0 \leq s \leq f}} \int_A d\mu(x)\ s(x), \qquad A \in \Sigma.$$

Theorem 1.7 (Beppo-Levi, Monotone convergence)**.** *Let $f_n : X \longrightarrow [0, +\infty)$ be a monotone non-decreasing sequence of non-negative measurable functions such that $f_n \xrightarrow[n\to\infty]{} f$ pointwise. Then,*

$$\lim_{n\to\infty} \int_A d\mu(x)\ f_n(x) = \int_A d\mu(x)\ f(x), \qquad \forall\, A \in \Sigma.$$

Corollary 1.8. *For any non-negative measurable function f there always exists a monotone non-decreasing sequence of non-negative simple functions $s_n\colon X \longrightarrow [0, +\infty)$ such that $s_n \xrightarrow[n\to\infty]{} f$ pointwise. Additionally, proposition 1.6 holds also for non-negative measurable functions.*

[6] The characteristic function $\mathbb{1}_A$ is μ-measurable iff $A \in \Sigma$.

[7] Here we adopt the convention $0 \cdot (+\infty) = 0$.

Definition 1.14 (Lebesgue Integral)**.** For any real-valued measurable function $f\colon X \longrightarrow \mathbb{R}$, s.t.

$$\int_X d\mu(x)\ \max\{f(x),0\} < +\infty, \quad \text{and} \quad \int_X d\mu(x)\ \max\{-f(x),0\} < +\infty,$$

one can define for any $A \in \Sigma$

$$\int_A d\mu(x)\ f(x) := \int_A d\mu(x)\ \max\{f(x),0\} - \int_A d\mu(x)\ \max\{-f(x),0\}.$$

Similarly, in case f is complex-valued one can consider separately the integration of the real and the imaginary part of f. We say f is **integrable** in case $\int_X d\mu(x)\ |f(x)| < +\infty$.

Proposition 1.9. *Apart from point iv), proposition 1.6 holds also for integrable functions. Additionally, given $f,g\colon X \longrightarrow \mathbb{C}$ integrable one has for all $A \in \Sigma$*

$$\left|\int_A d\mu(x)\ f(x)\right| \leq \int_A d\mu(x)\ |f(x)|.$$

Proposition 1.10. *Let $f\colon X \longrightarrow \mathbb{C}$ be measurable. Then*

$$\int_X d\mu(x)\ |f(x)| = 0 \quad \Longleftrightarrow \quad f(x) = 0 \quad \mu\text{-}a.e.$$

Moreover, in case f is either non-negative or integrable

$$\mu(A) = 0 \quad \Longrightarrow \quad \int_A d\mu(x)\ f(x) = 0.$$

Remark 1.6. *Notice that the integral does not change if we add to the domain of integration a set of zero measure or if we modify the value of the integrand along a set of zero measure. In particular, two functions equal a.e. have the same integral.*

Theorem 1.11 (Generalized Fatou's lemma)**.** *If $f_n\colon X \longrightarrow \mathbb{R}$ is a sequence of real-valued measurable functions and $g\colon X \longrightarrow \mathbb{R}$ some integrable function, then for all $A \in \Sigma$*

$$\int_A d\mu(x)\ \liminf_{n\to\infty} f_n(x) \leq \liminf_{n\to\infty} \int_A d\mu(x)\ f_n(x), \qquad \text{if } f_n \geq g;$$
$$\limsup_{n\to\infty} \int_A d\mu(x)\ f_n(x) \leq \int_A d\mu(x)\ \limsup_{n\to\infty} f_n(x), \qquad \text{if } f_n \leq g.$$

Theorem 1.12 (Fatou-Lebesgue, Uniform Dominated Convergence)**.** *Let $f_n\colon X \longrightarrow \mathbb{C}$ be a sequence of complex-valued measurable functions such that $f_n \xrightarrow[n\to\infty]{} f$ pointwise for some measurable $f\colon X \longrightarrow \mathbb{C}$. Then, if there exists $g\colon X \longrightarrow \mathbb{C}$ integrable such that $|f_n| \leq g$, one has for all $A \in \Sigma$*

$$\lim_{n\to\infty} \int_A d\mu(x)\ |f_n(x) - f(x)| = 0, \quad \text{hence} \quad \lim_{n\to\infty} \int_A d\mu(x)\ f_n(x) = \int_A d\mu(x)\ f(x).$$

Theorem 1.13 (Riesz-Markov-Kakutani). *Let X be a locally compact*[8] *Hausdorff space and a functional*[9] $\ell : C_c(X, \mathbb{C}) \longrightarrow \mathbb{C}$ *such that* $\ell(f) \geq 0$ *if* $f \geq 0$. *Then, there exists a unique Borel measure* μ *satisfying*

$$\ell(f) = \int_X d\mu(x)\ f(x), \qquad \forall\, f \in C_c(X, \mathbb{C}),$$

with μ σ*-finite, outer regular and inner regular for the open sets (or for the Borel sets with finite measure) and such that* $(X, \mathfrak{B}(X), \mu)$ *is complete.*

Remark 1.7. *In case* ℓ *is the Riemann integral for (piece-wise) continuous functions, the previous theorem gives rise to the definition of a particular measure* μ *called* **Lebesgue measure**. *This tells us that all tools from calculus like integration by parts or integration by substitution are readily available for the Lebesgue integral on* $\mathbb{R}$.

Theorem 1.14. *Given a non-decreasing,* right-continuous[10] *function* $f : \mathbb{R} \longrightarrow \mathbb{R}$, *there exists a unique regular Borel measure* $\mu_f : \mathfrak{B}(\mathbb{R}) \longrightarrow [0, +\infty]$ *satisfying*

$$\mu_f\big((a, b]\big) = f(b) - f(a), \quad a < b \quad \text{and} \quad \mu_f(\{x\}) = f(x) - \lim_{\epsilon \to 0^+} f(x - \epsilon).$$

Two distinct functions provide the same measure iff they differ by a constant.

Remark 1.8. *In the previous theorem,* μ_f *is called a* Lebesgue-Stieltjes measure. *Notice that the value of the measure* μ_f *at the singleton* $\{x\}$ *is zero iff* f *is continuous at* x. *Moreover, in case* $f : x \longmapsto x$, *then* μ_f *is the Lebesgue measure. Additionally, suppose* f *to be the Heaviside step function (equal to* 1 *for non-negative* x *and to* 0 *in case* x *negative). Then,* μ_f *in this case is the* Dirac measure *at* 0, *since* $\mu_f(A) = 1$ *in case* $0 \in A$, *Borel set, while* $\mu_f(A) = 0$ *otherwise.*

Proposition 1.15. *Let* (X, Σ, μ) *be a measure space and* $Y \subseteq \mathbb{R}$. *Then, consider* $f : X \times Y \longrightarrow \mathbb{C}$ *s.t.* $x \longmapsto f(x, y)$ *is integrable* $\forall\, y \in Y$ *and* $y \longmapsto f(x, y)$ *is differentiable* $\mu-$*a.e. There holds*

$$F(y) = \int_A d\mu(x)\ f(x, y)$$

is differentiable in Y *if there exists an integrable function* $g : X \longrightarrow \mathbb{C}$ *s.t.* $\left|\frac{\partial}{\partial_y} f(\cdot, y)\right| \leq g$. *Moreover,* $x \longmapsto \frac{\partial}{\partial_y} f(x, y)$ *is* μ*-measurable* $\forall\, y \in Y$ *and*

$$\tfrac{d}{dy} F(y) = \int_A d\mu(x)\ \tfrac{\partial}{\partial_y} f(x, y).$$

Theorem 1.16 (Radon-Nikodym). *Let* $\mu, \nu : \Sigma \longrightarrow [0 + \infty]$ *two* σ*-finite measures. One has* $\nu \ll \mu$ *iff there exists a non-negative measurable function* $f : X \longrightarrow [0, +\infty)$ *such that*

$$\nu(A) = \int_A d\mu(x)\ f(x), \qquad \forall\, A \in \Sigma.$$

The function f *is determined uniquely* $\mu-$ *a.e. and is called the* **Radon-Nikodym derivative** $\frac{d\nu}{d\mu}$ *of* ν *with respect to* μ.

[8] For any point $p \in X$ there exist G open and K compact such that $p \in G \subset K$.

[9] We denote by $C_c(X; \mathbb{C})$ the set of complex-valued continuous functions on X with compact support.

[10] For a right-continuous function there holds $f(x) = \lim_{\epsilon \to 0^+} f(x + \epsilon)$ for all $x \in \mathbb{R}$.

Theorem 1.17 (Lebesgue Decomposition)**.** *Let $\mu, \nu : \Sigma \longrightarrow [0, +\infty]$ be two σ-finite measures. Then, ν can be uniquely decomposed as $\nu = \nu_{\mathrm{ac}} + \nu_{\mathrm{sing}}$ with $\nu_{\mathrm{ac}} \ll \mu$ and $\nu_{\mathrm{sing}} \perp \mu$.*

Theorem 1.18 (Refinement of Lebesgue Decomposition)**.** *Let $\lambda : \mathfrak{B}(X) \longrightarrow [0, +\infty]$ be the Lebesgue measure on $X \subseteq \mathbb{R}$ and ν another regular measure on X. Then, $\nu = \nu_{\mathrm{ac}} + \nu_{\mathrm{sc}} + \nu_{\mathrm{pp}}$ with $\nu_{\mathrm{ac}} \ll \lambda$, while $\nu_{\mathrm{sc}} \perp \lambda, \nu_{\mathrm{pp}} \perp \lambda$ and $\nu_{\mathrm{sc}} \perp \nu_{\mathrm{pp}}$, where ν_{pp} is a pure-point measure,* i.e. *$\nu_{\mathrm{pp}} = \sum_{n \in \mathbb{N}} a_n \delta_{x_n}$, where $a_n \geq 0$ and δ_{x_n} is the Dirac measure centered at $x_n \in X$.*

Remark 1.9. *Last theorem highlights that from one hand ν_{pp} is a* discrete measure *(or* pure point *measure), while ν_{sc} must be continuous (since $\nu_{\mathrm{sc}} \perp \nu_{\mathrm{pp}}$), namely it has non-zero values only on uncountable sets of Lebesgue measure zero (since $\nu_{\mathrm{sc}} \perp \lambda$). An example of a singularly continuous measure is the* Cantor measure.

Proposition 1.19. *Let μ, ν, λ three σ-finite measures on the same σ-algebra.*

- *If $\nu \ll \lambda$ and $\mu \ll \lambda$ one has $\frac{d(\nu+\mu)}{d\lambda} = \frac{d\nu}{d\lambda} + \frac{d\mu}{d\lambda}, \quad \lambda -$ a.e.*
- *If $\nu \ll \mu \ll \lambda$ one has $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}, \quad \lambda -$ a.e.* $\longleftarrow$ *chain rule*

In particular, in case $\nu \ll \mu$ and $\mu \ll \nu$, we have

$$\frac{d\nu}{d\mu} = \left(\frac{d\mu}{d\nu}\right)^{-1}, \quad \mu - a.e.$$

1.2 Banach Spaces

Definition 1.15 (Normed Space)**.** Given X a vector space over $\mathbb{C}$, we say that $(X, \|\cdot\|_X)$ is a **normed space** if the space X is equipped with a *norm*, *i.e.* a map $\|\cdot\|_X : X \longrightarrow [0, +\infty)$ fulfilling

i) $\|f\|_X = 0 \iff f = 0,$

ii) $\|\alpha f\|_X = |\alpha| \, \|f\|_X \,, \qquad \forall\, \alpha \in \mathbb{C},$

iii) $\|f + g\|_X \leq \|f\|_X + \|g\|_X \,.$

If condition 1. does not hold $\|\cdot\|_X$ is said a *semi-norm*.

Proposition 1.20 (Inverse Triangular Inequality)**.** *Given a normed space X, for any $f, g \in X$ there holds*

$$\|f - g\|_X \geq |\, \|f\|_X - \|g\|_X|.$$

Remark 1.10. *Every normed space X can be understood as a* metric space*, by introducing the distance induced by the norm*

$$\mathrm{dist}(f, g) := \|f - g\|_X \,.$$

As a consequence, one can induce a topology *on X as the set of the open balls*

$$\tau = \{\mathcal{B}_R(f_0) \mid f_0 \in X,\, R > 0\}, \qquad \mathcal{B}_R(f_0) := \{f \in X \mid \|f - f_0\|_X < R\}\,.$$

Notice that any metric space is a Hausdorff *topological space.*

Remark 1.11. *By the means of remark 1.10, a normed space is naturally equipped with a notion of* convergence. *Given $f \in X$ and a sequence $\{f_n\}_{n\in\mathbb{N}} \subset X$ converging to f in X as n grows means*

$$\|f_n - f\|_X \xrightarrow[n\to\infty]{} 0.$$

Notice that one can adopt the same notion of convergence for a semi-normed space, but the limit is not unique in that case.

Proposition 1.21. *Given $\{f_n\}_{n\in\mathbb{N}} \subset X$ a Cauchy sequence in the normed space X one has $\{\|f_n\|_X\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ is a converging sequence.*

Definition 1.16 (Banach Space)**.** A normed space which is complete according to its norm is said a **Banach space**, namely, every *Cauchy sequence*[11] admits limit in X.

Remark 1.12. *Owing to Proposition 1.20, the norm is continuous in a Banach space, namely*

$$\forall\,\{f_n\}_{n\in\mathbb{N}} \subset X \;\text{ s.t. }\; f_n \xrightarrow[n\to\infty]{} f \in X, \quad \text{one has} \quad \|f_n\|_X \xrightarrow[n\to\infty]{} \|f\|_X\,.$$

However, the converse is false: a sequence of vectors whose norm converge is not necessary convergent in X.

Definition 1.17 (Density)**.** Let D be a proper subset of X, Banach. D is said to be **dense** in X if

$$\forall\, f \in X \quad \exists\,\{f_n\}_{n\in\mathbb{N}} \subset D \text{ s.t. } f_n \xrightarrow[n\to\infty]{} f.$$

Consider a complex vector space X endowed with two distinct norms $\|\cdot\|_1$ and $\|\cdot\|_2$. If there exists $c > 0$ such that $\|f\|_1 \le c\,\|f\|_2$ for any $f \in X$, we say that $\|\cdot\|_2$ is *stronger* than $\|\cdot\|_1$. In particular, a Cauchy sequence in $(X, \|\cdot\|_2)$ is also Cauchy in $(X, \|\cdot\|_1)$ and any dense subspace of $(X, \|\cdot\|_2)$ is also dense in $(X, \|\cdot\|_1)$. Two norms are called *equivalent* if there exists $c > 1$ s.t.

$$\frac{1}{c}\,\|f\|_2 \le \|f\|_1 \le c\,\|f\|_2\,, \qquad \forall\, f \in X.$$

Theorem 1.22. *Given X finite-dimensional normed space, all norms defined on X are equivalent.*

Proposition 1.23 (Absolute Convergence)**.** *Let X be a Banach space and suppose $\{f_n\}_{n\in\mathbb{N}} \subset X$ be a sequence satisfying $\sum_{n\in\mathbb{N}} \|f_n\|_X < +\infty$. Then $\sum_{n\in\mathbb{N}} f_n := \lim_{N\to\infty} \sum_{n=1}^{N} f_n$ exists.*

[11] All convergent sequences are also Cauchy, but the converse is not trivial in general!

Definition 1.18 (Completion)**.** Let X be an incomplete normed space and denote with X_C, X_{C_0} the vector spaces of the Cauchy sequences in X and the Cauchy sequences in X converging to 0, respectively. One defines $\bar{X}$ the **completion** of X according to its norm $\|\cdot\|_X$ as

$$\bar{X} := {}^{X_C}\!/_{X_{C_0}},$$

namely, we identify Cauchy sequences whose difference converges to zero. Additionally, given $\bar{x} = [\mathrm{c}] \in \bar{X}$, where $\mathrm{c} = \{c_n\}_{n\in\mathbb{N}} \in X_C$ stands for a representative of the equivalence class, the map $|\!|\!|\bar{x}|\!|\!| := \lim_{n\to\infty} \|c_n\|_X$ defines a norm in $\bar{X}$ (which does not depend on the equivalence class) and one has that $(\bar{X}, |\!|\!|\cdot|\!|\!|)$ is a Banach space.

Linear Operators

Definition 1.19 (Linear maps)**.** Given X, Y two normed spaces, we denote by $\mathscr{L}(X, Y)$ the set of linear maps between a subset of X and Y, namely $A\colon \mathfrak{D}(A) \subseteq X \longrightarrow Y$

$$A \in \mathscr{L}(X, Y) \iff A(\alpha f + \beta g) = \alpha(Af) + \beta(Ag) \in Y, \qquad \forall\, \alpha, \beta \in \mathbb{C},\ f, g \in \mathfrak{D}(A).$$

$\mathscr{L}(X)$ shall be a shortcut for $\mathscr{L}(X, X)$. For any $A \in \mathscr{L}(X, Y)$ we denote by

$\mathfrak{D}(A) \subseteq X$ the **domain** of A, namely a linear subset in which A is well-defined,
$\operatorname{ran}(A)$ the **range** of A, namely $\operatorname{ran}(A) := \{g \in Y \mid \exists\, f \in \mathfrak{D}(A) \text{ s.t. } Af = g\}$,
$\ker(A) := \{f \in \mathfrak{D}(A) \mid Af = 0\}$ denotes the **kernel** (or null space) of A.

Moreover, the vector space $\mathscr{L}(X, Y)$ can be equipped with the **operator norm** defined as follows

$$\|A\|_{\mathscr{L}(X,Y)} := \sup_{\substack{f\in\mathfrak{D}(A):\\ \|f\|_X=1}} \|Af\|_Y .$$

Definition 1.20 (Bounded Operators)**.** We denote by $\mathscr{B}(X, Y)$ the space of linear maps between a subspace of X and Y (both normed spaces), which are bounded according to the norm $\|\cdot\|_{\mathscr{L}(X,Y)}$. Also in this case, $\mathscr{B}(X)$ shall correspond to $\mathscr{B}(X, X)$.

Theorem 1.24. *Given X a finite-dimensional normed space, one has $\mathscr{L}(X, Y) = \mathscr{B}(X, Y)$.*

Proposition 1.25. *$\mathscr{B}(X, Y)$ is a Banach space with respect to $\|\cdot\|_{\mathscr{L}(X,Y)}$ if Y is Banach.*

According to the previous definition we have that the space $\mathscr{B}(X, \mathbb{C})$ corresponds to the set of linear and bounded **functionals** defined on X. This space is called the *dual* of X and it is also sometimes denoted by X^*. Moreover, proposition 1.25 implies X^* is a Banach, even if X is not.

Remark 1.13. *Given X a normed space, another natural notion of convergence in X arises at this point. We say that a sequence $\{f_n\}_{n\in\mathbb{N}} \subset X$* **converges weakly** *to $f \in X$ if*

$$|\ell(f_n) - \ell(f)| \xrightarrow[n\to\infty]{} 0, \qquad \forall\, \ell \in X^*.$$

In this case the so-called weak topology *is naturally induced*

$$\tau_w = \{\mathcal{B}_R^w(f_0) \mid f_0 \in X,\ R > 0\}, \qquad \mathcal{B}_R^w(f_0) := \{f \in X \mid |\ell(f) - \ell(f_0)| < R, \quad \forall\, \ell \in X^*\}.$$

Clearly, this is a weaker notion of convergence, since for all $\ell \in X^*$

$$\|f_n - f\|_X \xrightarrow[n\to\infty]{} 0 \qquad \Longrightarrow \qquad |\ell(f_n) - \ell(f)| \le \|\ell\|_{\mathscr{L}(X,\mathbb{C})} \|f_n - f\|_X \xrightarrow[n\to\infty]{} 0.$$

Proposition 1.26. *Given* $A \in \mathscr{L}(Z,Y)$ *and* $B \in \mathscr{L}(X,Z)$ *one has* $AB := A \circ B \in \mathscr{L}(X,Y)$ *and*

$$\|AB\|_{\mathscr{L}(X,Y)} \le \|A\|_{\mathscr{L}(Z,Y)} \|B\|_{\mathscr{L}(X,Z)}\,.$$

In particular, one has that *the product of two bounded operators is bounded.*

Definition 1.21 (Banach Algebra)**.** A given X Banach space equipped with a product is called a **Banach algebra** if such a product fulfils for any $a, b, c \in X$

- associativity: $(ab)c = a(bc), \quad \alpha(ab) = (\alpha a)b = a(\alpha b), \quad \forall\, \alpha \in \mathbb{C}$;
- distributivity: $(a+b)c = ac + bc, \quad a(b+c) = ab + ac$;
- continuity: $\|ab\|_X \le \|a\|_X \|b\|_X$.

Notice that this product is *not commutative* in general: $ab \neq ba$.

An example of Banach algebra is the space of bounded operators $\mathscr{B}(X)$ with X Banach, endowed with the product by composition. This kind of Banach algebra has an identity element

$$\mathbb{1}_X : f \longmapsto f, \quad \|\mathbb{1}_X\|_{\mathscr{L}(X)} = 1.$$

Proposition 1.27. *Let the function* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *be represented by a power series with radius of convergence*[12] $R > 0$

$$f(z) = \sum_{j\in\mathbb{N}_0} f_j z^j, \qquad |z| < R.$$

Moreover, let $A \in \mathscr{B}(X)$ *be s.t.* $\|A\|_{\mathscr{L}(X)} < R$*. Then*[13]*, because of propositions 1.23 and 1.26*

$$\sum_{j\in\mathbb{N}_0} f_j A^j =: f(A) \in \mathscr{B}(X)\,.$$

Definition 1.22 (Invertibility)**.** We call a densely-defined, injective map $A \in \mathscr{L}(X,Y)$

- **invertible** if $\mathrm{ran}(A)$ is dense in Y. In this case there exists a unique densely-defined injective map $A^{-1} \in \mathscr{L}(Y,X)$ such that $\mathfrak{D}(A^{-1}) = \mathrm{ran}(A)$, $\mathrm{ran}(A^{-1}) = \mathfrak{D}(A)$ and

$$AA^{-1}\psi = \psi, \quad \forall\, \psi \in \mathrm{ran}(A), \qquad A^{-1}A\phi = \phi, \qquad \forall\, \phi \in \mathfrak{D}(A);$$

- **boundedly invertible** if there exists a unique injective operator $A^{-1} \in \mathscr{B}(Y,X)$ such that $\mathrm{ran}(A^{-1}) = \mathfrak{D}(A)$ and

$$AA^{-1} = \mathbb{1}_Y, \qquad A^{-1}A\phi = \phi, \qquad \forall\, \phi \in \mathfrak{D}(A).$$

[12] We remind that such a series converges absolutely if z is in a compact contained in the open disk of radius R.
[13] A map in $\mathscr{L}(X)$ raised to the zero power is by definition the identity map $\mathbb{1}_X$.

Proposition 1.28. *Given $A \in \mathscr{L}(X,Y)$ densely-defined, one has*

i) A *is* injective *iff* $\ker(A) = \{0\}$. *Moreover,* $\ker(A)$ *is closed according to the topology induced by* $\|\cdot\|_X$ *if* $A \in \mathscr{B}(X,Y)$*;*

ii) A *is boundedly invertible iff* $\mathrm{ran}(A)$ *is dense and there is some* $c > 0$ *s.t.* $\inf_{\psi \in \mathfrak{D}(A)} \|A\psi\|_Y \geq c$*;*

iii) A *is boundedly invertible if there exists some* $B \in \mathscr{L}(X,Y)$ *boundedly invertible satisfying*

$$\sup_{\substack{\psi \in \mathfrak{D}(A) \cap \mathfrak{D}(B): \\ \|\psi\|_X = 1}} \|A\psi - B\psi\|_Y \left\|B^{-1}\right\|_{\mathscr{L}(Y,X)} < 1.$$

Definition 1.23 (Continuity)**.** $A \in \mathscr{L}(X,Y)$ is **continuous** if, given $\{f_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}(A)$, one has

$$f_n \xrightarrow[n\to\infty]{} f \text{ in } X, \text{ with } f \in \mathfrak{D}(A) \quad \Longrightarrow \quad Af_n \xrightarrow[n\to\infty]{} Af \text{ in } Y.$$

Proposition 1.29 (Boundedness is Continuity)**.** *A linear map is continuous iff it is bounded.*

Definition 1.24. Given $A, \tilde{A} \in \mathscr{L}(X,Y)$ we say that $\tilde{A}$ is an **extension** of A if

- $\tilde{A}f = Af, \qquad \forall\, f \in \mathfrak{D}(A)$;
- $\mathfrak{D}(A) \subseteq \mathfrak{D}(\tilde{A})$.

In this case we denote $A \subseteq \tilde{A}$. It is clear that $A = \tilde{A}$ iff $A \subseteq \tilde{A}$ and $\tilde{A} \subseteq A$.

Theorem 1.30 (BLT - Bounded, Linear Transform)**.** *Given* $A \in \mathscr{B}(X,Y)$*, with* Y *a Banach space and* $\mathfrak{D}(A)$ *dense in* X*, there exists a unique extension* $\tilde{A} \in \mathscr{B}(X,Y)$ *s.t.* $\mathfrak{D}(\tilde{A}) = X$ *and* $\|A\|_{\mathscr{L}(X,Y)} = \|\tilde{A}\|_{\mathscr{L}(X,Y)}$.

Remark 1.14. *According to theorem 1.30, there is no ambiguity in providing a densely-defined and bounded operator, since there exists only one possible norm-preserving extension everywhere-defined.*

Theorem 1.31 (Banach-Steinhaus)**.** *Let* X *be a Banach space and* Y *a normed space. Given a family of bounded operators* $F \subset \mathscr{B}(X,Y)$ *fulfilling* $\sup_{A\in F} \|Af\|_Y < +\infty$ *for any fixed* $f \in X$*, then*

$$\sup_{A\in F} \|A\|_{\mathscr{L}(X,Y)} < +\infty.$$

1.3 Hilbert Spaces

Definition 1.25 (Sesquilinear Forms)**.** Let $\mathfrak{H}$ be a vector space over $\mathbb{C}$. A map $s : \mathfrak{H} \times \mathfrak{H} \longrightarrow \mathbb{C}$ is said a *sesquilinear form* if it is *anti-linear* in its first variable[14] and linear in the second one, *i.e.*

[14] This is the convention adopted by physicists.

- $s(\alpha_1 f_1 + \alpha_2 f_2, g) = \bar{\alpha}_1 s(f_1, g) + \bar{\alpha}_2 s(f_2, g), \qquad \forall\, f_1, f_2, g \in \mathfrak{H},\ \alpha_1, \alpha_2 \in \mathbb{C};$
- $s(f, \alpha_1 g_1 + \alpha_2 g_2) = \alpha_1 s(f, g_1) + \alpha_2 s(f, g_2), \qquad \forall\, f, g_1, g_2 \in \mathfrak{H},\ \alpha_1, \alpha_2 \in \mathbb{C}.$

Definition 1.26 (Inner Product)**.** A sesquilinear form in $\mathfrak{H}$ is said

- **positive** if $\quad s(f,f) > 0, \qquad \forall\, f \in \mathfrak{H} \setminus \{0\};$
- **symmetric** if $\quad s(f,g) = \overline{s(g,f)}, \qquad \forall\, f, g \in \mathfrak{H}.$

A positive and symmetric sesquilinear form $\langle \cdot, \cdot \rangle_{\mathfrak{H}}$ is called **inner product**.

Definition 1.27 (pre-Hilbert Space)**.** A complex vector space that is endowed with an inner product is said a *pre-Hilbert space*.

Notice that any pre-Hilbert space is also normed, since one always has the induced norm

$$\|\cdot\|_{\mathfrak{H}} = \sqrt{\langle \cdot, \cdot \rangle_{\mathfrak{H}}}.$$

Proposition 1.32 (Cauchy-Schwarz-Bunjakowski inequality)**.** *Let $\mathfrak{H}$ be a pre-Hilbert space and $f, g \in \mathfrak{H}$. Then,*

$$|\langle f, g \rangle_{\mathfrak{H}}| \le \|f\|_{\mathfrak{H}} \|g\|_{\mathfrak{H}}.$$

Moreover, equality holds iff $f = \alpha g$ for some $\alpha \in \mathbb{C}$ or $g = 0$.

Theorem 1.33 (Jordan-Von Neumann)**.** *Let X be a normed space. Then, X is a pre-Hilbert space (namely, there exists an inner product associated with $\|\cdot\|_X$) iff the* parallelogram identity *holds,* i.e.

$$\|f+g\|_X^2 + \|f-g\|_X^2 = 2\|f\|_X^2 + 2\|g\|_X^2, \qquad \forall\, f, g \in X.$$

In this case the inner product is defined via the polarization identity

$$\langle f, g \rangle_X := \frac{1}{4}\left(\|f+g\|_X^2 - \|f-g\|_X^2 + i\,\|f-ig\|_X^2 - i\,\|f+ig\|_X^2\right).$$

Definition 1.28 (Hilbert Spaces)**.** Given $\mathfrak{H}$ a pre-Hilbert space, it is called **Hilbert** space if it is complete according to the norm induced by its inner product.

Theorem 1.34 (Riesz)**.** *Given $\mathfrak{H}$ a Hilbert space, for any linear and continuous functional $\ell \in \mathfrak{H}^*$ there exists a unique vector $\varphi_\ell \in \mathfrak{H}$ s.t. for any $\psi \in \mathfrak{H}$*

$$\ell(\psi) = \langle \varphi_\ell, \psi \rangle_{\mathfrak{H}}, \qquad \|\varphi_\ell\|_{\mathfrak{H}} = \|\ell\|_{\mathscr{L}(\mathfrak{H},\mathbb{C})}.$$

Remark 1.15. *Notice that, owing to proposition 1.32, given a convergent sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathfrak{H}$ in a Hilbert space with $f_n \xrightarrow[n\to\infty]{} f \in \mathfrak{H}$, we also have $\lim_{n\to\infty} \langle f_n, g \rangle_{\mathfrak{H}} = \langle f, g \rangle_{\mathfrak{H}}$, namely, the map $f \longmapsto \langle f, g \rangle_{\mathfrak{H}}$ is continuous*[15] *for any $g \in \mathfrak{H}$.*
Moreover, because of theorem 1.34, the weak convergence in a Hilbert space is represented in terms of inner products

$$\lim_{n\to\infty} \ell(\psi_n) = \ell(\psi), \quad \forall\, \ell \in \mathfrak{H}^* \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} \langle \varphi, \psi_n \rangle_{\mathfrak{H}} = \langle \varphi, \psi \rangle_{\mathfrak{H}}, \quad \forall\, \varphi \in \mathfrak{H}.$$

[15] Clearly the same is true for any map sending $g \longmapsto \langle f, g \rangle_{\mathfrak{H}}$ with fixed $f \in \mathfrak{H}$.

We denote a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{H}$ weakly converging to $\psi \in \mathfrak{H}$ in the following way

$$\psi_n \xrightarrow[n\to\infty]{} \psi.$$

Proposition 1.35. *Given $\mathfrak{H}$ Hilbert space, let $\psi \in \mathfrak{H}$ and $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{H}$ s.t. $\psi_n \xrightarrow[n\to\infty]{} \psi$. Then,*

i) $\|\psi\|_{\mathfrak{H}} \leq \liminf_{n\to\infty} \|\psi_n\|_{\mathfrak{H}}$; $\longleftarrow$ *The norm is lower semi-continuous in the weak topology*

ii) $\sup_{n\in\mathbb{N}} \|\psi_n\|_{\mathfrak{H}} < +\infty$;

iii) $\psi_n \xrightarrow[n\to\infty]{} \psi \quad \Longleftrightarrow \quad \limsup_{n\to\infty} \|\psi_n\|_{\mathfrak{H}} \leq \|\psi\|_{\mathfrak{H}}$;

iv) if $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathfrak{H}$ is such that $\phi_n \xrightarrow[n\to\infty]{} \phi \in \mathfrak{H}$, one has $\lim_{n\to\infty} \langle \psi_n, \phi_n \rangle_{\mathfrak{H}} = \langle \psi, \phi \rangle_{\mathfrak{H}}$.

Orthogonal Subspaces

Definition 1.29 (Orthogonal Complement)**.** Let M be a proper subset of a complex Hilbert space $\mathfrak{H}$. We denote its *orthogonal complement* by $M^\perp := \{\psi \in \mathfrak{H} \mid \langle \psi, \varphi \rangle_{\mathfrak{H}} = 0, \quad \forall \varphi \in M\}$.

Remark 1.16. *Let M be a proper subset of $\mathfrak{H}$, complex Hilbert space. Then*

- *$M^\perp$ is closed according to the topology induced by $\|\cdot\|_{\mathfrak{H}}$ because of the continuity of the inner product;*
- *$M^\perp$ defines a Hilbert subspace of $\mathfrak{H}$;*
- *M is dense in $\mathfrak{H}$ iff $M^\perp = \{0\}$;*
- *$M^{\perp\perp} = \overline{M}$, namely the closure of M according to the norm $\|\cdot\|_{\mathfrak{H}}$.*

Exploiting this last property one can also prove that M is closed iff every weakly converging Cauchy sequence in M has limit in M.

Theorem 1.36. *Suppose $\mathfrak{H}$ a complex Hilbert space and $M \subset \mathfrak{H}$ closed. Then, there exists a unique decomposition*[16] *of any vector $\psi \in \mathfrak{H}$ so that*

$$\psi = \psi_{/\!/} + \psi_\perp, \qquad \psi_{/\!/} \in M,\ \psi_\perp \in M^\perp.$$

Additionally,

$$\min_{\varphi\in M} \|\psi - \varphi\|_{\mathfrak{H}} = \|\psi - \psi_{/\!/}\|_{\mathfrak{H}} = \|\psi_\perp\|_{\mathfrak{H}}.$$

Definition 1.30 (Orthogonal Projections)**.** We say that $P \in \mathscr{B}(\mathfrak{H})$ is an **orthogonal projection** if

$$P^2 = P, \qquad \langle P\psi, \varphi \rangle_{\mathfrak{H}} = \langle \psi, P\varphi \rangle_{\mathfrak{H}}, \quad \forall \varphi, \psi \in \mathfrak{H}.$$

Proposition 1.37. *Suppose $P \in \mathscr{B}(\mathfrak{H})$ orthogonal projection with $P \neq 0$ and set $M = \mathrm{ran}(P)$. Then,*

i) $\|P\|_{\mathscr{L}(\mathfrak{H})} = 1$;

[16] In this situation we write $\mathfrak{H} = M \oplus M^\perp$.

ii) $P\psi = \psi$ *for all* $\psi \in M$ *and* M *is closed;*

iii) $\varphi \in M^\perp$ *implies* $P\varphi \in M^\perp$ *and thus* $P\varphi \in M \cap M^\perp = \{0\}$.

Definition 1.31 (Direct Sum)**.** Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be two complex Hilbert spaces. We define their (orthogonal) direct sum $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ as the space composed of couples $(\psi_1, \psi_2) \in \mathfrak{H}_1 \times \mathfrak{H}_2$ endowed with the inner product

$$\langle (\psi_1, \psi_2),\, (\varphi_1, \varphi_2)\rangle_{\mathfrak{H}_1 \oplus \mathfrak{H}_2} = \langle \psi_1,\, \varphi_1\rangle_{\mathfrak{H}_1} + \langle \psi_2,\, \varphi_2\rangle_{\mathfrak{H}_2}.$$

Unsurprisingly, $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ is a Hilbert space. It is a common use to write $\psi_1 + \psi_2$ instead of (ψ_1, ψ_2) thinking of $\mathfrak{H}_1$ and $\mathfrak{H}_2$ as two orthogonal and complementary Hilbert subspaces of the bigger Hilbert space $\mathfrak{H}_1 \oplus \mathfrak{H}_2$.

More generally, given $\{\mathfrak{H}_n\}_{n\in\mathbb{N}}$ a set of (at most) countable complex Hilbert spaces, we define

$$\bigoplus_{n\in\mathbb{N}} \mathfrak{H}_n := \left\{ \sum_{n\in\mathbb{N}} \psi_n, \quad \psi_n \in \mathfrak{H}_n \,\middle|\, \sum_{n\in\mathbb{N}} \|\psi_n\|^2_{\mathfrak{H}_n} < +\infty \right\}$$

where the inner product is

$$\langle \sum_{j\in\mathbb{N}} \varphi_j,\, \sum_{k\in\mathbb{N}} \psi_k \rangle_{\oplus} := \sum_{n\in\mathbb{N}} \langle \varphi_n,\, \psi_n\rangle_{\mathfrak{H}_n}$$

Definition 1.32 (Tensor Product)**.** Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be two complex Hilbert spaces. Let $\mathscr{F}_n(\mathfrak{H}_1, \mathfrak{H}_2)$ be given by the set of linear combinations of n couples in $\mathfrak{H}_1 \times \mathfrak{H}_2$

$$\mathscr{F}_n(\mathfrak{H}_1, \mathfrak{H}_2) = \left\{ \sum_{j=1}^{n} \alpha_j (\psi_j, \varphi_j) \,\middle|\, (\psi_j, \varphi_j) \in \mathfrak{H}_1 \times \mathfrak{H}_2,\ \alpha_j \in \mathbb{C} \right\}.$$

Then consider the quotient

$$\bigcup_{n\in\mathbb{N}} \mathscr{F}_n(\mathfrak{H}_1, \mathfrak{H}_2)\big/\sim\,,$$

where the equivalence is described in the following

- $(\psi_1 + \psi_2, \varphi) \sim (\psi_1, \varphi) + (\psi_2, \varphi)$;
- $(\psi, \varphi_1 + \varphi_2) \sim (\psi, \varphi_1) + (\psi, \varphi_2)$;
- $(\alpha\psi, \varphi) \sim \alpha(\psi, \varphi) \sim (\psi, \alpha\varphi), \qquad \alpha \in \mathbb{C}$.

We define the tensor product $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ as the completion of $\bigcup_{n\in\mathbb{N}} \mathscr{F}_n(\mathfrak{H}_1, \mathfrak{H}_2)\big/\sim$ according to the norm induced by the inner product

$$\langle (\psi_1, \psi_2),\, (\varphi_1, \varphi_2)\rangle_{\mathfrak{H}_1 \otimes \mathfrak{H}_2} = \langle \psi_1,\, \varphi_1\rangle_{\mathfrak{H}_1} \langle \psi_2,\, \varphi_2\rangle_{\mathfrak{H}_2}.$$

A couple $(\psi, \varphi) \in \mathfrak{H}_1 \otimes \mathfrak{H}_2$ is denoted by $\psi \otimes \varphi$.

Remark 1.17. *Notice that, as simple cases, one has* $\bigoplus_{n\in\mathbb{N}} \mathbb{C} = \ell_2(\mathbb{N})$ *and* $\mathfrak{H} \otimes \mathbb{C}^n = \mathfrak{H}^n$*. Moreover, an equality* $\psi \otimes \varphi = \psi' \otimes \varphi'$ *holds when there is some* $\alpha \in \mathbb{C} \setminus \{0\}$ *such that* $\psi = \alpha\psi'$ *and* $\varphi = \alpha^{-1}\varphi'$.

Definition 1.33 (Unitary Operator)**.** A bijective map $U \in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ is said a **unitary operator** between two complex Hilbert spaces if

$$\langle U\varphi,\, U\psi\rangle_{\mathfrak{H}_2} = \langle \varphi,\, \psi\rangle_{\mathfrak{H}_1}, \qquad \forall\, \varphi, \psi \in \mathfrak{H}_1,$$

or, equivalently (due to the polarization identity), if

$$\|U\psi\|_{\mathfrak{H}_2} = \|\psi\|_{\mathfrak{H}_1}, \qquad \forall\, \psi \in \mathfrak{H}_1.$$

In this case $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are said to be unitarily equivalent through U.

Proposition 1.38. *Given $\mathfrak{H}_1, \mathfrak{H}_2$ complex Hilbert spaces and $M \subseteq \mathfrak{H}_1$, one has*

$$UM^{\perp} = (UM)^{\perp}, \qquad \forall\, U \in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_2) \;\; \textit{unitary.}$$

Remark 1.18. *Because of the previous proposition one has that, whenever two Hilbert spaces are unitarily equivalent, each orthogonal subspace in $\mathfrak{H}_1$ has its own unitarily equivalent representation in $\mathfrak{H}_2$, so that the structure of the orthogonal components is preserved.*

Complete Orthonormal Systems

Lemma 1.39. *Let $\{\varphi_j\}_{j=1}^n$ be an orthonormal set*[17] *in a complex Hilbert space $\mathfrak{H}$. Then for any $\psi \in \mathfrak{H}$ one has*

$$\psi = \psi_{/\!/} + \psi_{\perp}, \qquad \psi_{/\!/} = \sum_{j=1}^{n} \langle \varphi_j,\, \psi\rangle_{\mathfrak{H}}\, \varphi_j, \qquad \langle \varphi_j,\, \psi_{\perp}\rangle_{\mathfrak{H}} = 0, \quad \forall\, j \in \{1, \dots, n\}.$$

Additionally, $\|\psi\|_{\mathfrak{H}}^2 = \|\psi_{\perp}\|_{\mathfrak{H}}^2 + \sum_{j=1}^{n} |\langle \varphi_j,\, \psi\rangle_{\mathfrak{H}}|^2.$

Remark 1.19. *Notice that in the previous lemma, for any $\phi \in \operatorname{span}\{\varphi_j\}$ one has*

$$\|\psi - \phi\|_{\mathfrak{H}} \geq \|\psi_{\perp}\|_{\mathfrak{H}}$$

since equality is attained for $\phi = \psi_{/\!/}$ because of theorem 1.36.
Moreover, there holds the so called **Bessel inequality**

$$\|\psi\|_{\mathfrak{H}}^2 \geq \sum_{j=1}^{n} |\langle \varphi_j,\, \psi\rangle_{\mathfrak{H}}|^2.$$

This implies that even[18] *in case $n \longrightarrow \infty$, the series $\sum_{j=1}^{\infty} |\langle \varphi_j,\, \psi\rangle_{\mathfrak{H}}|^2$ converges. In particular, $\|\sum_{j=m}^{n} \langle \varphi_j,\, \psi\rangle_{\mathfrak{H}}\, \varphi_j\|_{\mathfrak{H}}^2 = \sum_{j=m}^{n} |\langle \varphi_j,\, \psi\rangle_{\mathfrak{H}}|^2$ for any $n, m \in \mathbb{N}$, hence $\{\sum_{j=1}^{n} \langle \varphi_j,\, \psi\rangle_{\mathfrak{H}}\, \varphi_j\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{H}$ iff $\{\sum_{j=1}^{n} |\langle \varphi_j,\, \psi\rangle_{\mathfrak{H}}|^2\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}_+$.*
In other words, $\sum_{j=1}^{\infty} \langle \varphi_j,\, \psi\rangle_{\mathfrak{H}}\, \varphi_j$ is a well-defined vector in $\mathfrak{H}$.

[17] In the sense that $\langle \varphi_j,\, \varphi_k\rangle_{\mathfrak{H}} = \delta_{jk}, \quad \forall\, j, k \in \{1, \dots, n\}$.

[18] Actually, the Bessel inequality implies that $\sum_{j\in J} |\langle \varphi_j,\, \psi\rangle_{\mathfrak{H}}|^2$ converges for any interval J. Indeed, in this case, $\langle \varphi_j,\, \psi\rangle_{\mathfrak{H}} \neq 0$ at most for a countable number of indices $j \in J$.

Theorem 1.40 (Complete Orthonormal System)**.** *Given $\{\varphi_j\}_{j\in\mathbb{N}}$ an orthonormal set in the complex Hilbert space $\mathfrak{H}$, the following statements are equivalent*

i) $\mathrm{span}\{\varphi_j\}_{j\in\mathbb{N}}$ *is dense in* $\mathfrak{H}$*;*

ii) $\forall\,\psi\in\mathfrak{H}$ *one has* $\psi=\sum_{j\in\mathbb{N}}\langle\varphi_j,\,\psi\rangle_{\mathfrak{H}}\,\varphi_j\,;$ $\longleftarrow$ $\{\langle\varphi_j,\psi\rangle_{\mathfrak{H}}\}_{j\in\mathbb{N}}\in\ell_2(\mathbb{N})$ *are the* Fourier coefficients.

iii) $\forall\,\phi,\psi\in\mathfrak{H}$ *one has* $\langle\phi,\,\psi\rangle_{\mathfrak{H}}=\sum_{j\in\mathbb{N}}\langle\phi,\,\varphi_j\rangle_{\mathfrak{H}}\langle\varphi_j,\,\psi\rangle_{\mathfrak{H}};$ $\longleftarrow$ Parseval equality.

iv) $\langle\varphi_j,\,\psi\rangle=0,\quad\forall\,j\in\mathbb{N}\implies\psi=0.$

The previous theorem also holds for an orthonormal set $\{\varphi_j\}_{j\in J}$ with J interval.
A complete orthonormal system is also called an **orthonormal basis**.

Theorem 1.41. *Every Hilbert space has an orthonormal basis. If a basis is countable, then every other possible basis is. The dimension of a Hilbert space is the number of elements composing the basis.*

Definition 1.34 (Separability)**.** A Hilbert space with a countable orthonormal basis is **separable**.

Remark 1.20. *Every infinite-dimensional, separable and complex Hilbert space $\mathfrak{H}$ is unitarily equivalent to $\ell_2(\mathbb{N})$. Indeed, let $\{\varphi_j\}_{j\in\mathbb{N}}\subset\mathfrak{H}$ be an orthonormal basis, then define the operator*

$$U:\mathfrak{H}\longrightarrow\ell_2(\mathbb{N}),\qquad U:\psi\longmapsto\{\langle\varphi_j,\,\psi\rangle\}_{j\in\mathbb{N}}.$$

One can prove that U is a bijection and there holds $\|U\psi\|^2_{\ell_2(\mathbb{N})}=\sum_{j\in\mathbb{N}}|\langle\varphi_j,\,\psi\rangle_{\mathfrak{H}}|^2=\|\psi\|^2_{\mathfrak{H}}$.

Proposition 1.42. *Given an orthonormal basis $\{\varphi_j\}_{j\in\mathbb{N}}$ in a complex, separable Hilbert space $\mathfrak{H}$, one has that any operator $A\in\mathscr{B}(\mathfrak{H})$ is uniquely characterized by its* matrix-elements $A_{ij}:=\langle\varphi_i,\,A\varphi_j\rangle_{\mathfrak{H}}$ *since for all $\psi\in\mathfrak{H}$*

$$A\psi=\sum_{j\in\mathbb{N}}a_j(\psi)\,\varphi_j,\qquad \textit{where}\ \ a_j(\psi):=\sum_{k\in\mathbb{N}}A_{jk}\,\langle\varphi_k,\,\psi\rangle_{\mathfrak{H}}.$$

However, $\mathscr{B}(\mathfrak{H})$ is not separable if $\dim\mathfrak{H}=+\infty$ (we do not have a countable base for $\mathscr{B}(\mathfrak{H})$).

Proposition 1.43. *If $\{\varphi_j\}_{j\in\mathbb{N}}$ and $\{\tilde{\varphi}_k\}_{k\in\mathbb{N}}$ are two orthonormal bases for $\mathfrak{H}$ and $\tilde{\mathfrak{H}}$, respectively, then the set $\{\varphi_j\otimes\tilde{\varphi}_k\}_{(j,k)\in\mathbb{N}^2}$ is an orthonormal basis for $\mathfrak{H}\otimes\tilde{\mathfrak{H}}$.*

1.4 Operator Topologies

In this section we provide a brief description of the major families of bounded operators one could deal with, highlighting their main properties.

Definition 1.35 (Adjoint Operator)**.** Given two Hilbert spaces $\mathfrak{H}_1,\mathfrak{H}_2$ and a *bounded* operator $A\in\mathscr{B}(\mathfrak{H}_1,\mathfrak{H}_2)$, we define its adjoint $A^*\in\mathscr{B}(\mathfrak{H}_2,\mathfrak{H}_1)$ as

$$\langle\varphi,\,A^*\psi\rangle_{\mathfrak{H}_1}=\langle A\varphi,\,\psi\rangle_{\mathfrak{H}_2},\qquad\forall\,\varphi\in\mathfrak{H}_1,\psi\in\mathfrak{H}_2.$$

Proposition 1.44. *Let* $A, B \in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ *and* $C \in \mathscr{B}(\mathfrak{H}_2, \mathfrak{H}_3)$*. Then,*

i) $(A+B)^* = A^* + B^*, \qquad (\alpha A)^* = \bar{\alpha} A^*, \quad \forall\, \alpha \in \mathbb{C};$

ii) $A^{**} = A;$

iii) $(CA)^* = A^* C^* \in \mathscr{B}(\mathfrak{H}_3, \mathfrak{H}_1);$

iv) $\ker(A^*) = \operatorname{ran}(A)^{\perp};$

v) $\|A\|^2_{\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_2)} = \|A^*A\|_{\mathscr{L}(\mathfrak{H}_1)} = \|AA^*\|_{\mathscr{L}(\mathfrak{H}_2)}.$

Remark 1.21. *In particular, there also holds* $\|A\|_{\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_2)} = \|A^*\|_{\mathscr{L}(\mathfrak{H}_2,\mathfrak{H}_1)}$*. Indeed,*

$$\|A\|^2_{\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_2)} = \|AA^*\|_{\mathscr{L}(\mathfrak{H}_2)} \leq \|A\|_{\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_2)} \|A^*\|_{\mathscr{L}(\mathfrak{H}_2,\mathfrak{H}_1)} \quad \Longrightarrow \quad \|A\|_{\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_2)} \leq \|A^*\|_{\mathscr{L}(\mathfrak{H}_2,\mathfrak{H}_1)},$$

hence, in particular one also has $\|A^*\|_{\mathscr{L}(\mathfrak{H}_2,\mathfrak{H}_1)} \leq \|A^{**}\|_{\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_2)} = \|A\|_{\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_2)}$*, providing the result. This means that the anti-linear map* $^* : A \longmapsto A^*$ *is continuous in* $\mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_2)$*.*

Definition 1.36 ($C^\star$-algebra)**.** Given a Banach algebra $\mathfrak{a}$ and an involution * (*i.e.* a map equal to its inverse), we say that $(\mathfrak{a}, ^\star)$ is a $C^\star$-algebra if

- $(a+b)^\star = a^\star + b^\star, \qquad (\alpha a)^\star = \bar{\alpha}\, a^\star, \quad \forall\, \alpha \in \mathbb{C},$
- $(ab)^\star = b^\star a^\star,$
- $\|a\|^2_{\mathfrak{a}} = \|a^\star a\|_{\mathfrak{a}} = \|a a^\star\|_{\mathfrak{a}}.$

Definition 1.37. A sub-algebra of the $C^\star$-algebra $(\mathfrak{a}, ^\star)$ is said an **ideal** $\mathfrak{i} \subset \mathfrak{a}$ if

$$ab \in \mathfrak{i}, \quad ba \in \mathfrak{i}, \qquad \forall\, a \in \mathfrak{i},\, b \in \mathfrak{a}.$$

Moreover, if $\mathfrak{i}$ is closed under the involution it is said a *-ideal.
A *-homomorphism is a map $h\colon (\mathfrak{a}, ^{\star_1}) \longrightarrow (\mathfrak{b}, ^{\star_2})$ such that

$$h(ab) = h(a)h(b), \qquad h(a^{\star_1}) = h(a)^{\star_2}.$$

In particular, if there exists an identity element $e \in \mathfrak{a}$ and $\mathfrak{b}$ is *-homomorphic to $\mathfrak{a}$, then also $\mathfrak{b}$ must have an identity element, *e.g.* $h(e)$ (h is not necessarily injective).

Remark 1.22. *If there exists an identity element in the Banach algebra* $e \in \mathfrak{a} : ea = a, \quad \forall\, a \in \mathfrak{a}$ *(or neutral element), then* $ee^\star = e^\star$ *and* $(ee^\star)^\star = e$*, therefore* $e = e^\star$*. This also implies that for any invertible* $b \in \mathfrak{a}$ *one has* $b^\star$ *is invertible, since*

$$b^{-1}b = bb^{-1} = e \quad \Longrightarrow \quad b^\star (b^{-1})^\star = (b^{-1})^\star b^\star = e \quad \Longrightarrow \quad (b^{-1})^\star = (b^\star)^{-1}.$$

As an example, $\mathscr{B}(\mathfrak{H})$ is a $C^\star$-algebra if equipped with the anti-linear map $^* : A \longmapsto A^*$.

Definition 1.38. An element a of a $C^\star$-algebra $(\mathfrak{a}, ^\star)$ is said to be

- *normal*, if $\quad aa^\star = a^\star a,$
- *self-adjoint*, if $\quad a^\star = a,$
- *unitary*, if $\quad a^\star a = aa^\star = e,$

- *orthogonal projection*, if $\quad a^2 = a = a^\star$,
- *non-negative*, if $\quad \exists\, b \in \mathfrak{a} \,:\, a = b^\star b$.

Proposition 1.45 (Characterization of Normal Operators)**.** *Given $N \in \mathscr{B}(\mathfrak{H})$, one has*

$$NN^* = N^*N \quad \Longleftrightarrow \quad \|N\psi\|_{\mathscr{L}(\mathfrak{H})} = \|N^*\psi\|_{\mathscr{L}(\mathfrak{H})}\,, \quad \forall\, \psi \in \mathfrak{H}.$$

The same result can be generalized for $N \in \mathscr{L}(\mathfrak{H})$ in case $\mathfrak{D}(N) = \mathfrak{D}(N^*)$, provided a proper definition of the adjoint for densely-defined linear maps that shall be disclosed in section 2.1.

Definition 1.39 (Compact Operators)**.** A bounded operator $K \in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ is said **compact** if

$$\forall\, \{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{H}_1 \,:\, \psi_n \xrightarrow[n\to\infty]{} \psi \in \mathfrak{H}_1, \qquad K\psi_n \xrightarrow[n\to\infty]{} K\psi \quad \text{in } \mathfrak{H}_2.$$

Remark 1.23. *By definition, given $B_1 \in \mathscr{B}(\mathfrak{H}_2, \mathfrak{H}_3)\,, B_2 \in \mathscr{B}(\mathfrak{H}_3, \mathfrak{H}_1)$ two bounded operators and $K \in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ compact one has*

$$B_1K \in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_3) \;\; \textit{is compact}, \qquad KB_2 \in \mathscr{B}(\mathfrak{H}_3, \mathfrak{H}_2) \;\; \textit{is compact.}$$

For instance, the set of self-adjoint compact operators is a *-ideal of $(\mathscr{B}(\mathfrak{H}), ^*)$.

Theorem 1.46 (Canonical Form of Compact Operators)**.** *Let $K \in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ be compact. Then there exists a couple of orthonormal sets $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathfrak{H}_1, \{\varphi_n\}_{n\in\mathbb{N}} \subset \mathfrak{H}_2$ and positive numbers $\{s_n(K)\}_{n\in\mathbb{N}} \subset \ell_\infty(\mathbb{N})$ such that*

$$K = \sum_{n\in\mathbb{N}} s_n(K) \langle \phi_n, \cdot\rangle_{\mathfrak{H}_1} \varphi_n, \qquad K^* = \sum_{n\in\mathbb{N}} s_n(K) \langle \varphi_n, \cdot\rangle_{\mathfrak{H}_2} \phi_n.$$

Moreover, there holds $\|K\|_{\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_2)} = \max\limits_{n\in\mathbb{N}} s_n(K)$.

One has $K\phi_j = s_j(K)\varphi_j$ and $K^*\varphi_j = s_j(K)\phi_j$, hence $s_j(K)$, which are called the **singular values** of K, are defined as the square root of the eigenvalues of $KK^* \in \mathscr{B}(\mathfrak{H}_2)$ or $K^*K \in \mathscr{B}(\mathfrak{H}_1)$. A compact operator is said of **finite rank** in case its singular values are eventually zero.
In case $K \in \mathscr{B}(\mathfrak{H}_1)$ is self-adjoint one can choose $\varphi_n = \sigma_n \phi_n$, with $\sigma_n \in \{-1, 1\}$ so that $\sigma_j\, s_j(K)$ are the eigenvalues of K. Together with proposition 1.42 this argument proves the following result.

Proposition 1.47. *Let $A \in \mathscr{B}(\mathfrak{H})$ be self-adjoint. Then*

$$A \;\textit{ is compact} \quad \Longleftrightarrow \quad \textit{there exists an orthonormal basis of eigenvectors of } A.$$

Definition 1.40 (Operators Topologies)**.** Given $A \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ and a sequence of linear maps $A_n \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ with[19] $\mathfrak{D}(A) = \liminf\limits_{n\to\infty} \mathfrak{D}(A_n)$, we say that

- A_n converges **uniformly** to A, denoting $A_n \xrightarrow[n\to\infty]{} A$ if

$$\forall\, \epsilon > 0 \quad \exists\, N \in \mathbb{N} \,:\, \forall\, n \geq N \quad \sup\left\{ \|A_n\psi - A\psi\|_{\mathfrak{H}_2} \;\middle|\; \psi \in \bigcap_{n\geq N} \mathfrak{D}(A_n),\; \|\psi\|_{\mathfrak{H}_1} = 1 \right\} < \epsilon;$$

[19]Here, given a sequence of sets $\{S_n\}_{n\in\mathbb{N}}$ we mean $\liminf\limits_{n\to\infty} S_n := \bigcup\limits_{n\in\mathbb{N}} \bigcap\limits_{j\geq n} S_j$.

- A_n converges **strongly** to A, denoting $A_n \xrightarrow[n\to\infty]{s} A$ if

$$\forall\,\epsilon>0 \quad \exists\, N\in\mathbb{N} \,:\, \forall\, n\geq N \quad \|A_n\psi - A\psi\|_{\mathfrak{H}_2}<\epsilon, \qquad \forall\,\psi\in\bigcap_{n\geq N}\mathfrak{D}(A_n);$$

- A_n converges **weakly** to A, denoting $A_n \xrightarrow[n\to\infty]{w} A$ if

$$\forall\,\epsilon>0 \quad \exists\, N\in\mathbb{N} \,:\, \forall\, n\geq N \quad |\langle\varphi,\, A_n\psi - A\psi\rangle_{\mathfrak{H}_2}|<\epsilon, \qquad \forall\,\varphi\in\mathfrak{H}_2,\ \forall\,\psi\in\bigcap_{n\geq N}\mathfrak{D}(A_n).$$

Remark 1.24. *As suggested by the names, the uniform convergence implies strong convergence which also implies the weak one. Additionally,*

- *if* $\{A_n\}_{n\in\mathbb{N}}\subset\mathscr{B}(\mathfrak{H}_1,\mathfrak{H}_2)$ *and* $A_n \xrightarrow[n\to\infty]{} A\in\mathscr{B}(\mathfrak{H}_1,\mathfrak{H}_2)$*, as a consequence of proposition 1.20, one has* $\|A_n\|_{\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_2)} \xrightarrow[n\to\infty]{} \|A\|_{\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_2)}$*;*
- *similarly, given* $\{A_n\}_{n\in\mathbb{N}}\subset\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_2)$ *such that* $A_n \xrightarrow[n\to\infty]{s} A\in\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_2)$*, one has*

$$\|A_n\psi - A\psi\|^2_{\mathfrak{H}_2} = \|A_n\psi\|^2_{\mathfrak{H}_2} - 2\mathrm{Re}\,\langle A\psi,\, A_n\psi\rangle_{\mathfrak{H}_2} + \|A\psi\|^2_{\mathfrak{H}_2}\,, \qquad \forall\,\psi\in\mathfrak{D}(A)\cap\mathfrak{D}(A_n),$$

hence, since one also has $A_n \xrightarrow[n\to\infty]{w} A$*, the previous expression implies*

$$\forall\,\epsilon>0 \quad \exists\, N\in\mathbb{N} \,:\, \forall\, n\geq N \quad \big|\,\|A_n\psi\|_{\mathfrak{H}_2} - \|A\psi\|_{\mathfrak{H}_2}\big|<\epsilon, \qquad \forall\,\psi\in\bigcap_{n\geq N}\mathfrak{D}(A_n).$$

Remark 1.25. *Given* $\{A_n\}_{n\in\mathbb{N}}\subset\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_2)$ *s.t.* $A_n \xrightarrow[n\to\infty]{w} A\in\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_2)$ *and* $K\in\mathscr{B}(\mathfrak{H}_2,\mathfrak{H}_3)$ *compact, one has*

$$KA_n \xrightarrow[n\to\infty]{s} KA\in\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_3).$$

Proposition 1.48. *Consider* $\{A_n\}_{n\in\mathbb{N}}\subset\mathscr{B}(\mathfrak{H}_2,\mathfrak{H}_3)$ *such that* $A_n \xrightarrow[n\to\infty]{s} A\in\mathscr{B}(\mathfrak{H}_2,\mathfrak{H}_3)$ *and* $K\in\mathscr{B}(\mathfrak{H}_1,\mathfrak{H}_2)$*. Then*

$$A_nK \xrightarrow[n\to\infty]{} AK\in\mathscr{B}(\mathfrak{H}_1,\mathfrak{H}_3) \qquad\Longleftrightarrow\qquad K \ \text{is compact.}$$

Additionally, in case $\{A_n\}_{n\in\mathbb{N}}\subset\mathscr{B}(\mathfrak{H}_2)$ *are normal with* $A_n \xrightarrow[n\to\infty]{s} A\in\mathscr{B}(\mathfrak{H}_2)$*, one also has*

$$K^*A_n \xrightarrow[n\to\infty]{} K^*A\in\mathscr{B}(\mathfrak{H}_2,\mathfrak{H}_1) \qquad\Longleftrightarrow\qquad K \ \text{is compact.}$$

Proposition 1.49 (The Space of Compact Operators is Closed)**.** *Let* $\{K_n\}_{n\in\mathbb{N}}\subset\mathscr{B}(\mathfrak{H}_1,\mathfrak{H}_2)$ *be a sequence of compact operators such that* $K_n \xrightarrow[n\to\infty]{} K\in\mathscr{B}(\mathfrak{H}_1,\mathfrak{H}_2)$*. Then,* K *is compact.*

Corollary 1.50. *One can always write a compact operator as a limit of finite rank operators.*

Remark 1.26. *Suppose* $\{A_n\}_{n\in\mathbb{N}}\subset\mathscr{B}(\mathfrak{H}_1,\mathfrak{H}_2)$ *and* $A\in\mathscr{B}(\mathfrak{H}_1,\mathfrak{H}_2)$*. Then, the anti-linear map* $^*:A\longmapsto A^*$ *is continuous in the weak-operator topology, namely*

$$A_n \xrightarrow[n\to\infty]{w} A \qquad\Longrightarrow\qquad A_n^* \xrightarrow[n\to\infty]{w} A^*.$$

However, this is not the case for the strong-operator topology, unless we restrict ourselves to the set of normal operators *(see proposition 1.45).*

Proposition 1.51. *Let* $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathfrak{H}_1,\mathfrak{H}_2)$ *be a sequence of bounded operators. Then,*

i) if $\{A_n\}_{n\in\mathbb{N}}$ *is Cauchy in the weak-operator topology, then* $\sup_{n\in\mathbb{N}} \|A_n\|_{\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_2)} < +\infty$;

ii) $A_n \xrightarrow[n\to\infty]{w} A \quad \Longrightarrow \quad \|A\|_{\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_2)} \leq \liminf_{n\to\infty} \|A_n\|_{\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_2)}$;

iii) if $\sup_{n\in\mathbb{N}} \|A_n\|_{\mathscr{L}(\mathfrak{H}_1,\mathfrak{H}_2)} < +\infty$ *and* $A_n\psi \xrightarrow[n\to\infty]{} A\psi$ *in* $\mathfrak{H}_2$ *for any* ψ *in a dense subspace of* $\mathfrak{H}_1$, *one has* $A_n \xrightarrow[n\to\infty]{s} A$.

Proposition 1.52. *Let* $\{A_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathfrak{H}_1,\mathfrak{H}_2)$ *and* $\{B_n\}_{n\in\mathbb{N}} \subset \mathscr{B}(\mathfrak{H}_3,\mathfrak{H}_1)$. *Then,*

i) $A_n \xrightarrow[n\to\infty]{s} A$ *and* $B_n \xrightarrow[n\to\infty]{s} B \quad \Longrightarrow \quad A_nB_n \xrightarrow[n\to\infty]{s} AB$;

ii) $A_n \xrightarrow[n\to\infty]{w} A$ *and* $B_n \xrightarrow[n\to\infty]{s} B \quad \Longrightarrow \quad A_nB_n \xrightarrow[n\to\infty]{w} AB$;

iii) $A_n \xrightarrow[n\to\infty]{} A$ *and* $B_n \xrightarrow[n\to\infty]{w} B \quad \Longrightarrow \quad A_nB_n \xrightarrow[n\to\infty]{w} AB$.

We can notice that the product by composition is continuous in the strong-operator topology (we already know that it is continuous in the uniform-operator topology too).

Definition 1.41 (One-Parameter, Strongly-Continuous Unitary Group)**.** A *one-parameter, strongly continuous unitary group* is defined as a family of *unitary operators* $\{U(t)\}_{t\in\mathbb{R}} \subset \mathscr{B}(\mathfrak{H})$ satisfying

- $U(0) = \mathbb{1}_{\mathfrak{H}}$;
- $U(t+s) = U(t)U(s) = U(s)U(t)$; $\longleftarrow$ hence $U(t)^{-1} = U(-t)$, by picking $s=-t$.
- $U(t) \xrightarrow[t\to t_0]{s} U(t_0)$.

The above definition identifies a group, since

- there exists the neutral element $U(0)$;
- the operation of the group, namely the product by composition, is associative;
- for any element $U(t)$ there exists its inverse $U(-t)$.

To such a (*abelian*) group there can always be associate an **infinitesimal generator**

$$\mathcal{G}\psi := i \lim_{t\to 0} \frac{U(t)-\mathbb{1}}{t}\psi, \qquad \forall\, \psi \in \mathfrak{D}(\mathcal{G}) = \left\{ \phi \in \mathfrak{H} \mid \exists \lim_{t\to 0} \tfrac{U(t)\phi-\phi}{t} \right\}.$$

Remark 1.27. *Assume for simplicity that* $\mathcal{G} \in \mathscr{B}(\mathfrak{H})$. *Then, it is straightforward to see that* $\mathcal{G}$ *is* self-adjoint *(we shall provide the notion of self-adjointness for unbounded operators in the next chapter).*

Theorem 1.53 (Stone's theorem)**.** *Suppose* $U(\cdot)$ *is a strongly continuous, one-parameter unitary group on* $\mathfrak{H}$. *Then, the associated infinitesimal generator* A *is densely defined and self-adjoint and* $U(t) = e^{-itA}$ *for all* $t \in \mathbb{R}$.

Remark 1.28. *The Stone's theorem ensures that, given a generator, there is only one corresponding strongly continuous, one-parameter unitary group.*

Corollary 1.54. *Suppose* $U(t) = e^{-itA}$ *leaves invariant a dense subset* $\mathfrak{D} \subset \mathfrak{D}(A)$. *Then,* A *is essentially self-adjoint on* $\mathfrak{D}$.

Definition 1.42 (Unitary Representations)**.** Given a group $(G,\cdot)$, we say that $\rho\colon G \longrightarrow \mathscr{B}(\mathfrak{H})$ is a *unitary-representation* of G in $\mathfrak{H}$ if[20]

- $\rho_g \in \mathscr{B}(\mathfrak{H})$ is a unitary operator for all $g \in G$;
- $\rho_e = \mathbb{1}_{\mathfrak{H}}$, if e is the neutral element of G (namely, $e \cdot g = g \cdot e = g, \quad \forall\, g \in G$);
- $\rho_s \rho_t = \rho_{s\cdot t}, \qquad \forall\, s,t \in G.$ $\qquad \longleftarrow$ hence $\rho_{s^{-1}} = \rho_s^{-1}$

Remark 1.29. *Notice that, given the (abelian) group of symmetry of translations in the real axis,* i.e. $(\mathbb{R},+)$*, one has that a one-parameter, strongly-continuous unitary group* $\{U(t)\}_{t\in\mathbb{R}} \subset \mathscr{B}(\mathfrak{H})$ *is, by definition, a unitary-representation of* $(\mathbb{R},+)$ *in* $\mathfrak{H}$*.*

Definition 1.43 (Equivalent Representations)**.** We say that two unitary-representations of the same group $\rho\colon G \longrightarrow \mathscr{B}(\mathfrak{H})$, $\rho'\colon G \longrightarrow \mathscr{B}(\mathfrak{H}')$ are *equivalent* if

$$\exists\, U \in \mathscr{B}(\mathfrak{H},\mathfrak{H}') \ \text{ unitary operator s.t.} \quad \rho'_g U = U \rho_g, \qquad \forall\, g \in G.$$

Definition 1.44 (Irreducible Representations)**.** A unitary-representation ρ of a group $(G,\cdot)$ in the complex Hilbert space $\mathfrak{H}$ is said *reducible* if there exists a proper Hilbert subspace $X \subsetneq \mathfrak{H}$ s.t.

$$\rho_g \psi \in X, \qquad \forall\, \psi \in X,\ g \in G.$$

Similarly, ρ is said **irreducible** if it is not reducible.

Remark 1.30. *A given reducible unitary-representation* $\rho\colon G \longrightarrow \mathscr{B}(\mathfrak{H})$ *can be written in terms of its*[21] *(at most countably many) irreducible unitary-representations* $\{\varrho^j\}$ *as follows*

$$\rho_g = \sum_{j\in\mathbb{N}} \varrho^j_g P_j, \qquad \forall\, g \in G,$$

with $\{P_j\}_{j\in\mathbb{N}}$ *orthogonal projections satisfying* $P_j P_k = \delta_{jk} P_j$ *and* $\sum_{j\in\mathbb{N}} P_j = \mathbb{1}_{\mathfrak{H}}$ *(if* $\{\varrho^j\}$ *are finitely many,* P_j *is eventually the zero operator in this notation).*
In other words, because of proposition 1.38, the irreducible unitary-representations induce a decomposition of $\mathfrak{H}$ *in terms of orthogonal subspaces*

$$\mathfrak{H} = \bigoplus_{j\in\mathbb{N}} \mathfrak{H}_j, \qquad \mathfrak{H}_j := \operatorname{ran}(P_j).$$

[20] In case G has infinite elements and it is a Hausdorff topological group, we also require the unitary representation to be a strongly-continuous homomorphism: if $\{g_n\}_{n\in\mathbb{N}} \subset G$ is s.t. $g_n \xrightarrow[n\to\infty]{} g \in G$, then $\rho_{g_n} \xrightarrow[n\to\infty]{s} \rho_g$.

[21] Such a decomposition is not unique!

2. Axioms of Quantum Mechanics

Here we provide the Von Neumann (axiomatic) formulation of Quantum Mechanics (1955).

1. **Pure states**. Given $\mathfrak{H}$ a complex and separable Hilbert space, the *pure state* associated with an isolated physical system is represented, at a fixed time, by a vector (or unit ray)
$$|\psi\rangle \in {}^{\{\phi\in\mathfrak{H}\mid\ \|\phi\|_{\mathfrak{H}}=1\}}\!/_{\sim}, \qquad \text{with}$$
$$\phi_1 \sim \phi_2 \qquad \Longleftrightarrow \qquad \exists\, \theta \in [0, 2\pi) :\ \phi_1 = e^{i\theta}\phi_2.$$

2. **Observables**. Every observable a is represented by a linear map $A \in \mathscr{L}(\mathfrak{H})$ defined *maximally* on a dense subset $\mathfrak{D}(A)$. The expectation value for a measurement of a, when the system is in the pure state $|\psi\rangle$, $\psi \in \mathfrak{D}(A)$ is[1]
$$\mathbb{E}_\psi[a] = \langle\psi|A|\psi\rangle := \langle\psi,\, A\psi\rangle_{\mathfrak{H}} \in \mathbb{R}.$$

3. **Dynamics**. The time-evolution is implemented by a strongly-continuous, one-parameter unitary group $\{U(t)\}_{t\in\mathbb{R}}$ whose generator corresponds to the observable associated with the *energy* of the system.

4. **Measurement**. When a measurement of an observable a is performed on a pure state $|\phi\rangle$, $\phi \in \mathfrak{D}(A)$ and the result of such a measurement is the number $\lambda \in \mathbb{R}$, then the system will *collapse* in a pure state $|\varphi\rangle$, $\varphi \in \mathfrak{D}(A)$ which satisfies
$$A\varphi = \lambda\varphi.$$

Remark 2.1. *One can also provide a description of quantum systems in a so-called* mixed state*, where the knowledge of the system itself is not maximal.*

Remark 2.2. *We shall see that the second axiom actually identifies the class of densely-defined,* self-adjoint *operators.*
At this point we just mention that requiring $\langle\psi,\, A\psi\rangle_{\mathfrak{H}} \in \mathbb{R}$ *means*
$$\operatorname{Im}\langle\psi,\, A\psi\rangle_{\mathfrak{H}} = \operatorname{Im}\overline{\langle A\psi,\, \psi\rangle}_{\mathfrak{H}} = 0,$$
hence
$$\langle\psi,\, A\psi\rangle_{\mathfrak{H}} = \langle A\psi,\, \psi\rangle_{\mathfrak{H}}.$$

[1] According to the *Dirac notation* $\langle\psi| := |\bar{\psi}\rangle$, and $\langle\psi|\psi\rangle := \|\psi\|^2_{\mathfrak{H}} = 1$ does not depend on the equivalence class.

Remark 2.3. *Let $\mathcal{H}, \mathfrak{D}(\mathcal{H}) \in \mathscr{L}(\mathfrak{H})$ be the (time-independent) observable associated with the energy of a system in the pure initial state $|\psi_0\rangle$, $\psi_0 \in \mathfrak{D}(\mathcal{H})$. Moreover, let $\{U(t)\}_{t\in\mathbb{R}}$ be the strongly-continuous one-parameter unitary group describing the dynamics. Then the* **Schrödinger equation**

$$\boxed{i\,\tfrac{\mathrm{d}}{\mathrm{d}t}|\psi_t\rangle = \mathcal{H}|\psi_t\rangle\,, \qquad \psi_t \in \mathfrak{D}(\mathcal{H})}$$

is solved (uniquely) by $|\psi_t\rangle = U(t)|\psi_0\rangle$ for all $t \in \mathbb{R}$. Indeed, by construction

$$iU'(t)\psi_0 = i\lim_{s\to 0}\frac{U(t+s)\psi_0 - U(t)\psi_0}{s} = i\lim_{s\to 0}\frac{U(s)-\mathbb{1}_{\mathfrak{H}}}{s}U(t)\psi_0 = \mathcal{H}U(t)\psi_0.$$

Conversely, because of Stone's theorem we know the dynamics is given by $U(t) = e^{-i\mathcal{H}t}$, so that for any initial pure state[2]*$|\psi_0\rangle$, $\psi_0 \in \mathfrak{H}$ we have the time-evolution given by $|\psi_t\rangle = U(t)|\psi_0\rangle$. We will provide the meaning of a function of an unbounded operator in section 2.4.*
Notice that the dynamics is a unitary representation of the symmetry associated with the group of time-translations (we are indeed assuming a time-independent Hamiltonian).

Remark 2.4. *The last axiom is an abdication of the theory in describing universally the phenomena of reality, since it makes distinction between the quantum (microscopic) world and the measuring apparatus (which is macroscopic and outside the theory).*

In light of remark 2.2, a further understanding of the notion of self-adjointness in $\mathscr{L}(\mathfrak{H}) \setminus \mathscr{B}(\mathfrak{H})$ is required. However, for unbounded linear maps we don't have the BLT theorem 1.30, therefore $A \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ is not ambiguous only in a given domain of definition $\mathfrak{D}(A)$.
We denote by $A, \mathfrak{D}(A)$ such an unbounded operator (sometimes it is written as $A \restriction \mathfrak{D}(A)$).

2.1 Unbounded Operators

Definition 2.1 (Symmetric Operator)**.** A densely-defined operator $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ is said **symmetric** (or hermitian) if

$$\langle\varphi,\, A\psi\rangle_{\mathfrak{H}} = \langle A\varphi,\, \psi\rangle_{\mathfrak{H}}, \qquad \forall\,\varphi,\psi \in \mathfrak{D}(A).$$

Proposition 2.1. *$A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ is symmetric iff $\langle\psi,\, A\psi\rangle_{\mathfrak{H}} \in \mathbb{R}$ for each $\psi \in \mathfrak{D}(A)$.*

However, a symmetric operator cannot be associated with an observable, since we still need to require maximal definition (namely, such a symmetric operator must not have a proper extension).

[2] From one hand a pure state satisfying the Schrödinger equation at time t must be in $\mathfrak{D}(\mathcal{H})$, whereas the dynamics define the evolution in all $\mathfrak{H}$ (since it is bounded).

CLOSEDNESS

Definition 2.2 (Closed Operator)**.** Given a densely-defined $T, \mathfrak{D}(T) \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ we say that $T, \mathfrak{D}(T)$ is **closed** if $\forall \{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}(T)$ such that $\psi_n \xrightarrow[n\to\infty]{} \psi$ in $\mathfrak{H}_1$ one has

$$\|T\psi_n - \varphi\|_{\mathfrak{H}_2} \xrightarrow[n\to\infty]{} 0 \text{ for some } \varphi \in \mathfrak{H}_2 \qquad \Longrightarrow \qquad \psi \in \mathfrak{D}(T) \ \wedge \ \varphi = T\psi.$$

In other words, $T, \mathfrak{D}(T)$ is closed if for any convergent sequence in $\mathfrak{H}_1$ that makes convergent the sequence $\{T\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{H}_2$, one finds the limit of ψ_n in $\mathfrak{D}(T)$ (in principle it is only in $\mathfrak{H}_1$).
This is the *closest property to continuity* one can demand for unbounded operators.

Definition 2.3 (Closable Operator)**.** An operator $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ is **closable** if there exists an extension $\tilde{A}, \mathfrak{D}(\tilde{A}) \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ that is closed.

Definition 2.4 (Graph)**.** Given $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$, we define its **graph set**[3] as

$$\mathscr{G}(A) := \{(\psi, A\psi) \in \mathfrak{H}_1 \times \mathfrak{H}_2 \mid \psi \in \mathfrak{D}(A)\}.$$

One can also introduce the **graph norm** in the subspace $\mathfrak{D}(A)$ as

$$\|\psi\|^2_{\mathscr{G}(A)} := \|\psi\|^2_{\mathfrak{H}_1} + \|A\psi\|^2_{\mathfrak{H}_2}, \qquad \forall \psi \in \mathfrak{D}(A).$$

Definition 2.5 (Closure of an Operator)**.** Given a closable operator $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ we define its closure $\bar{A}, \mathfrak{D}(\bar{A}) \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ as the unique closed extension of $A, \mathfrak{D}(A)$ satisfying

$$A \subseteq \bar{A}, \qquad \mathscr{G}(\bar{A}) = \overline{\mathscr{G}(A)}.$$

Here the topology that selects the closed sets in $\mathfrak{H}_1 \times \mathfrak{H}_2$ is the one induced by the graph norm.

Proposition 2.2. *Any closed operator has closed graph and closed kernel.*

Proposition 2.3. *Given $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ closed and $B \in \mathscr{B}(\mathfrak{H}_3, \mathfrak{H}_1)$. Then, the operator $AB, \mathfrak{D}(AB) \in \mathscr{L}(\mathfrak{H}_3, \mathfrak{H}_2)$ is closed, with $\mathfrak{D}(AB) = \{\psi \in \mathfrak{H}_3 \mid B\psi \in \mathfrak{D}(A)\}$.*
Moreover, if $C \in \mathscr{B}(\mathfrak{H}_2, \mathfrak{H}_3)$ is boundedly-invertible, $CA, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_3)$ is closed.

Theorem 2.4 (Closed Graph)**.** *Given the operator $A, \mathfrak{H}_1 \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ one has*

$$A \in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_2) \qquad \Longleftrightarrow \qquad A, \mathfrak{H}_1 \text{ is closed.}$$

There are indeed pathological examples for which an unbounded operator can be everywhere-defined, if it is not closed. For instance, consider the separable Hilbert space $\mathfrak{H}$ with $\{\varphi_j\}_{j\in\mathbb{N}}$ an orthonormal basis and P_{even} the projection onto $\operatorname{span}\{\varphi_j\}_{j\in 2\mathbb{N}}$. Then let $A, \mathfrak{H} \in \mathscr{L}(\mathfrak{H})$ be given by

$$A\psi = \begin{cases} \sum_{k=1}^{\ell} b_k\, a_k\, \varphi_{b_k}, & \text{if } P_{\text{even}}\psi = \sum_{k=1}^{\ell} a_k \varphi_{b_k} \text{ with } \ell \in \mathbb{N},\ \{a_k\}_{k=1}^{\ell} \subset \mathbb{C},\ \{b_k\}_{k=1}^{\ell} \subset 2\mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

In other words, if ψ has a non-zero projection in the space of finite linear combinations of even elements of the basis, A acts multiplying each term by its label, while it returns zero otherwise. Since $A\varphi_{2n} = 2n\varphi_{2n}$ it is clear that $\|A\varphi_{2n}\|_{\mathfrak{H}} \xrightarrow[n\to\infty]{} +\infty$, hence $A, \mathfrak{H}$ is unbounded.

[3] Notice that $(\mathscr{G}(A), \|\cdot\|_{\mathscr{G}(A)}) \subset \mathfrak{D}(A) \oplus \operatorname{ran}(A)$.

Definition 2.6 (Adjoint Operator)**.** Given a densely-defined operator $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$, its adjoint $A^*, \mathfrak{D}(A^*) \in \mathscr{L}(\mathfrak{H}_2, \mathfrak{H}_1)$ is defined by

$$\mathfrak{D}(A^*) := \left\{ \psi \in \mathfrak{H}_2 \,\middle|\, \exists! \; \Psi \in \mathfrak{H}_1 : \; \langle \psi, \, A\varphi \rangle_{\mathfrak{H}_2} = \langle \Psi, \, \varphi \rangle_{\mathfrak{H}_1}, \quad \forall \varphi \in \mathfrak{D}(A) \right\};$$
$$A^*\psi = \Psi.$$

This definition is not well-posed if $\mathfrak{D}(A)$ is not dense in $\mathfrak{H}_1$, since there would be an orthogonal complement of $\mathfrak{D}(A)$ which makes ambiguous the definition of $A^*\psi = \Psi + \varphi_\perp$ with $\varphi_\perp \in \mathfrak{D}(A)^\perp$.

Remark 2.5. *Observe that, given $A, \mathfrak{D}(A)$, $B, \mathfrak{D}(B) \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ densely-defined with $A \subseteq B$, the condition in the definition of the domain for the adjoint operator must be stricter (holding in a larger set) for B rather than A, resulting in a smaller domain $B^* \subseteq A^*$.*

Proposition 2.5 (Closed Adjoint)**.** *For any densely-defined $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ one has that its adjoint $A^*, \mathfrak{D}(A^*) \in \mathscr{L}(\mathfrak{H}_2, \mathfrak{H}_1)$ is closed. Moreover, $A, \mathfrak{D}(A)$ is closable iff $A^*, \mathfrak{D}(A^*)$ is densely-defined, in which case one has $\bar{A} = A^{**}$ and $(\bar{A})^* = A^*$.*

Remark 2.6. *We know that the adjoint of a bounded operator is bounded. Moreover, since $A^{**} = \bar{A}$ one has for any densely-defined and closable $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ that*

$$\bar{A} \in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_2) \iff A^* \in \mathscr{B}(\mathfrak{H}_2, \mathfrak{H}_1) \,.$$

Proposition 2.6. *Any normal operator $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ (namely, such that $\mathfrak{D}(A) = \mathfrak{D}(A^*)$ and $\|A\psi\|_{\mathfrak{H}} = \|A^*\psi\|_{\mathfrak{H}}$ for all $\psi \in \mathfrak{D}(A)$) is closed.*

Proposition 2.7. *Given $A, \mathfrak{D}(A)$, $B, \mathfrak{D}(B) \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ and $C, \mathfrak{D}(C) \in \mathscr{L}(\mathfrak{H}_2, \mathfrak{H}_3)$ all densely-defined with $\mathfrak{D}(CA) = \{\psi \in \mathfrak{D}(A) \mid A\psi \in \mathfrak{D}(C)\}$ dense in $\mathfrak{H}_1$, one has*[4]

i) $A^* + B^* \subseteq (A+B)^*, \qquad (\alpha A)^* = \bar{\alpha} A^*, \quad \forall \alpha \in \mathbb{C};$ $\qquad \longleftarrow \; \mathfrak{D}(A+B) = \mathfrak{D}(A) \cap \mathfrak{D}(B)$

ii) $A^* C^* \subseteq (CA)^*$;

iii) $\ker(A^*) = \operatorname{ran}(A)^\perp$;

iv) if $A, \mathfrak{D}(A)$ is invertible, then $A^*, \mathfrak{D}(A^*)$ is invertible and $(A^*)^{-1} = (A^{-1})^*$.
Additionally, if $A, \mathfrak{D}(A)$ is also closable with $\ker(\bar{A}) = \{0\}$, then $(\bar{A})^{-1} = \overline{A^{-1}}$.

Remark 2.7. *Proposition 2.7 iv) implies that the inverse of an invertible, closed operator is closed.*

Proposition 2.8. *Let $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ be closed and $z \in \mathbb{C}$. Suppose there exists $c > 0$ such that*

$$\|(A - z)\psi\|_{\mathfrak{H}} > c \, \|\psi\|_{\mathfrak{H}} \,, \qquad \forall \psi \in \mathfrak{D}(A).$$

Then, $\operatorname{ran}(A - z)$ is a closed subspace of $\mathfrak{H}$.

Proposition 2.9. *Let $S, \mathfrak{D}(S) \in \mathscr{L}(\mathfrak{H})$ be a symmetric operator. Then, $S, \mathfrak{D}(S)$ is closable, $\bar{S}, \mathfrak{D}(\bar{S})$ is also symmetric and there holds*

$$S \subseteq \bar{S} \subseteq S^* . \qquad \longleftarrow \; (S^* \text{ may not be symmetric!})$$

[4]In points *i)* and *ii)* equality holds in case B and C are bounded.

In light of the previous proposition, we know that everywhere-defined symmetric operators must be closed and therefore, thanks to theorem 2.4 one has

Proposition 2.10 (Hellinger-Toeplitz)**.** *An everywhere-defined symmetric operator is bounded.*

Self-Adjointness

Definition 2.7 (Self-Adjoint Operator)**.** We say that a symmetric operator $S, \mathfrak{D}(S) \in \mathscr{L}(\mathfrak{H})$ is **self-adjoint** if $S = S^*$. Additionally, we say that $S, \mathfrak{D}(S)$ is *essentially self-adjoint* if $S \subseteq \bar{S} = S^*$.

Proposition 2.11. *A self-adjoint operator is maximally defined.*

Indeed, there is no way of finding a proper extension for a self-adjoint operator that is symmetric since, assuming there exists a symmetric extension $\tilde{S}, \mathfrak{D}(\tilde{S}) \in \mathscr{L}(\mathfrak{H})$ of the self-adjoint operator $S, \mathfrak{D}(S) \in \mathscr{L}(\mathfrak{H})$, one has

$$S \subseteq \tilde{S} \subseteq \tilde{S}^* \subseteq S^* = S \qquad \Longrightarrow \qquad S = \tilde{S}.$$

Definition 2.8 (Positivity)**.** A symmetric operator $S, \mathfrak{D}(S) \in \mathscr{L}(\mathfrak{H})$ is *non-negative* (or positive semi-definite) if

$$\langle \psi,\, S\psi \rangle_{\mathfrak{H}} \geq 0, \qquad \forall\, \psi \in \mathfrak{D}(S).$$

$S, \mathfrak{D}(S)$ is said *lower-bounded* by $\gamma \in \mathbb{R}$, or $S \geq \gamma$, if $S - \gamma \mathbb{1}_{\mathfrak{H}}, \mathfrak{D}(S)$ is non-negative. Additionally, $S, \mathfrak{D}(S)$ is *positive-definite* if there exists $\gamma > 0$ s.t. $S \geq \gamma$.

Remark 2.8. *Because of proposition 2.7 iv), one has that if a self-adjoint operator is invertible* (i.e. *injective in this case), then also its inverse is self-adjoint.*

Theorem 2.12. *Given $S, \mathfrak{D}(S) \in \mathscr{L}(\mathfrak{H})$ a symmetric operator, it is essentially self-adjoint iff one has that $S - z, \mathfrak{D}(S)$ and $S - \bar{z}, \mathfrak{D}(S)$ are boundedly invertible for some $z \in \mathbb{C} \smallsetminus \mathbb{R}$. Moreover,*

$$\operatorname{ran}(\bar{S} - z) = \overline{\operatorname{ran}(S - z)} \qquad \textit{and} \qquad S, \mathfrak{D}(S) \;\textit{ is closed} \quad \Longleftrightarrow \quad \operatorname{ran}(S - z) \;\textit{ is closed}.$$

If $S \geq a$ for some $a \in \mathbb{R}$, z can also be chosen in the interval $(-\infty, a)$.

In particular, a positive operator is essentially self-adjoint iff it is boundedly invertible.

Proposition 2.13. *Suppose $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ densely-defined and closable operator. Then $A^*\bar{A}, \mathfrak{D}(A^*\bar{A}) \in \mathscr{L}(\mathfrak{H}_1)$ is self-adjoint with $\mathfrak{D}(A^*\bar{A}) = \{\psi \in \mathfrak{D}(\bar{A}) \mid \bar{A}\psi \in \mathfrak{D}(A^*)\}$.*

Theorem 2.14. *Let $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ and $U \in \mathscr{B}(\mathfrak{H}, \mathfrak{H}')$ unitary. Setting $A', \mathfrak{D}(A') \in \mathscr{L}(\mathfrak{H}')$ as*

$$\mathfrak{D}(A') = \{\psi \in \mathfrak{H}' \mid U^*\psi \in \mathfrak{D}(A)\} = U\mathfrak{D}(A), \qquad A'\psi = UAU^*\psi,$$

one has $A, \mathfrak{D}(A)$ is (essentially) self-adjoint iff $A', \mathfrak{D}(A')$ is (essentially) self-adjoint.

Theorem 2.15 (Criterion for self-adjointness)**.** *Let $S, \mathfrak{D}(S) \in \mathscr{L}(\mathfrak{H})$ be a symmetric operator. Then, if there exists $z \in \mathbb{C} \smallsetminus \mathbb{R}$ s.t.*

i) $\ker(S^* - \bar{z}) = \ker(S^* - z) = \{0\}$ *we have $S, \mathfrak{D}(S)$ is essentially self-adjoint;*

ii) $\operatorname{ran}(S - \bar{z}) = \operatorname{ran}(S - z) = \mathfrak{H}$ *we have $S, \mathfrak{D}(S)$ is self-adjoint.*

If $S \geq a$ for some $a \in \mathbb{R}$, z can also be chosen in the interval $(-\infty, a)$.

The point *i)* of this theorem can be rewritten in light of proposition 2.7 *iii)* and remark 1.16 yielding

$$S, \mathfrak{D}(S) \text{ is essentially self-adjoint if } \overline{\operatorname{ran}(S - \bar{z})} = \overline{\operatorname{ran}(S - z)} = \mathfrak{H} \text{ for some } z \in \mathbb{C} \smallsetminus \mathbb{R}.$$

Corollary 2.16. *Given $S, \mathfrak{D}(S) \in \mathscr{L}(\mathfrak{H})$ symmetric, if there exists an orthonormal basis of eigenvectors of S, then S is essentially self-adjoint.*

Definition 2.9 (Relative Boundedness)**.** Given $A, \mathfrak{D}(A)$, $B, \mathfrak{D}(B) \in \mathscr{L}(\mathfrak{H})$ with $\mathfrak{D}(A) \subseteq \mathfrak{D}(B)$, we say that $B, \mathfrak{D}(B)$ is A-bounded (or *relatively bounded* with respect to $A, \mathfrak{D}(A)$) if there exist $a, b > 0$ s.t.

$$\|B\psi\|_{\mathfrak{H}} \leq a \|A\psi\|_{\mathfrak{H}} + b \|\psi\|_{\mathfrak{H}}, \qquad \forall\, \psi \in \mathfrak{D}(A).$$

The infimum of the values of a for which the previous upper bound holds is called *A-bound* of B.

Theorem 2.17 (Kato-Rellich)**.** *Let $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ be (essentially) self-adjoint and $B, \mathfrak{D}(B) \in \mathscr{L}(\mathfrak{H})$ symmetric and A-bounded with A-bound $a < 1$. Then, the operator $A + B, \mathfrak{D}(A)$ is (essentially) self-adjoint. In this case, $B, \mathfrak{D}(B)$ is called a* Kato-small perturbation *of $A, \mathfrak{D}(A)$. In case $A, \mathfrak{D}(A)$ is bounded from below by $\gamma \in \mathbb{R}$, then $A + B, \mathfrak{D}(A)$ is bounded from below by $\gamma - \max\{a|\gamma| + b, \frac{b}{1-a}\}$.*

In order to obtain an observable from a symmetric operator $S, \mathfrak{D}(S) \in \mathscr{L}(\mathfrak{H})$, we need to find an extension of $S, \mathfrak{D}(S)$ that is self-adjoint. In case $S, \mathfrak{D}(S)$ is essentially self-adjoint, by definition we know that $\bar{S}, \mathfrak{D}(\bar{S})$ is the *unique* self-adjoint extension one could find, otherwise there might be several (possibly infinite) distinct self-adjoint extensions for $S, \mathfrak{D}(S)$, or none at all. In the following we provide some sufficient conditions for the existence of such self-adjoint extensions.

Definition 2.10 (Deficiency indices)**.** Given a symmetric operator $S, \mathfrak{D}(S) \in \mathscr{L}(\mathfrak{H})$, we define its *deficiency indices* $\eta_{\pm}(S)$ as

$$\eta_{\pm}(S) := \dim \operatorname{ran}(S \pm i)^{\perp} = \dim \ker(S^* \mp i).$$

In particular, in case $\eta_+(S) = \eta_-(S) = 0$, owing to theorem 2.15 we know that $S, \mathfrak{D}(S)$ is essentially self-adjoint.

Theorem 2.18. *Given a symmetric operator $S, \mathfrak{D}(S) \in \mathscr{L}(\mathfrak{H})$, there exists at least one self-adjoint extension for $S, \mathfrak{D}(S)$ if its deficiency indices $\eta_{\pm}(S)$ are equal to each other.*

Definition 2.11 (Conjugation Map)**.** An anti-linear involution is said a *conjugation* if it is an isometry. Namely, $C : \mathfrak{H} \longrightarrow \mathfrak{H}$ is a conjugation if

- $C(\alpha\psi + \beta\varphi) = \bar{\alpha}C\psi + \bar{\beta}C\varphi, \qquad \alpha, \beta \in \mathbb{C},\ \varphi, \psi \in \mathfrak{H};$
- $C^2 = \mathbb{1}_{\mathfrak{H}};$ $\longleftarrow$ this implies C is surjective: $\forall\,\psi\in\mathfrak{H}\ \exists\,\varphi = C\psi\in\mathfrak{H} : \psi = C\varphi$
- $\langle C\varphi,\, C\psi\rangle_{\mathfrak{H}} = \langle\varphi,\, \psi\rangle_{\mathfrak{H}}, \qquad \forall\,\varphi, \psi \in \mathfrak{H}.$

Additionally, an operator $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ is said C-*real*, with respect to the conjugation C if

- $C\mathfrak{D}(A) \subseteq \mathfrak{D}(A);$ $\longleftarrow$ actually $C^2 = \mathbb{1}_{\mathfrak{H}}$ implies $\mathfrak{D}(A) = C^2\mathfrak{D}(A) \subseteq C\mathfrak{D}(A)$, thus $\mathfrak{D}(A) = C\mathfrak{D}(A)$
- $AC\psi = CA\psi, \qquad \forall\,\psi \in \mathfrak{D}(A).$

Theorem 2.19. *Let $S, \mathfrak{D}(S) \in \mathscr{L}(\mathfrak{H})$ be a symmetric operator which is C-real for some conjugation C. Then, there exists at least one self-adjoint extension for $S, \mathfrak{D}(S)$.*

Theorem 2.20 (Friedrichs Extension)**.** *Let $S, \mathfrak{D}(S) \in \mathscr{L}(\mathfrak{H})$ be symmetric and lower-bounded with $S \geq \gamma \in \mathbb{R}$. Then, there exists a unique self-adjoint extension which is lower bounded by γ.*

From the physical point of view, axiom 2 associate the expectation of an observable in a pure state with a specific *quadratic form*, that is a sesquilinear form evaluated with the same vector in both its arguments. This suggests that one could rephrase the construction in terms of these objects instead of self-adjoint operators. However, to this end, it is required to understand in which case it is possible to associate a quadratic form to an observable. This shall be the content of the next section.

2.2 Quadratic Forms

Definition 2.12 (Quadratic Forms)**.** Given $\mathfrak{H}$ a complex Hilbert space, a map $q : \mathfrak{Q} \subseteq \mathfrak{H} \longrightarrow \mathbb{C}$ is called a **quadratic form** if

- $q[\alpha\psi] = |\alpha|^2 q[\psi], \qquad \alpha \in \mathbb{C},\ \psi \in \mathfrak{Q};$
- $q[\psi + \varphi] + q[\psi - \varphi] = 2q[\psi] + 2q[\varphi], \qquad \forall\,\varphi, \psi \in \mathfrak{Q}.$

If $q[\psi] \in \mathbb{R}$ for all $\psi \in \mathfrak{Q}$ and $\mathfrak{Q}$ is dense in $\mathfrak{H}$, we say that q is **hermitian**.
Moreover, in case there exists $\gamma \in \mathbb{R}$ such that $q[\psi] \geq \gamma\,\|\psi\|^2_{\mathfrak{H}}$ for all $\psi \in \mathfrak{Q}$, we say that the hermitian quadratic form q is **lower-bounded** by γ.

Remark 2.9. *Since a quadratic form q satisfies the parallelogram law, one can always associate with it a sesquilinear form s_q via the polarization identity. Additionally, such a sesquilinear form must be symmetric in case q is hermitian. Finally, in case q is lower-bounded by $\gamma \in \mathbb{R}$ we can define an inner product in $\mathfrak{Q}$ given by*

$$\langle\varphi,\, \psi\rangle_q = s_q(\varphi, \psi) + (1 - \gamma)\langle\varphi,\, \psi\rangle_{\mathfrak{H}}.$$

Consequently, this inner product induce the norm

$$\|\cdot\|^2_q = q[\cdot] + (1 - \gamma)\,\|\cdot\|^2_{\mathfrak{H}}.$$

The completion of $\mathfrak{Q}$ with respect to $\|\cdot\|_q$ shall be denoted by $\mathfrak{H}_q$.
Observe that $\|\cdot\|_q$ is stronger than $\|\cdot\|_{\mathfrak{H}}$ in $\mathfrak{Q}$.

Definition 2.13 (Form Domain)**.** Given $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ self-adjoint with $A \geq \gamma \in \mathbb{R}$ we set

$$q_A : \mathfrak{Q}(A) \subseteq \mathfrak{H} \longrightarrow \mathbb{R} \quad \text{as} \quad q_A : \psi \longmapsto \langle \sqrt{A-\gamma}\,\psi,\, \sqrt{A-\gamma}\,\psi \rangle_{\mathfrak{H}} + \gamma \, \|\psi\|^2_{\mathfrak{H}}$$

and we call $\mathfrak{Q}(A) = \mathfrak{D}(\sqrt{A-\gamma}\,) \supseteq \mathfrak{D}(A)$ the **form domain** of A.

Definition 2.14. A quadratic form $q : \mathfrak{Q} \subseteq \mathfrak{H} \longrightarrow \mathbb{C}$ is *bounded* in case

$$\sup_{\substack{\psi \in \mathfrak{Q}: \\ \|\psi\|_{\mathfrak{H}} = 1}} |q[\psi]| < +\infty.$$

Clearly, in case q is hermitian and bounded, one has that $\|\cdot\|_q$ is equivalent to $\|\cdot\|_{\mathfrak{H}}$.

Definition 2.15 (Relative Form-Boundedness)**.** Given $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ a self-adjoint operator with $A \geq \gamma \in \mathbb{R}$, a quadratic form $q : \mathfrak{Q} \subseteq \mathfrak{H} \longrightarrow \mathbb{C}$, with $\mathfrak{Q}(A) \subseteq \mathfrak{Q}$ is called **relatively form-bounded** with respect to q_A if there exist $a, b > 0$ such that

$$|q[\psi]| \leq a\, q_A[\psi] + (b - a\gamma)\, \|\psi\|^2_{\mathfrak{H}}, \qquad \forall\, \psi \in \mathfrak{Q}(A).$$

The infimum of the values of a for which the previous upper bound holds is said *relative bound*.

Definition 2.16. A lower-bounded quadratic form $q : \mathfrak{Q} \subseteq \mathfrak{H} \longrightarrow \mathbb{R}$ is **closable** if for every Cauchy sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{Q}$ with respect to $\|\cdot\|_q$ one has[5]

$$\psi_n \xrightarrow[n\to\infty]{} 0 \text{ in } \mathfrak{H} \quad \Longrightarrow \quad \|\psi_n\|_q \xrightarrow[n\to\infty]{} 0.$$

In case a quadratic form q is closable, one has its *closure* $\bar{q} : \mathfrak{H}_q \longrightarrow \mathbb{R}$ defined as

$$\bar{q}[\psi] = \lim_{n\to\infty} q[\psi_n], \qquad \forall\, \{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{Q} \;:\; \|\psi_n - \psi_m\|_q \xrightarrow[n,m\to\infty]{} 0, \quad \|\psi_n - \psi\|_{\mathfrak{H}} \xrightarrow[n\to\infty]{} 0.$$

Remark 2.10. *We stress that closability ensures that $\mathfrak{H}_q$ is an actual subspace of $\mathfrak{H}$. Indeed, consider a Cauchy sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathfrak{Q}$ with respect to $\|\cdot\|_q$ (hence $\lim_{n\to\infty} \|\phi_n\|_q$ exists) converging to $\Phi \in \mathfrak{H}_q$. We know that ϕ_n has limit in $\mathfrak{H}$, since $\|\cdot\|_q$ is stronger than $\|\cdot\|_{\mathfrak{H}}$*, e.g. $\phi_n \xrightarrow[n\to\infty]{} \phi$, *therefore one has*

$$\|\Phi\|^2_{\mathfrak{H}_q} = \lim_{n\to\infty} \|\phi_n\|^2_q = \lim_{n\to\infty} q[\phi_n] + (1-\gamma)\, \|\phi\|^2_{\mathfrak{H}}.$$

Now, by way of contradiction suppose $\mathfrak{H}_q \supsetneq \mathfrak{H}$. This means we can find another Cauchy sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathfrak{Q}$ with respect to $\|\cdot\|_q$ such that

$$\|\phi\|^2_{\mathfrak{H}_q} = \lim_{n\to\infty} \|\varphi_n\|^2_q = \lim_{n\to\infty} q[\varphi_n] + (1-\gamma)\, \|\phi\|^2_{\mathfrak{H}}.$$

Indeed, $\varphi_n \xrightarrow[n\to\infty]{} \phi$ in $\mathfrak{H}$ since $\|\cdot\|_{\mathfrak{H}_q}$ is stronger than $\|\cdot\|_{\mathfrak{H}}$ in $\mathfrak{H} \cap \mathfrak{H}_q$. Therefore, it is clear that $\|\varphi_n - \phi_n\|_{\mathfrak{H}} \xrightarrow[n\to\infty]{} 0$ and, because of closability, $\|\varphi_n - \phi_n\|_q \xrightarrow[n\to\infty]{} 0$. This means the two Cauchy sequences $\{\varphi_n\}_{n\in\mathbb{N}}$ and $\{\phi_n\}_{n\in\mathbb{N}}$ are equivalent, namely they have the same limit $\Phi = \phi \in \mathfrak{H}$. We have just shown that $\mathfrak{H}_q \subseteq \mathfrak{H}$, but this is a contradiction. Hence the hypothesis $\mathfrak{H}_q \supsetneq \mathfrak{H}$ is false.

[5] Since $\|\cdot\|_q$ is stronger than $\|\cdot\|_{\mathfrak{H}}$ in $\mathfrak{Q}$, one has $\|\psi_n\|_q \xrightarrow[n\to\infty]{} 0$ implies $\|\psi_n\|_{\mathfrak{H}} \xrightarrow[n\to\infty]{} 0$, but not the converse!

Definition 2.17 (Closedness)**.** A closable quadratic form $q: \mathfrak{Q} \subseteq \mathfrak{H} \longrightarrow \mathbb{R}$ is **closed** if $\mathfrak{Q}$ is a Banach space with respect to $\|\cdot\|_q$ or, equivalently, if $\mathfrak{H}_q = \mathfrak{Q}$.

By construction, given a closable $q: \mathfrak{Q} \subseteq \mathfrak{H} \longrightarrow \mathbb{R}$ one has $\bar{q}: \mathfrak{H}_q \subseteq \mathfrak{H} \longrightarrow \mathbb{R}$ is closed, since $\mathfrak{H}_q$ is complete with respect to $\|\cdot\|_{\bar{q}}$.

Theorem 2.21. *Given a self-adjoint operator $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ such that $A \geq \gamma \in \mathbb{R}$ and a hermitian quadratic form $q: \mathfrak{Q} \subseteq \mathfrak{H} \longrightarrow \mathbb{R}$ that is relatively form-bounded with respect to q_A with relative bound $a < 1$. Then, $q_A + q: \mathfrak{Q}(A) \subseteq \mathfrak{H} \longrightarrow \mathbb{R}$ is closed and bounded from below.*

Theorem 2.22 (KLMN- Kato, Lions, Lax, Milgram, Nelson)**.** *Let $q: \mathfrak{Q} \subseteq \mathfrak{H} \longrightarrow \mathbb{R}$ be a lower-bounded and closed quadratic form. Then, denoting by s_q the sesquilinear form associated with q, there exists a unique self-adjoint operator $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ such that*

$$\begin{aligned} &\mathfrak{D}(A) = \{\psi \in \mathfrak{Q} \mid \exists!\ \Psi \in \mathfrak{H} \ :\ s_q(\varphi, \psi) = \langle \varphi,\ \Psi \rangle_{\mathfrak{H}}, \quad \forall\, \varphi \in \mathfrak{Q}\}, \\ &A\psi = \Psi. \end{aligned}$$

In particular, under the hypothesis of this theorem one has $q[\psi] = \langle \psi,\ A\psi \rangle_{\mathfrak{H}}$ for any $\psi \in \mathfrak{D}(A)$ (here $\mathfrak{Q} \supseteq \mathfrak{D}(A)$ is the form domain of A). Additionally, it is also true the converse: any self-adjoint and lower-bounded operator is uniquely associated with a closed and bounded from below quadratic form.

Corollary 2.23. *Let q be a quadratic form as in theorem 2.22. If q is also bounded, one obtains that $A, \mathfrak{D}(A)$ is bounded as well, with*

$$\sup_{\substack{\psi \in \mathfrak{Q}: \\ \|\psi\|_{\mathfrak{H}} = 1}} |q[\psi]| = \|A\|_{\mathscr{L}(\mathfrak{H})}.$$

Remark 2.11. *In particular, corollary 2.23 implies that for self-adjoint operators there holds*

$$\|A\|_{\mathscr{L}(\mathfrak{H})} = \sup_{\substack{\psi \in \mathfrak{D}(A): \\ \|\psi\|_{\mathfrak{H}} = 1}} |\langle \psi,\ A\psi \rangle_{\mathfrak{H}}|.$$

In conclusion, any observable can be described by a lower-bounded and closed quadratic form.

Proposition 2.24. *Given $A_n, \mathfrak{D}(A_n) \in \mathscr{L}(\mathfrak{H})$ a sequence of symmetric operators such that*

$$\forall\, \epsilon > 0 \quad \exists\, N \in \mathbb{N} : \forall\, n \geq N \quad |q_{A_n}[\psi] - q[\psi]| < \epsilon, \qquad \forall\, \psi \in \bigcap_{n \geq N} \mathfrak{Q}(A_n)$$

for some lower-bounded and closed quadratic form $q : \liminf_{n \to \infty} \mathfrak{Q}(A_n) \subseteq \mathfrak{H} \longrightarrow \mathbb{R}$, one has $A_n \xrightarrow[n \to \infty]{w} A$, namely, the unique self-adjoint and lower-bounded operator associated with q.

2.3 Resolvents and Spectra

Definition 2.18 (Resolvent Set)**.** Let $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ be a closed operator. The **resolvent set** is

$$\varrho(A) := \{z \in \mathbb{C} \mid A - z, \mathfrak{D}(A) \text{ is boundedly invertible}\}.$$

Moreover, we set the **resolvent** map $\mathcal{R}_A : \varrho(A) \longrightarrow \mathscr{B}(\mathfrak{H})$ as

$$\mathcal{R}_A : z \longmapsto (A - z)^{-1}.$$

Remark 2.12. *We stress that, because of remark 2.7, for any value $z \in \mathbb{C}$ s.t. $A - z, \mathfrak{D}(A)$ is invertible, one has $(A - z)^{-1}, \operatorname{ran}(A - z) \in \mathscr{L}(\mathfrak{H})$ is closed. Hence, in order to have a bounded inverse, it is enough to check that $A - z$ is bijective, in light of theorem 2.4.*

Proposition 2.25 (Resolvent Identities)**.** *Let $A, \mathfrak{D}$, $B, \mathfrak{D} \in \mathscr{L}(\mathfrak{H})$ closed. Then,*

i) $\mathcal{R}_A(z)^* = \mathcal{R}_{A^*}(\bar{z}), \qquad \forall\, z \in \varrho(A) = \{z \in \mathbb{C} \mid \bar{z} \in \varrho(A^*)\};$

ii) $\mathcal{R}_A(z) - \mathcal{R}_A(w) = (z - w)\mathcal{R}_A(z)\mathcal{R}_A(w) = (z - w)\mathcal{R}_A(w)\mathcal{R}_A(z), \qquad \forall\, z, w \in \varrho(A);$

iii) $\mathcal{R}_A(z) - \mathcal{R}_B(z) = \mathcal{R}_A(z)(B - A)\mathcal{R}_B(z) = \mathcal{R}_B(z)(B - A)\mathcal{R}_A(z), \qquad \forall\, z \in \varrho(A) \cap \varrho(B).$

Points *ii)* and *iii)* are called first and second resolvent identity, respectively.

Definition 2.19 (Spectrum)**.** Let $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ be closed. The **spectrum** is defined as

$$\sigma(A) = \mathbb{C} \setminus \varrho(A).$$

In particular, we decompose the spectrum as follows: $\sigma(A) = \sigma_{\mathrm{p}}(A) \cup \sigma_{\mathrm{c}}(A) \cup \sigma_{\mathrm{res}}(A)$, with

- $\sigma_{\mathrm{p}}(A) := \{z \in \mathbb{C} \mid \exists\, \psi \in \mathfrak{D}(A) : A\psi = z\psi,\ \psi \neq 0\};$ $\longleftarrow$ $A - z, \mathfrak{D}(A)$ is not injective
- $\sigma_{\mathrm{c}}(A) := \{z \in \mathbb{C} \mid \ker(A - z) = \{0\},\ \operatorname{ran}(A - z) \subsetneq \mathfrak{H} \text{ is dense}\};$ $\longleftarrow$ $\inf\limits_{\substack{\psi \in \mathfrak{D}(A): \\ \|\psi\|_{\mathfrak{H}} = 1}} \|(A - z)\psi\|_{\mathfrak{H}} = 0$
- $\sigma_{\mathrm{res}}(A) := \{z \in \mathbb{C} \mid \ker(A - z) = \{0\},\ \operatorname{ran}(A - z) \subsetneq \mathfrak{H} \text{ not dense}\}.$

We call them, respectively, *point, continuous and residual* spectra.
Additionally, in case $\dim\ker(A - z) = n \geq 1$, $z \in \sigma_{\mathrm{p}}(A)$ is called **eigenvalue** of $A, \mathfrak{D}(A)$ with multiplicity n (in case $n = 1$ it is a *simple eigenvalue*) and any non-null $\psi \in \mathfrak{D}(A)$ satisfying $A\psi = z\psi$ is an **eigenfunction** associated with z. In case $\|\psi\|_{\mathfrak{H}} = 1$, the unit ray $|\psi\rangle$ is called an **eigenstate** of $A, \mathfrak{D}(A)$.

Remark 2.13. *If $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ is self-adjoint, we have $\sigma_{\mathrm{res}}(A) = \emptyset$, owing to proposition 2.7.*

Theorem 2.26 (Unitary Equivalence)**.** *Let $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ closed and $U \in \mathscr{B}(\mathfrak{H}, \mathfrak{H}')$ unitary. Setting $A', \mathfrak{D}(A') \in \mathscr{L}(\mathfrak{H}')$ the closed operator given by*

$$\mathfrak{D}(A') = \{\psi \in \mathfrak{H}' \mid U^*\psi \in \mathfrak{D}(A)\} = U\mathfrak{D}(A), \qquad A'\psi = UAU^*\psi,$$

one has $\sigma(A) = \sigma(A')$ and $\sigma_{\mathrm{p}}(A) = \sigma_{\mathrm{p}}(A')$.

Definition 2.20. Given a self-adjoint operator $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ we say that

- $\sigma_{\text{disc}}(A)$ is the set of *isolated* eigenvalues with finite multiplicity;
- $\sigma_{\text{ess}}(A) = \sigma(A) \setminus \sigma_{\text{disc}}(A)$.

In other words, $\sigma_{\text{ess}}(A)$ is the union between the continuous spectrum, the set of accumulation points in $\sigma_{\text{p}}(A)$ and eigenvalues with infinite multiplicity. Moreover, $\sigma_{\text{ess}}(A)$ is a closed set.

Proposition 2.27. *Let $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ be closed. Then,*

i) *the resolvent set $\varrho(A)$ is open in $\mathbb{C}$ (hence, $\sigma(A)$ is closed);*

ii) $\|\mathcal{R}_A(z)\|_{\mathscr{L}(\mathfrak{H})} \geq \frac{1}{\text{dist}(z, \sigma(A))}, \qquad \forall\, z \in \varrho(A);$

iii) $\mathcal{R}_A(z) = \sum_{n \in \mathbb{N}_0} (z - z_0)^n \mathcal{R}_A(z_0)^{n+1}, \qquad \forall\, z, z_0 \in \varrho(A) : \|\mathcal{R}_A(z_0)\|_{\mathscr{L}(\mathfrak{H})} < \frac{1}{|z - z_0|};$

iv) *if $A, \mathfrak{D}(A)$ is bounded, then $\{z \in \mathbb{C} \mid \|A\|_{\mathscr{L}(\mathfrak{H})} < |z|\} \subset \varrho(A)$ and for such values of z one has the Neumann series*

$$\mathcal{R}_A(z) = -\sum_{j \in \mathbb{N}_0} \frac{A^j}{z^{j+1}}.$$

Proposition 2.28. *Let $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ be closed and invertible and $B \in \mathscr{B}(\mathfrak{H})$. Then,*

- $\sigma(A^{-1}) \setminus \{0\} = (\sigma(A) \setminus \{0\})^{-1}, \qquad A\psi = z\psi \iff A^{-1}\psi = z^{-1}\psi, \quad z \in \sigma_{\text{p}}(A) \setminus \{0\};$
- $\mathcal{R}_{BB^*}(z) = \frac{1}{z}(B\mathcal{R}_{B^*B}(z)B^* - 1), \quad \mathcal{R}_{B^*B}(z) = \frac{1}{z}(B^*\mathcal{R}_{BB^*}(z)B - 1)$ *and* $\sigma(BB^*) \setminus \{0\} = \sigma(B^*B) \setminus \{0\}$.

Definition 2.21 (Compatible Observables)**.** We say two bounded operators $A, B \in \mathscr{B}(\mathfrak{H})$ **commute** if $AB - BA =: [A, B] = 0$. In case $A, \mathfrak{D}(A)$ and $B, \mathfrak{D}(B)$ are unbounded, we say they commute if there exist some $z_1 \in \varrho(A)$ and $z_2 \in \varrho(B)$ s.t.

$$[\mathcal{R}_A(z_1), \mathcal{R}_B(z_2)] = [\mathcal{R}_{A^*}(\bar{z}_1), \mathcal{R}_B(z_2)] = [\mathcal{R}_A(z_1), \mathcal{R}_{B^*}(\bar{z}_2)] = 0.$$

Additionally, in case both $A, \mathfrak{D}(A)$ and $B, \mathfrak{D}(B)$ are self-adjoint, they are said **compatible**.

Definition 2.22 (Singular Weyl Sequence)**.** Given $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ self-adjoint operator, we say $\{\psi_n\}_{n \in \mathbb{N}} \subset \mathfrak{D}(A)$ with $\|\psi_n\|_{\mathfrak{H}} = 1$ for all $n \in \mathbb{N}$ is a *Weyl sequence* for $A, \mathfrak{D}(A)$ if

$$\|A\psi_n - z\psi_n\|_{\mathfrak{H}} \xrightarrow[n \to \infty]{} 0, \qquad \text{for some } z \in \mathbb{C}.$$

It is called a **singular Weyl sequence** if additionally there holds $\psi_n \xrightarrow[n \to \infty]{} 0$.

Proposition 2.29. *If there exists a Weyl sequence for $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$, self-adjoint operator, with parameter $z \in \mathbb{C}$, then one has $z \in \sigma(A)$.*
Moreover, if such a Weyl sequence is singular, then $z \in \sigma_{\text{ess}}(A)$.

Proposition 2.30. *Suppose $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ is closed with $\varrho(A) \neq \emptyset$ and $B, \mathfrak{D}(B) \in \mathscr{L}(\mathfrak{H})$ closable. Then the following are equivalent*

i) *$B, \mathfrak{D}(B)$ is A-bounded;*

ii) $\mathfrak{D}(A) \subseteq \mathfrak{D}(B)$;

iii) *$B\mathcal{R}_A(z)$ is bounded for one (and hence for all) $z \in \varrho(A)$.*

Moreover, if $B, \mathfrak{D}(B)$ *is* A*-bounded, its* A*-bound is not larger than* $\inf_{z\in\varrho(A)} \|B\mathcal{R}_A(z)\|_{\mathfrak{H}}$.

Definition 2.23 (Relative Compactness)**.** Let $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ be a closed operator with resolvent set $\varrho(A) \neq \emptyset$. An operator $B, \mathfrak{D}(B) \in \mathscr{L}(\mathfrak{H})$ is called *relatively* A*-compact* if

- $\mathfrak{D}(B) \supseteq \mathfrak{D}(A)$;
- $B\mathcal{R}_A(z) \in \mathscr{B}(\mathfrak{H})$ is compact for some (and hence for all) $z \in \varrho(A)$.

Clearly, a compact operator is relatively compact with respect to any other operator.

Proposition 2.31. *Let* $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ *be self-adjoint and* $K, \mathfrak{D}(K) \in \mathscr{L}(\mathfrak{H})$ *relatively compact with respect to* $A, \mathfrak{D}(A)$*. Then,* $K, \mathfrak{D}(K)$ *is* A*-bounded with* A*-bound equal to zero*[6].

Proposition 2.32. *Let* $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ *be self-adjoint and* $B, \mathfrak{D}(B) \in \mathscr{L}(\mathfrak{H})$ *symmetric and* A*-bounded with* A*-bound less than* 1*. If* $K, \mathfrak{D}(K) \in \mathscr{L}(\mathfrak{H})$ *is relatively compact with respect to* $A, \mathfrak{D}(A)$*, then it is also relatively compact with respect to* $A+B, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$.

Theorem 2.33 (Weyl)**.** *Let* $A, \mathfrak{D}(A)$*,* $B, \mathfrak{D}(B) \in \mathscr{L}(\mathfrak{H})$ *be self-adjoint operators. If*

$$\mathcal{R}_A(z) - \mathcal{R}_B(z) \in \mathscr{B}(\mathfrak{H}) \text{ is compact for some } z \in \varrho(A) \cap \varrho(B),$$

Then, $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$.

Remark 2.14. *Let* $K, \mathfrak{D}(K) \in \mathscr{L}(\mathfrak{H})$ *be a self-adjoint operator which is relatively compact with respect to* $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$*. Then, because of the second resolvent identity one has*

$$\mathcal{R}_{A+K}(z) - \mathcal{R}_A(z) \in \mathscr{B}(\mathfrak{H}) \text{ is compact for all } z \in \varrho(A+K) \cap \varrho(A).$$

Thus, this is a particular case for which the Weyl's theorem applies. In other words, the essential spectrum of a self-adjoint operator is invariant under relatively compact perturbations.

Corollary 2.34. *Let* $S, \mathfrak{D}(S) \in \mathscr{L}(\mathfrak{H})$ *a symmetric operator with* finite *deficiency indices equal to each other. Then, all its self-adjoint extensions have the same essential spectrum.*

Proposition 2.35. *Let* $P \in \mathscr{B}(\mathfrak{H})$ *be an orthogonal projection. Then* P *is positive semi-definite and, if* $P \neq \mathbb{1}_{\mathfrak{H}}$ *and* $P \neq 0$*, one has* $\sigma(P) = \sigma_{\text{p}}(P) = \{0, 1\}$.

Proposition 2.36. *Let* $U \in \mathscr{B}(\mathfrak{H})$ *be unitary. Then,* $\sigma(U) \subseteq \{z \in \mathbb{C} \mid |z| = 1\}$ *and eigenfunctions corresponding to distinct eigenvalues are orthogonal.*

Proposition 2.37 (Riesz-Schauder)**.** *Let* $K \in \mathscr{B}(\mathfrak{H})$ *be compact. Then,* $\sigma(K) \setminus \{0\} = \sigma_{\text{disc}}(K)$*, such eigenvalues are either finite or countable with* $\{0\}$ *as accumulation point,* i.e. $\sigma_{\text{ess}}(K) \subseteq \{0\}$ *(it might be empty).*

[6] This does not imply that $K, \mathfrak{D}(K)$ is bounded! It could happen that $\|K\psi\|_{\mathfrak{H}} \leq \epsilon \|A\psi\|_{\mathfrak{H}} + \frac{1}{\epsilon}\|\psi\|_{\mathfrak{H}}, \quad \forall \epsilon > 0$.

Proposition 2.38. *Given a real-valued μ-measurable function $f: X \longrightarrow \mathbb{R}$, the* multiplication operator[7] $M_f, \mathfrak{D}(M_f) \in \mathscr{L}(L^2(X, d\mu))$ *is self-adjoint (and bounded in case f is essentially bounded) and*

$$\sigma(M_f) = \big\{\lambda \in \mathbb{R} \,\big|\, \mu\big(\{x \in X : \; |f(x) - \lambda| < \epsilon\}\big) > 0, \quad \forall\, \epsilon > 0\big\}.$$

Additionally, if $\mu \ll \nu$, with ν the Lebesgue measure, one has $\sigma(M_f) = \sigma_{\text{ess}}(M_f)$.

Theorem 2.39. *Let $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ be symmetric. Then,*

i) all eigenvalues are real and the corresponding eigenfunctions are orthogonal;

ii) $A, \mathfrak{D}(A)$ is self-adjoint iff $\sigma(A) \subseteq \mathbb{R}$;

iii) $A, \mathfrak{D}(A)$ is self-adjoint and $A \geq \gamma \in \mathbb{R} \quad \Longleftrightarrow \quad \sigma(A) \subseteq [\gamma, +\infty)$;

iv) if $A, \mathfrak{D}(A)$ is self-adjoint, $\|\mathcal{R}_A(z)\|_{\mathscr{L}(\mathfrak{H})} = \frac{1}{\text{dist}(z, \sigma(A))}, \quad \forall\, z \in \varrho(A)$. Moreover,

$$\inf \sigma(A) = \inf_{\substack{\psi \in \mathfrak{D}(A): \\ \|\psi\|_{\mathfrak{H}} = 1}} \langle \psi,\, A\psi \rangle_{\mathfrak{H}}, \qquad \sup \sigma(A) = \sup_{\substack{\psi \in \mathfrak{D}(A): \\ \|\psi\|_{\mathfrak{H}} = 1}} \langle \psi,\, A\psi \rangle_{\mathfrak{H}}.$$

Remark 2.15. *In case an eigenvalue $\lambda_0 \in \mathbb{R}$ is not simple, two distinct corresponding eigenfunctions does not have to be orthogonal to each other, however, exploiting the Gram-Schmidt technique, one can always provide an orthonormal set of eigenfunctions spanning* $\ker(A - \lambda_0)$.

Further Operator Topologies

Definition 2.24 (Norm- and Strong-Resolvent Convergence)**.** Given a sequence of self-adjoint operators $A_n, \mathfrak{D}(A_n) \in \mathscr{L}(\mathfrak{H})$ and $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ self-adjoint we say

- A_n converges to A in the **norm resolvent** sense if $\mathcal{R}_{A_n}(z) \xrightarrow[n \to \infty]{} \mathcal{R}_A(z)$ for one $z \in \Gamma$,
- A_n converges to A in the **strong resolvent** sense if $\mathcal{R}_{A_n}(z) \xrightarrow[n \to \infty]{s} \mathcal{R}_A(z)$ for one $z \in \Gamma$,

where $\Gamma := \mathbb{C} \smallsetminus \big[\sigma(A) \cup \bigcup_{n \in \mathbb{N}} \sigma(A_n)\big]$.

Remark 2.16. *We stress that uniform convergence implies norm resolvent convergence whereas strong convergence implies strong resolvent convergence.*
Moreover, given $z \in \Gamma \smallsetminus \mathbb{R}$ such that $\mathcal{R}_{A_n}(z) \xrightarrow[n \to \infty]{w} \mathcal{R}_A(z)$, one also has $\mathcal{R}_{A_n}(z) \xrightarrow[n \to \infty]{s} \mathcal{R}_A(z)$.
In conclusion, if a sequence of operators converges in norm (or strong) resolvent sense for one $z_0 \in \Gamma$, than it converges for all $z \in \Gamma$.

Proposition 2.40. *Suppose $A_n, \mathfrak{D}(A_n) \in \mathscr{L}(\mathfrak{H})$ self-adjoint operators converge in the strong resolvent sense to $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$. Then,*

$$e^{-itA_n} \xrightarrow[n \to \infty]{s} e^{itA}, \qquad t \in \mathbb{R}.$$

Proposition 2.41. *Let $A_n, \mathfrak{D} \in \mathscr{L}(\mathfrak{H})$ a sequence of self-adjoint operators and $A, \mathfrak{D} \in \mathscr{L}(\mathfrak{H})$ self-adjoint. Then, A_n converges to A in the norm resolvent sense if there exist two sequences of positive numbers $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ converging to zero such that*

$$\|(A - A_n)\psi\|_{\mathfrak{H}} \leq a_n \|\psi\|_{\mathfrak{H}} + b_n \|A\psi\|_{\mathfrak{H}}, \qquad \forall\, \psi \in \mathfrak{D}.$$

[7] $(M_f \psi)(x) = f(x)\psi(x), \quad \forall\, \psi \in \mathfrak{D}(M_f) = \big\{\psi \in L^2(X, d\mu) \,\big|\, f\psi \in L^2(X, d\mu)\big\}$.

Theorem 2.42. *Let $A_n, \mathfrak{D}(A_n) \in \mathscr{L}(\mathfrak{H})$ be a sequence of self-adjoint operators. If $A_n, \mathfrak{D}(A_n)$ converges to a self-adjoint operator $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ in the strong resolvent sense, one has*

$$\sigma(A) \subseteq \lim_{n\to\infty} \sigma(A_n),$$

where equality holds in case $A_n, \mathfrak{D}(A_n)$ converges to $A, \mathfrak{D}(A)$ in the norm resolvent sense.

2.4 Spectral Theorem

In this section, our goal is to give a meaning to $f(A), \mathfrak{D}(f(A))$ for some function f and an unbounded self-adjoint operator $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$, so that, provided a time-independent Hamiltonian $\mathcal{H}, \mathfrak{D}(\mathcal{H}) \in \mathscr{L}(\mathfrak{H})$ describing the energy of a system in the initial pure state $|\psi_0\rangle$, $\psi_0 \in \mathfrak{H}$ one always has the unique solution $|\psi_t\rangle = e^{-i\mathcal{H}t}|\psi_0\rangle$ associated with the time evolution (ψ_t solves the Schrödinger equation if ψ_0 is in the dense subspace $\mathfrak{D}(\mathcal{H})$, which is invariant under the action of the dynamics, *i.e.* $e^{-i\mathcal{H}t}\mathfrak{D}(\mathcal{H}) = \mathfrak{D}(\mathcal{H})$).

Definition 2.25 (Projection-valued Measure)**.** Let $\mathfrak{B}$ denote the σ-algebra of Borel sets in $\mathbb{R}$. A *projection-valued measure* is a map

$$P \colon \mathfrak{B} \longrightarrow \mathscr{B}(\mathfrak{H}), \qquad P : \Omega \longmapsto P(\Omega), \text{ orthogonal projection}$$

satisfying

- $P(\mathbb{R}) = \mathbb{1}_{\mathfrak{H}}$;
- given $\{\Omega_n\}_{n\in\mathbb{N}} \subset \mathfrak{B}$ pairwise disjoint, one has

$$\sum_{k=1}^{n} P(\Omega_k) \xrightarrow[n\to\infty]{s} P(\Omega), \qquad \text{with } \Omega = \bigcup_{n\in\mathbb{N}} \Omega_n \in \mathfrak{B}. \qquad \longleftarrow \text{ strong } \sigma\text{-additivity}$$

Remark 2.17. *Instead of strong convergence one could just require weak convergence, since a sequence of projections weakly converging to a projection, converges also strongly*

$$P_n \xrightarrow[n\to\infty]{w} P \implies \|P_n\psi - P\psi\|^2_{\mathfrak{H}} = \langle\psi,\, P_n\psi\rangle_{\mathfrak{H}} + \langle\psi,\, P\psi\rangle_{\mathfrak{H}} - 2\mathrm{Re}\langle P\psi,\, P_n\psi\rangle_{\mathfrak{H}} \xrightarrow[n\to\infty]{} 0.$$

Proposition 2.43. *A projection-valued measure satisfies the following properties*

i) $P(\emptyset) = 0$;

ii) $P(\mathbb{R} \smallsetminus \Omega) = \mathbb{1}_{\mathfrak{H}} - P(\Omega), \qquad \forall\, \Omega \in \mathfrak{B}$;

iii) $P(\Omega_1 \cup \Omega_2) + P(\Omega_1 \cap \Omega_2) = P(\Omega_1) + P(\Omega_2), \qquad \forall\, \Omega_1, \Omega_2 \in \mathfrak{B}$;

iv) $P(\Omega_1)P(\Omega_2) = P(\Omega_2)P(\Omega_1) = P(\Omega_1 \cap \Omega_2), \qquad \forall\, \Omega_1, \Omega_2 \in \mathfrak{B}$;

v) $\Omega_1, \Omega_2 \in \mathfrak{B} \,:\, \Omega_1 \subseteq \Omega_2 \implies P(\Omega_2) - P(\Omega_1) = P(\Omega_2 \smallsetminus \Omega_1) \geq 0.$ $\longleftarrow$ *monotonicity*

Remark 2.18. *All maps $\mu_\psi : \mathfrak{B} \longrightarrow \mathbb{R}_+$ given by $\mu_\psi(\cdot) := \langle\psi,\, P(\cdot)\psi\rangle_{\mathfrak{H}} = \|P(\cdot)\psi\|^2_{\mathfrak{H}}$ define a finite regular (see proposition 1.1) Borel measure on $\mathbb{R}$ for a given $\psi \in \mathfrak{H}$ with $\mu_\psi(\mathbb{R}) = \|\psi\|^2_{\mathfrak{H}}$.*

Definition 2.26 (Spectral Family - Resolution of the Identity)**.** Given a projection-valued measure P, we call the map $E_P : \mathbb{R} \longrightarrow \mathscr{B}(\mathfrak{H})$ defined by $E_P : \lambda \longmapsto P((-\infty, \lambda])$ a *resolution of the identity* and each operator $E_P(\lambda)$ a *spectral projection*.

Proposition 2.44. *A resolution of the identity fulfils*

i) $E_P(\lambda)$ *is an orthogonal projection for each* $\lambda \in \mathbb{R}$*;*

ii) $E_P(\lambda_1) \leq E_P(\lambda_2)$ *for* $\lambda_1 \leq \lambda_2$*;*

iii) $E_P(\lambda + \epsilon) \xrightarrow[\epsilon \to 0^+]{s} E_P(\lambda), \qquad \lambda \in \mathbb{R};$ $\qquad \longleftarrow$ *strong right-continuity*

iv) $E_P(\lambda) \xrightarrow[\lambda \to +\infty]{s} \mathbb{1}_{\mathfrak{H}}, \quad E_P(\lambda) \xrightarrow[\lambda \to -\infty]{s} 0.$

Remark 2.19. *We stress that the regular Borel measure* $\mu_\psi(\cdot) = \langle \psi,\, P(\cdot)\psi \rangle_{\mathfrak{H}}$ *can be thought of as the Lebesgue-Stieltjes measure associated with the non-decreasing, right-continuous function* $\lambda \longmapsto \langle \psi,\, E_P(\lambda)\psi \rangle_{\mathfrak{H}}$*, since* $\mu_\psi\big((a,b]\big) = \langle \psi,\, P\big((-\infty, b]\big)\psi \rangle_{\mathfrak{H}} - \langle \psi,\, P\big((-\infty, a]\big)\psi \rangle_{\mathfrak{H}}$*.*
μ_ψ *is the* **spectral measure** *associated with* P*.*

Definition 2.27 (Functional Calculus - Simple functions)**.** Given a simple function $s : \mathbb{R} \longrightarrow \mathbb{C}$ and a projection-valued measure P, we define its *functional calculus* as the map P satisfying

$$P(s) \equiv \int_{\mathbb{R}} dE_P(\lambda)\; s(\lambda) := \sum_{k=1}^{n} \alpha_k P(\Omega_k), \qquad s : x \longmapsto \sum_{k=1}^{n} \alpha_k \mathbb{1}_{\Omega_k}(x), \quad \Omega_k \in \mathfrak{B}.$$

In particular, for any $\Omega \in \mathfrak{B}$ we have $P(\mathbb{1}_\Omega) = P(\Omega)$.

Remark 2.20. *Given a projection-valued measure* P*, there holds*

$$\langle \psi,\, P(s)\psi \rangle_{\mathfrak{H}} = \sum_{k=1}^{n} \alpha_k \, \|P(\Omega_k)\psi\|_{\mathfrak{H}}^2 = \sum_{k=1}^{n} \alpha_k \, \mu_\psi(\Omega_k) = \int_{\mathbb{R}} d\mu_\psi(\lambda)\; s(\lambda),$$

$$\|P(s)\psi\|_{\mathfrak{H}}^2 = \sum_{k=1}^{n} |\alpha_k|^2 \, \|P(\Omega_k)\psi\|_{\mathfrak{H}}^2 = \int_{\mathbb{R}} d\mu_\psi(\lambda)\; |s(\lambda)|^2. \qquad \longleftarrow \quad \{\Omega_k\}_{k=1}^{n} \text{ are pairwise disjoint}$$

Equipping the vector space of simple functions with the sup *norm, we infer that the linear map* P *is continuous between the normed space of simple functions and the bounded operators. Indeed,*

$$\|P\| = \sup_{\|s\|_\infty = 1} \|P(s)\|_{\mathscr{L}(\mathfrak{H})} = \sup_{\|s\|_\infty = 1} \sup_{\|\psi\|_{\mathfrak{H}} = 1} \|P(s)\psi\|_{\mathfrak{H}} = 1.$$

Since the normed space of simple functions is dense in the Banach space of complex-valued, bounded Borel functions, denoted by $B(\mathbb{R}, \mathbb{C})$, owing to theorem 1.30 one has the following.

Definition 2.28 (Functional Calculus - Bounded Borel functions)**.** Given a projection-valued measure P and its functional calculus for simple functions, we define $P : B(\mathbb{R}, \mathbb{C}) \longrightarrow \mathscr{B}(\mathfrak{H})$ as its unique extension to a bounded linear operator in $\mathscr{L}(B(\mathbb{R}, \mathbb{C}), \mathscr{B}(\mathfrak{H}))$ with norm 1.

There still holds for any $f \in B(\mathbb{R},\mathbb{C})$

$$\begin{aligned}\langle\psi,\, P(f)\psi\rangle_{\mathfrak{H}} &= \int_{\mathbb{R}} d\mu_\psi(\lambda)\ f(\lambda),\\ \|P(f)\psi\|^2_{\mathfrak{H}} &= \int_{\mathbb{R}} d\mu_\psi(\lambda)\ |f(\lambda)|^2.\end{aligned}$$

Theorem 2.45. *Given P a projection-valued measure, one has its functional calculus for complex-valued, bounded Borel functions $P(f) = \int_{\mathbb{R}} dE_P(\lambda)\ f(\lambda)$ satisfying*

$$\begin{aligned}P(f)^* &= P(\bar{f}), &&\forall\, f \in B(\mathbb{R},\mathbb{C});\\ \langle P(g)\psi,\, P(f)\psi\rangle_{\mathfrak{H}} &= \int_{\mathbb{R}} d\mu_\psi(\lambda)\ \overline{g(\lambda)}\, f(\lambda), &&\forall\, g, f \in B(\mathbb{R},\mathbb{C}).\end{aligned}$$

Additionally, if $f_n \xrightarrow[n\to\infty]{} f$ pointwise and $\{\|f_n\|_\infty\}_{n\in\mathbb{N}}$ is a bounded sequence, one has

$$P(f_n) \xrightarrow[n\to\infty]{s} P(f).$$

Remark 2.21. *This theorem highlights that the functional calculus for bounded Borel functions of a projection-valued measure is a *-homomorphism between the $C^\star$-algebra of $B(\mathbb{R},\mathbb{C})$ with involution given by the complex conjugation and the $C^\star$-algebra $(\mathscr{B}(\mathfrak{H})\,,{}^*)$. As a consequence, one has for any $\Omega \in \mathfrak{B}$*

$$\mu_{P(f)\psi}(\Omega) = \langle P(f)\psi,\, P(\Omega)P(f)\psi\rangle_{\mathfrak{H}} = \int_{\Omega} d\mu_\psi(\lambda)\ |f(\lambda)|^2 \quad\Longrightarrow\quad d\mu_{P(f)\psi} = |f|^2 d\mu_\psi.$$

Additionally, the functional calculus P maps non-negative functions in non-negative operators.

Next we want to define the functional calculus of a projection-valued measure for unbounded Borel functions. Since we expect the resulting operator to be unbounded, we need to set a domain of definition

$$\mathfrak{D}(P(f)) = \big\{\psi \in \mathfrak{H} \,\big|\, f \in L^2(\mathbb{R}, d\mu_\psi)\big\}.$$

This is a linear subspace of $\mathfrak{H}$ since, given $\psi,\varphi \in \mathfrak{D}(P(f))$ one also has $\alpha\psi+\beta\varphi \in \mathfrak{D}(P(f))$ for all $\alpha,\beta \in \mathbb{C}$ since

$$\begin{aligned}\mu_{\alpha\psi+\beta\varphi}(\Omega) = \|P(\Omega)(\alpha\psi+\beta\varphi)\|^2_{\mathfrak{H}} &\le 2|\alpha|^2\,\|P(\Omega)\psi\|^2_{\mathfrak{H}} + 2|\beta|^2\,\|P(\Omega)\varphi\|^2_{\mathfrak{H}}\\ &= 2|\alpha|^2\mu_\psi(\Omega) + 2|\beta|^2\mu_\varphi(\Omega),\end{aligned}$$

hence $f \in L^2(\mathbb{R}, d\mu_\psi) \cap L^2(\mathbb{R}, d\mu_\varphi)$ implies $f \in L^2(\mathbb{R}, d\mu_{\alpha\psi+\beta\varphi})$. Notice that in case f is bounded $\mathfrak{D}(P(f)) = \mathfrak{H}$ since μ_ψ is finite. Furthermore, $\mathfrak{D}(P(f))$ is also dense in $\mathfrak{H}$.
Indeed, let $\psi \in \mathfrak{H}$ and $\Omega_n = \{\lambda \in \mathbb{R} \mid |f(\lambda)| \le n\} \in \mathfrak{B}$. Then, $\psi_n := P(\Omega_n)\psi \in \mathfrak{D}(P(f))$ for all $n \in \mathbb{N}$ since $d\mu_{\psi_n} = \mathbb{1}_{\Omega_n} d\mu_\psi$, hence $\|f\|_{L^2(\mathbb{R},\, d\mu_{\psi_n})} \le n\,\|\psi\|_{\mathfrak{H}}$. Moreover, one also has

$$\begin{aligned}\|\psi-\psi_n\|^2_{\mathfrak{H}} = \|(\mathbb{1}_{\mathfrak{H}} - P(\Omega_n))\psi\|^2_{\mathfrak{H}} = \|P(\mathbb{R}\smallsetminus\Omega_n)\psi\|^2_{\mathfrak{H}} = \mu_\psi(\mathbb{R}\smallsetminus\Omega_n),\\ \mu_\psi(\mathbb{R}\smallsetminus\Omega_n) \xrightarrow[n\to\infty]{} \mu_\psi\Big(\bigcap_{k\in\mathbb{N}} \mathbb{R}\smallsetminus\Omega_k\Big) = \mu_\psi\Big(\mathbb{R}\smallsetminus\bigcup_{k\in\mathbb{N}}\Omega_k\Big) = 0.\end{aligned}$$

Finally, for any Borel function f, set $f_n = \mathbb{1}_{\Omega_n} f \in B(\mathbb{R}, \mathbb{C})$ which defines a Cauchy sequence in $L^2(\mathbb{R}, d\mu_\psi)$ for any given $\psi \in \mathfrak{D}(P(f))$. Therefore, the sequence of vectors $\{P(f_n)\psi\}_{n\in\mathbb{N}}$ is Cauchy in $\mathfrak{H}$, since

$$\|(P(f_n) - P(f_m))\psi\|^2_{\mathfrak{H}} = \|P(f_n - f_m)\psi\|^2_{\mathfrak{H}} = \int_{\mathbb{R}} d\mu_\psi(\lambda)\ |f_n(\lambda) - f_m(\lambda)|^2 \xrightarrow[n,m\to\infty]{} 0.$$

Hence $\lim_{n\to\infty} P(f_n)\psi$ exists in $\mathfrak{H}$ by completeness.

Definition 2.29 (Functional Calculus)**.** Given a projection-valued measure P, for any f complex-valued Borel function we define its *functional calculus* $P(f), \mathfrak{D}(P(f))$ as the strong limit of $P(f_n)$, provided $\{f_n\}_{n\in\mathbb{N}} \subset B(\mathbb{R}, \mathbb{C})$ s.t. $\|f_n - f\|_{L^2(\mathbb{R}, d\mu_\psi)} \xrightarrow[n\to\infty]{} 0$ for all $\psi \in \mathfrak{D}(P(f))$.

Theorem 2.46. *For every Borel function f, the densely defined linear operator*

$$P(f) = \int_{\mathbb{R}} dE_P(\lambda)\ f(\lambda), \qquad \mathfrak{D}(P(f)) = \left\{\psi \in \mathfrak{H} \,\middle|\, f \in L^2(\mathbb{R}, d\mu_\psi)\right\}$$

is normal (hence closed) and satisfies for all $\psi \in \mathfrak{D}(P(f))$

$$\|P(f)\psi\|^2_{\mathfrak{H}} = \int_{\mathbb{R}} d\mu_\psi(\lambda)\ |f(\lambda)|^2, \qquad \langle\psi,\ P(f)\psi\rangle_{\mathfrak{H}} = \int_{\mathbb{R}} d\mu_\psi(\lambda)\ f(\lambda).$$

Additionally, for any f, g Borel functions and $\alpha, \beta \in \mathbb{C}$

i) $P(f)^* = P(\bar{f})$;

ii) $\alpha P(f) + \beta P(g) \subseteq P(\alpha f + \beta g), \qquad \mathfrak{D}(\alpha P(f) + \beta P(g)) = \mathfrak{D}(P(|f| + |g|))$;

iii) $P(f)P(g) \subseteq P(fg), \qquad \mathfrak{D}(P(f)P(g)) = \mathfrak{D}(P(fg)) \cap \mathfrak{D}(P(g))$.

Notice that in case f is real-valued, $P(f), \mathfrak{D}(P(f)) \in \mathscr{L}(\mathfrak{H})$ is a self-adjoint operator.

Now, let $F_{A;\psi}(z) = \langle\psi,\ \mathcal{R}_A(z)\psi\rangle_{\mathfrak{H}}$ for a given self-adjoint operator $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$. This function is holomorphic for $z \in \varrho(A)$ and satisfies

$$F_{A;\psi}(\bar{z}) = \overline{F_{A;\psi}(z)}, \qquad |F_{A;\psi}(z)| \leq \frac{\|\psi\|^2_{\mathfrak{H}}}{|\mathrm{Im}\, z|} \quad \text{and} \quad \mathrm{Im}\, F_{A;\psi}(z) = \mathrm{Im}\, z\, \|\mathcal{R}_A(z)\psi\|^2_{\mathfrak{H}}\,.$$

In particular, $F_{A;\psi}$ is a *Herglotz–Nevanlinna* function[8].

Lemma 2.47. *Any Herglotz–Nevanlinna function F such that $|F(z)| < \frac{M}{|\mathrm{Im}\, z|}$ for some $M > 0$ can be written uniquely as the* Borel transform *of a finite Borel measure μ_F (with $\mu_F(\mathbb{R}) < M$) given by the* Stieltjes inversion *formula, namely*

$$F(z) = \int_{\mathbb{R}} d\mu_F(\lambda)\ \frac{1}{\lambda - z}, \qquad \longleftarrow \ \textit{Borel transform}$$

$$\mu_F\big((a,b]\big) = \lim_{\delta\to 0^+} \lim_{\epsilon\to 0^+} \frac{1}{\pi} \int_{a+\delta}^{b+\delta} d\lambda\ \mathrm{Im}\, F(\lambda + i\epsilon), \qquad \mu_F(\{\lambda\}) = \lim_{\epsilon\to 0^+} \epsilon\, \mathrm{Im}\, F(\lambda + i\epsilon).$$

According to this lemma, for each $\psi \in \mathfrak{H}$ there exists a unique finite Borel measure $\mu_{A;\psi}$ s.t.

$$\langle\psi,\ \mathcal{R}_A(z)\psi\rangle_{\mathfrak{H}} = \int_{\mathbb{R}} d\mu_{A;\psi}(\lambda)\ \frac{1}{\lambda - z}.$$

[8]Such functions are holomorphic on the open upper half-plane with non-negative imaginary part.

Lemma 2.48. *Given* $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ *self-adjoint and* $\Omega \in \mathfrak{B}$*, the quantity*

$$\mu_{A;\psi}(\Omega) \equiv \int_{\mathbb{R}} d\mu_{A;\psi}(\lambda)\, \mathbb{1}_\Omega(\lambda) =: \langle \psi,\, P_A(\Omega)\psi \rangle_{\mathfrak{H}}$$

is a spectral measure, with projection-valued measure P_A.

Theorem 2.49 (Spectral theorem)**.** *For any self-adjoint operator* $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ *there exists a unique projection-valued measure* P_A *such that*

$$q_A[\psi] = \int_{\mathbb{R}} d\mu_{A;\psi}(\lambda)\, \lambda, \qquad A = \int_{\mathbb{R}} dE_{P_A}(\lambda)\, \lambda$$

where the form domain of A *is* $\mathfrak{Q}(A) = \mathfrak{D}(|A|^{1/2}) = \left\{ \psi \in \mathfrak{H} \,\middle|\, \int_{\mathbb{R}} d\mu_{A;\psi}(\lambda)\, |\lambda| < +\infty \right\}$ *and the domain can be rewritten as* $\mathfrak{D}(A) = \left\{ \psi \in \mathfrak{H} \,\middle|\, \int_{\mathbb{R}} d\mu_{A;\psi}(\lambda)\, \lambda^2 < +\infty \right\}$.

Proposition 2.50. *Let* $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ *be self-adjoint. Then,* $\operatorname{ran}(P_A(\{\lambda_0\})) = \ker(A - \lambda_0)$.

Theorem 2.51. *The spectrum of* $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ *self-adjoint is given by*

$$\sigma(A) = \{\lambda \in \mathbb{R} \mid P_A\big((\lambda - \epsilon, \lambda + \epsilon)\big) \neq 0, \quad \forall \epsilon > 0\}.$$

Additionally,

$$\sigma_{\text{disc}}(A) = \{\lambda \in \sigma_{\text{p}}(A) \mid \exists\, \epsilon > 0 \,:\, \dim \operatorname{ran}(P_A(\lambda - \epsilon, \lambda + \epsilon)) < +\infty\},$$
$$\sigma_{\text{ess}}(A) = \{\lambda \in \mathbb{R} \mid \dim \operatorname{ran}(P_A(\lambda - \epsilon, \lambda + \epsilon)) = +\infty, \quad \forall \epsilon > 0\}.$$

In particular, $P_A\big((a, b)\big) = 0$ if and only if $(a, b) \subseteq \varrho(A)$.

Corollary 2.52. *We have* $P_A(\sigma(A)) = \mathbb{1}_{\mathfrak{H}}$ *and* $P_A(\mathbb{R} \cap \varrho(A)) = 0$.

Remark 2.22. *We stress that the spectral theorem implies* $P_A(\lambda \longmapsto \lambda) = A$*. If we now assume* $A \in \mathscr{B}(\mathfrak{H})$ *, then we know* $\sigma(A)$ *is bounded* (i.e. *compact),* $\mathfrak{D}(P_A(\lambda \longmapsto \lambda)) = \mathfrak{H}$ *and*

$$A^2 = P_A(\lambda \longmapsto \lambda) P_A(\lambda \longmapsto \lambda) = P_A(\lambda \longmapsto \lambda^2),$$

since P_A *is a* **-homomorphism (indeed* $f|_{\sigma(A)}$ *is bounded even if* $f : \lambda \longmapsto \lambda$ *is not). Hence, for any polynomial* p *we know that* $P_A(p) = p(A)$*. By the* Stone–Weierstrass *theorem, the set of polynomials is dense (according to the* sup*-norm) in the space of real-valued continuous functions, so that for any continuous function* f *we can define the bounded, self-adjoint operator* $f(A)$ *as*

$$\langle \psi,\, f(A)\psi \rangle_{\mathfrak{H}} = \langle \psi,\, P_A(f)\psi \rangle_{\mathfrak{H}} = \int_{\sigma(A)} d\mu_{A;\psi}(\lambda)\, f(\lambda), \qquad \forall \psi \in \mathfrak{H}.$$

Guided by these motivations for any $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ *and* f *complex-valued Borel function, we define the operator* $f(A) := P_A(f)$*, with* $\mathfrak{D}(f(A)) = \{\psi \in \mathfrak{H} \mid f \in L^2(\sigma(A), d\mu_{A;\psi})\}$.

Theorem 2.53 (Spectral Mapping - Bounded Borel functions). *Let $f:\mathbb{R}\longrightarrow\mathbb{C}$ be a bounded Borel function and $A,\mathfrak{D}(A)\in\mathscr{L}(\mathfrak{H})$ a self-adjoint operator. Then, $f(A)\in\mathscr{B}(\mathfrak{H})$ and*

$$\sigma(f(A))=f(\sigma(A)),\qquad \|f(A)\|_{\mathscr{L}(\mathfrak{H})}=\|f\|_{L^\infty(\sigma(A))}\,.$$

Proposition 2.54. *If two self-adjoint operators $A,\mathfrak{D}(A)\in\mathscr{L}(\mathfrak{H})$ and $B,\mathfrak{D}(B)\in\mathscr{L}(\mathfrak{H})$ are compatible then $[f(A),g(B)]=0$ for any f,g bounded Borel functions.*
Additionally, the following are equivalent

i) $A,\mathfrak{D}(A)$ and $B,\mathfrak{D}(B)$ are compatible;
ii) $[e^{-itA},e^{-itB}]=0$ for all $t\in\mathbb{R}$;
iii) $A,\mathfrak{D}(A)$ commutes with e^{-itB} for all $t\in\mathbb{R}$;
iv) $B,\mathfrak{D}(B)$ commutes with e^{-itA} for all $t\in\mathbb{R}$.

Assuming B bounded (not necessarily self-adjoint), it commutes with $A,\mathfrak{D}(A)$ iff $BA\subseteq AB$. In this case one has $Bf(A)\subseteq f(A)B$ for any f Borel function, where equality holds in case f is bounded.

Proposition 2.55. *Let $A,\mathfrak{D}(A)$ and $B,\mathfrak{D}(B)$ be self-adjoint operators in $\mathscr{L}(\mathfrak{H})$ with non-empty point spectrum. A and B are compatible if and only if they have a common orthonormal basis of eigenfunctions.*

We would like to extend theorem 2.53 for unbounded Borel functions. To this end, let us consider

$$\mathfrak{H}_\psi:=\{P(g)\psi\mid g\in L^2(\mathbb{R},d\mu_\psi)\}\subseteq\mathfrak{H}$$

which is a closed subspace since $L^2(\mathbb{R},d\mu_\psi)$ is complete and $\psi_n=P(g_n)\psi$ converges in $\mathfrak{H}$ if and only if g_n converges in $L^2(\mathbb{R},d\mu_\psi)$.

Lemma 2.56. *Given $\Pi_\psi\in\mathscr{B}(\mathfrak{H})$ the projection onto $\mathfrak{H}_\psi$, one has*

$$\Pi_\psi P(f)\subseteq P(f)\Pi_\psi,\qquad \Pi_\psi P(f)\Pi_\psi=P(f)\Pi_\psi.$$

Remark 2.23. *Observe that $\Pi_\psi\mathfrak{D}(P(f))=\mathfrak{D}(P(f))\cap\mathfrak{H}_\psi$ and*

$$\begin{aligned}\mathfrak{D}(P(f))\cap\mathfrak{H}_\psi&=\{P(g)\psi\in\mathfrak{H}\mid g\in L^2(\mathbb{R},d\mu_\psi),\ f\in L^2(\mathbb{R},d\mu_{P(g)\psi})\}\\&=\{P(g)\psi\in\mathfrak{H}\mid g\in L^2(\mathbb{R},d\mu_\psi),\ f\in L^2(\mathbb{R},|g|^2d\mu_\psi)\}\\&=\{P(g)\psi\in\mathfrak{H}\mid g\in L^2(\mathbb{R},d\mu_\psi),\ fg\in L^2(\mathbb{R},d\mu_\psi)\}.\end{aligned}$$

This means that for any $\varphi\in\Pi_\psi\mathfrak{D}(P(f))$ one has a $g_\varphi\in L^2(\mathbb{R},d\mu_\psi)$ such that $g_\varphi f\in L^2(\mathbb{R},d\mu_\psi)$ and

$$P(f)\varphi=P(f)P(g_\varphi)\psi=P(fg_\varphi)\psi\in\mathfrak{H}_\psi.$$

We introduce the unitary operator

$$\begin{aligned}U_\psi:\mathfrak{H}_\psi&\longrightarrow L^2(\mathbb{R},d\mu_\psi)\\P(g)\psi&\longmapsto g.\end{aligned}$$

One has $U_\psi \Pi_\psi \mathfrak{D}(P(f)) = \{g \in L^2(\mathbb{R}, d\mu_\psi) \mid fg \in L^2(\mathbb{R}, d\mu_\psi)\} = \mathfrak{D}(M_f)$, thus

$$U_\psi P(f) \Pi_\psi = M_f U_\psi,$$

where $M_f, \mathfrak{D}(M_f) \in \mathscr{L}(L^2(\mathbb{R}, d\mu_\psi))$ is the multiplication operator by the function f.
In case $\mathfrak{H}_\psi = \mathfrak{H}$ one has that the operator $P(f), \mathfrak{D}(P(f))$ is unitary equivalent to the multiplication operator by f. In such a case ψ is said **cyclic**. Otherwise, a set $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{H}$ is called a **spectral basis** if $\mathfrak{H} = \bigoplus_{n\in\mathbb{N}} \mathfrak{H}_{\psi_n}$, where $\mathfrak{H}_{\psi_n} \perp \mathfrak{H}_{\psi_m}$ for all $n \neq m$.

Theorem 2.57. *For every projection-valued measure P, in an infinite dimensional, separable Hilbert space there exists a spectral basis $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{H}$ (at most countable). Moreover, one has the unitary operator*

$$U : \mathfrak{H} \longrightarrow \bigoplus_{n\in\mathbb{N}} L^2(\mathbb{R}, d\mu_{\psi_n})$$

$$U\Pi_{\psi_n} = U_{\psi_n}, \quad UP(f) = M_f U,$$

$$U\mathfrak{D}(P(f)) = \mathfrak{D}(M_f) = \Big\{\{g_n\}_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} L^2(\mathbb{R}, d\mu_{\psi_n}) \;\Big|\; \{fg_n\}_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} L^2(\mathbb{R}, d\mu_{\psi_n})\Big\}.$$

Definition 2.30 (Maximal Spectral Measure)**.** Given a spectral basis $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{H}$, a spectral measure μ_ψ such that $\mu_{\psi_n} \ll \mu_\psi$ for all $n \in \mathbb{N}$ is called a *maximal spectral measure* and $\psi \in \mathfrak{H}$ is the corresponding *maximal spectral vector*.

Lemma 2.58. *For every self-adjoint operator there exists a maximal spectral measure.*

Theorem 2.59 (Spectral Mapping)**.** *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a Borel function and $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ a self-adjoint operator. Then, denoting by μ its maximal spectral measure*

$$\sigma(f(A)) = \big\{z \in \mathbb{C} \;\big|\; \mu\big(f^{-1}(\mathcal{B}_\epsilon(z))\big) > 0, \quad \forall \epsilon > 0\big\}, \qquad \mathcal{B}_\epsilon(z) = \{w \in \mathbb{C} \mid |w - z| < \epsilon\}.$$

In particular,

$$\sigma(f(A)) \subseteq \overline{f(\sigma(A))},$$

where equality holds if f is continuous and the closure can be dropped in case, in addition, $\sigma(A)$ is bounded or $|f(\lambda)| \xrightarrow[\lambda\to+\infty]{} +\infty$.

Definition 2.31. Given a spectral measure $\mu_\psi : \mathfrak{B} \longrightarrow [0, +\infty]$ associated with some projection-valued measure on $\mathfrak{H}$, we define (see theorem 1.18)

- $\mathfrak{H}_{\text{ac}} = \{\psi \in \mathfrak{H} \mid \mu_\psi \text{ is absolutely continuous}\}$;
- $\mathfrak{H}_{\text{sc}} = \{\psi \in \mathfrak{H} \mid \mu_\psi \text{ is singularly continuous}\}$;
- $\mathfrak{H}_{\text{pp}} = \{\psi \in \mathfrak{H} \mid \mu_\psi \text{ is pure point}\}$.

Proposition 2.60. *One has $\mathfrak{H} = \mathfrak{H}_{\text{ac}} \oplus \mathfrak{H}_{\text{sc}} \oplus \mathfrak{H}_{\text{pp}}$.*

Definition 2.32 (Spectral Types)**.** Given a self-adjoint operator $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ and a maximal spectral measure μ_ψ (with ψ maximal spectral vector) we set

- the absolutely continuous spectrum $\sigma_{\mathrm{ac}}(A) = \sigma(A|_{\mathfrak{H}_{\mathrm{ac}}})$;
- the singularly continuous spectrum $\sigma_{\mathrm{sc}}(A) = \sigma(A|_{\mathfrak{H}_{\mathrm{sc}}})$;
- the pure point spectrum $\sigma_{\mathrm{pp}}(A) = \sigma(A|_{\mathfrak{H}_{\mathrm{pp}}})$.

In particular, proposition 2.60 implies that two unitarily equivalent self-adjoint operators preserve the classification of the spectrum given in definition 2.32.

Remark 2.24. *We stress that $\sigma_{\mathrm{pp}}(A) = \overline{\sigma_{\mathrm{p}}(A)}$ (it is the set of eigenvalues together with possible accumulation points) and $\sigma_{\mathrm{c}}(A) = \sigma_{\mathrm{ac}}(A) \cup \sigma_{\mathrm{sc}}(A) \cup (\sigma_{\mathrm{pp}}(A) \setminus \sigma_{\mathrm{p}}(A))$. Additionally, the essential spectrum is the union of the absolutely continuous spectrum, the singularly continuous spectrum and the elements in the pure point spectrum which are either isolated eigenvalues of infinite multiplicity or accumulation points. In summary,*

$$\boxed{\sigma_{\mathrm{disc}}(A) \subseteq \sigma_{\mathrm{p}}(A) \subseteq \sigma_{\mathrm{pp}}(A), \qquad \sigma_{\mathrm{ac}}(A) \cup \sigma_{\mathrm{sc}}(A) \subseteq \sigma_{\mathrm{c}}(A) \subseteq \sigma_{\mathrm{ess}}(A).}$$

Theorem 2.61 (RAGE theorem)**.** *Let $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ be self-adjoint and $\{K_n\}_{n\in\mathbb{N}} \subset \mathscr{L}(\mathfrak{H})$ a sequence of relatively compact operators with respect to $A, \mathfrak{D}(A)$ with $K_n \xrightarrow[n\to\infty]{s} \mathbb{1}_{\mathfrak{H}}$. Then,*

$$\lim_{n\to\infty} \lim_{T\to+\infty} \frac{1}{T}\int_0^T dt\ \left\|K_n e^{-itA}\psi\right\|_{\mathfrak{H}} = 0 \qquad \Longleftrightarrow \qquad \psi \in \mathfrak{H}_{\mathrm{ac}} \oplus \mathfrak{H}_{\mathrm{sc}},$$
$$\lim_{n\to\infty} \sup_{t\geq 0} \left\|(\mathbb{1}_{\mathfrak{H}} - K_n) e^{-itA}\psi\right\|_{\mathfrak{H}} = 0 \qquad \Longleftrightarrow \qquad \psi \in \mathfrak{H}_{\mathrm{pp}}.$$

Remark 2.25. *This theorem gives a qualitative characterization of a Hilbert space $\mathfrak{H}$ in terms of the maximal spectral measure of a self-adjoint operator acting on $\mathfrak{H}$. In particular, considering the Hamiltonian of a quantum system $\mathcal{H}, \mathfrak{D}(\mathcal{H}) \in \mathscr{L}(\mathfrak{H})$, the subspace $\mathfrak{H}_{\mathrm{pp}}$ is composed of* **bound states***, that's to say, vectors whose evolution is arbitrarily improbable outside a compact (think of $\mathfrak{H}$ as $L^2(\mathbb{R}^d)$ and K_n as the compact operators multiplying by $\mathbb{1}_{\mathcal{B}_n(0)}$). Conversely, for a* **scattering state** *one has an arbitrarily small (time-average) probability of staying in a compact region.*

Theorem 2.62 (Global Probabilistic Interpretation)**.** *Given a complex and separable Hilbert space $\mathfrak{H}$, for any self-adjoint operator $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$, there exists a measure space (X, Σ, ν), with ν σ-finite, a unitary operator $U : \mathfrak{H} \longrightarrow L^2(X, d\nu)$ and a real-valued, ν-measurable function h s.t.*

$$U\mathfrak{D}(A) = \{\psi \in L^2(X, d\nu) \mid h\psi \in L^2(X, d\nu)\}$$
$$UAU^*\psi = h\psi, \qquad \forall\, \psi \in U\mathfrak{D}(A).$$

Remark 2.26. *According to this theorem and the 2nd postulate, the expectation value of an observable a represented by the self-adjoint operator $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ is*

$$\mathbb{E}_\varphi[a] = \langle \varphi,\, A\varphi\rangle_{\mathfrak{H}} = \int_X d\nu(x)\ h(x)|\psi(x)|^2, \qquad U\varphi = \psi,$$
$$\mathbb{P}_\varphi(a \in \Omega) = \langle \varphi,\, \mathbb{1}_\Omega(A)\varphi\rangle_{\mathfrak{H}} = \int_X d\nu(x)\ \mathbb{1}_{h^{-1}(\Omega)}(x)|\psi(x)|^2 = \int_{h^{-1}(\Omega)} d\nu(x)\ |\psi(x)|^2, \qquad \Omega \in \mathfrak{B}.$$

This means that observables can be thought of as real random variables on the probability space $(X, \Sigma, |\psi|^2 d\nu)$ *(notice* $\|\varphi\|_{\mathfrak{H}} = 1$ *implies* $\|\psi\|_{L^2(X,\,d\nu)} = 1$*). This is the reason why sometimes it is convenient to treat the observables as time dependent rather than the states (*i.e. *the probability spaces). For instance, assuming* $\mathfrak{D}(\mathcal{H}) \subseteq \mathfrak{D}(A)$ *(then* $e^{-it\mathcal{H}}\mathfrak{D}(H) \subseteq \mathfrak{D}(A)$*) one can write*

$$A(t) := e^{it\mathcal{H}} A\, e^{-it\mathcal{H}}, \qquad \langle \psi_t,\, A\psi_t\rangle_{\mathfrak{H}} = \langle \psi_0,\, A(t)\,\psi_0\rangle_{\mathfrak{H}}, \qquad \forall\, \psi_0 \in \mathfrak{D}(H)$$

for $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ *self-adjoint. In physics literature this is known as the* **Heisenberg picture***.*

One can extend the same (global) probabilistic interpretation for a set of compatible observables.

Theorem 2.63 (Joint Probabilistic Representation)**.** *Given a complex and separable Hilbert space* $\mathfrak{H}$*, for any set of compatible self-adjoint operators* $\{A_i, \mathfrak{D}(A_i)\}_{i=1}^N \subset \mathscr{L}(\mathfrak{H})$*, there exist a measure space* (X, Σ, ν)*, with* ν σ*-finite, a unitary operator* $U : \mathfrak{H} \longrightarrow L^2(X, d\nu)$ *and a set of real-valued,* ν*-measurable functions* $\{h_i\}_{i=1}^N$ *s.t.*

$$U\mathfrak{D}(A_i) = \{\psi \in L^2(X, d\nu) \mid h_i\psi \in L^2(X, d\nu)\}$$
$$(UA_iU^*\psi)(x) = h_i(x)\psi(x), \qquad \forall\, \psi \in U\mathfrak{D}(A_i).$$

3. Observables in Quantum Mechanics

In this chapter we introduce some relevant observables associated with a quantum system in a pure state. First, we consider the cyclic group $\mathbb{Z}/_{2\mathbb{Z}} = \big(\{0,1\}, + \bmod 2\big)$. A unitary representation of this group in $L^2(\mathbb{R}^d)$ can be given by ρ_a, with $a \in \{0,1\}$ where $\rho_0 = \mathbb{1}$ while ρ_1 is the **parity operator** $\Pi : \psi(\boldsymbol{x}) \longmapsto \psi(-\boldsymbol{x})$. Such unitary operator satisfies, by construction, $\Pi^2 = \mathbb{1}$, hence it is self-adjoint (it is symmetric and everywhere-defined). In particular, $\sigma(\Pi) = \sigma_{\text{ess}}(\Pi) = \{-1, +1\}$.

Next, we take into account the group of translations given by $(\mathbb{R}^d, +)$ and two associated unitary representations in $L^2(\mathbb{R}^d)$

$$\rho \colon \mathbb{R}^d \longrightarrow \mathscr{B}\big(L^2(\mathbb{R}^d)\big), \qquad (\rho_{\boldsymbol{v}}\psi)(\boldsymbol{x}) = \psi(\boldsymbol{x} - \boldsymbol{v}),$$
$$\tilde{\rho} \colon \mathbb{R}^d \longrightarrow \mathscr{B}\big(L^2(\mathbb{R}^d)\big), \qquad (\tilde{\rho}_{\boldsymbol{v}}\psi)(\boldsymbol{x}) = e^{-i\boldsymbol{v}\cdot\boldsymbol{x}}\psi(\boldsymbol{x}).$$

In both cases, we can write these representations as follows

$$\rho_{\boldsymbol{v}} = \prod_{i=1}^{d} \rho_{v_i \boldsymbol{e}_i}, \qquad \tilde{\rho}_{\boldsymbol{v}} = \prod_{i=1}^{d} \tilde{\rho}_{v_i \boldsymbol{e}_i}, \qquad v_i = \boldsymbol{v} \cdot \mathbf{e}_i,$$

where $\{\boldsymbol{e}_i\}_{i=1}^d$ is the canonical basis of $\mathbb{R}^d$. One can check that both $\rho_{v_i \boldsymbol{e}_i}$ and $\tilde{\rho}_{v_i \boldsymbol{e}_i}$ are strongly-continuous, one-parameter unitary groups for $v_i \in \mathbb{R}$. Let's therefore compute their generators

$$P_i = i \lim_{v_i \to 0} \frac{\rho_{v_i \boldsymbol{e}_i} - \mathbb{1}}{v_i}\, \psi(\boldsymbol{x}) = -i\, \tfrac{\partial}{\partial x_i}\, \psi(\boldsymbol{x}),$$
$$Q_i = i \lim_{v_i \to 0} \frac{\tilde{\rho}_{v_i \boldsymbol{e}_i} - \mathbb{1}}{v_i}\, \psi(\boldsymbol{x}) = x_i\, \psi(\boldsymbol{x}).$$

The operator $\boldsymbol{P} = -i\nabla$ is known as the **momentum operator** and $\boldsymbol{Q}$ is the **position operator** (multiplying by the function $\boldsymbol{x} \longmapsto \boldsymbol{x}$). By the Stone's theorem we have for all $\boldsymbol{v} \in \mathbb{R}^d$

$$\rho_{\boldsymbol{v}} = e^{-i\boldsymbol{v}\cdot\boldsymbol{P}}, \quad \mathfrak{D}(\boldsymbol{P}) = H^1(\mathbb{R}^d),$$
$$\tilde{\rho}_{\boldsymbol{v}} = e^{-i\boldsymbol{v}\cdot\boldsymbol{Q}}, \quad \mathfrak{D}(Q) = \{\psi \in L^2(\mathbb{R}^d) \mid \boldsymbol{x}\,\psi(\boldsymbol{x}) \in L^2(\mathbb{R}^d, d\boldsymbol{x})\}.$$

One has $\sigma(Q_i) = \sigma_{\text{ac}}(Q_i) = \mathbb{R}$ and $\sigma(P_i) = \sigma_{\text{ac}}(P_i) = \mathbb{R}$. Additionally, for any $\psi \in L^2(\mathbb{R}^d)$ such that $\psi \in \mathfrak{D}(P_k Q_j) \cap \mathfrak{D}(Q_j P_k)$ there hold the so called **Weyl relations**

$$[Q_j, P_k]\,\psi(\boldsymbol{x}) = i\,\delta_{jk}\,\psi(\boldsymbol{x}).$$

Theorem 3.1 (Heisenberg Uncertainty Principle)**.** *Consider two symmetric operators* $A, \mathfrak{D}(A)$ *and* $B, \mathfrak{D}(B)$ *in* $\mathscr{L}(\mathfrak{H})$ *. Then, for any* $\psi \in \mathfrak{D}(AB) \cap \mathfrak{D}(BA)$ *one has*

$$\|(A - \langle A\rangle_\psi)\psi\|_{\mathfrak{H}}\, \|(B - \langle B\rangle_\psi)\psi\|_{\mathfrak{H}} \geq \tfrac{1}{2}|\langle\psi,\, [A,B]\psi\rangle_{\mathfrak{H}}|, \qquad \langle A\rangle_\psi := \langle\psi,\, A\psi\rangle_{\mathfrak{H}}.$$

Remark 3.1. *In the previous theorem, in case* A *and* B *are associated with two observables* a *and* b*, the result can be rephrased as*

$$\mathrm{Var}_\psi(a)\mathrm{Var}_\psi(b) \geq \tfrac{1}{4}\,\mathbb{E}_\psi[c]^2,$$

where c *is the observable associated with* $i[A,B]$*. This implies that two* incompatible observables *cannot be measured simultaneously with arbitrary precision in the pure state* $|\psi\rangle, \psi \in \mathfrak{H}$ *(the left-hand side cannot vanish) in case* $\mathbb{E}_\psi[c] \neq 0$*. In particular, this implies that in case the system is in the pure state* $|\varphi\rangle$*,* $\varphi \in \mathfrak{D}(AB) \cap \mathfrak{D}(BA)$*, with* φ *a given eigenstate of* $A, \mathfrak{D}(A)$ *(if there is any), then either* $\mathrm{Var}_\varphi(b) = +\infty$ *or* $\mathbb{E}_\varphi[c] = 0$*.*

However, ρ and $\tilde\rho$ are two equivalent unitary representations of the translations, since $\forall\, \boldsymbol{v} \in \mathbb{R}^d$

$$\mathcal{F}\rho_{\boldsymbol{v}} = \tilde\rho_{\boldsymbol{v}}\,\mathcal{F}, \qquad (\mathcal{F}\psi)(\boldsymbol{p}) = \tfrac{1}{(2\pi)^{d/2}} \lim_{n\to\infty} \int_{|\boldsymbol{x}|\leq n} d\boldsymbol{x}\; e^{-i\boldsymbol{x}\cdot\boldsymbol{p}}\psi(\boldsymbol{x}),$$

with $\mathcal{F}\colon L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$ the unitary *Fourier transform*.
In accordance with classical mechanics, if the pure state $|\psi\rangle$, $\psi \in L^2(\mathbb{R}^d)$ represents a quantum particle, its kinetic energy $\mathcal{H}_0$ is defined as $\frac{1}{2m}P^2$ with $m > 0$ the mass of such a particle. Clearly $\mathcal{H}_0$ and P_i are compatible observables and the momentum is conserved in a free motion.

Theorem 3.2 (Noether's theorem)**.** *If a self-adjoint operator* $A, \mathfrak{D}(A) \in \mathscr{L}(\mathfrak{H})$ *is compatible with the Hamiltonian of a quantum system* $\mathcal{H}, \mathfrak{D}(\mathcal{H}) \in \mathscr{L}(\mathfrak{H})$ *(thus* $e^{-it\mathcal{H}}A \subseteq A\,e^{-it\mathcal{H}}$*,* $\forall\, t \in \mathbb{R}$*), then* $\mathfrak{D}(A)$ *is invariant under* $e^{-it\mathcal{H}}$ *and* A *is a conserved quantity,* i.e. *for all* $\psi_0 \in \mathfrak{D}(A)$

$$\langle\psi_0,\, A\psi_0\rangle_{\mathfrak{H}} = \langle\psi_t,\, A\psi_t\rangle_{\mathfrak{H}}, \qquad \psi_t = e^{-it\mathcal{H}}\psi_0 \in \mathfrak{D}(A),\ t \in \mathbb{R}.$$

We stress that, given a pure state $|\psi_E\rangle$ with $\psi_E \in \mathfrak{D}(\mathcal{H})$ an eigenstate of the Hamiltonian with eigenvalue $E \in \mathbb{R}$, one has that its evolution in time is simply $e^{-it\mathcal{H}}|\psi_E\rangle = e^{-itE}|\psi_E\rangle \sim |\psi_E\rangle$. In other words, eigenstates of the Hamiltonian stay the same over time.
In general, a widely studied class of operators in Quantum Mechanics in $L^2(\mathbb{R}^{dN})$ is

$$\mathcal{H} = -\sum_{i=1}^{N} \frac{\Delta_{\boldsymbol{x}_i}}{2m_i} + \sum_{1\leq i<j\leq N} V_{ij}\,(|\boldsymbol{x}_i - \boldsymbol{x}_j|), \tag{$*$}$$

where the so-called *potentials* V_{ij} are the multiplication operators by the (radial) real functions V_{ij}. Operators of the form $(*)$ are called **Schrödinger operators**.

Theorem 3.3. *Given* $\mathcal{H}, H^2(\mathbb{R}^{dN}) \in \mathscr{L}\big(L^2(\mathbb{R}^{dN})\big)$ *a Schrödinger operator, in case each pairwise potential* $V_{ij} \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ *satisfies*

$$\max\{-V_{ij}, 0\} \in \begin{cases} (L^{d/2} + L^\infty)(\mathbb{R}^d), & \text{if } d \geq 3;\\ (L^{1+\epsilon} + L^\infty)(\mathbb{R}^2), & \text{for some } \epsilon > 0;\\ (L^1 + L^\infty)(\mathbb{R}^1). \end{cases}$$

Then, one has $\mathcal{H}, H^2(\mathbb{R}^{dN})$ *is self-adjoint and lower-bounded.*

Then, take account of the group $(\mathbb{R}_+, \cdot)$ and the unitary representation in $L^2(\mathbb{R}^d)$ given by

$$(\rho_\lambda \psi)(\boldsymbol{x}) = \lambda^{d/2}\psi(\lambda\boldsymbol{x}), \qquad \lambda \in \mathbb{R}_+.$$

Notice that $\rho_{e^{-s}}$ is a strongly-continuous, one-parameter unitary group for $s \in \mathbb{R}$. The associated generator is

$$D = \tfrac{1}{2}\left(\boldsymbol{Q}\cdot\boldsymbol{P} + \boldsymbol{P}\cdot\boldsymbol{Q}\right) = \boldsymbol{Q}\cdot\boldsymbol{P} - \tfrac{id}{2}\mathbb{1}, \qquad \mathfrak{D}(D) = \{\psi \in L^2(\mathbb{R}^d) \mid \boldsymbol{x}\cdot\nabla\psi(\boldsymbol{x}) \in L^2(\mathbb{R}^d, d\boldsymbol{x})\}.$$

D is known as the **dilation operator**.

Next, we consider the matrix group of rotations $SO(3)$ represented in $L^2(\mathbb{S}^2)$ as follows

$$\rho_R = \psi(R^{-1}\hat{\boldsymbol{x}}), \qquad \psi \in L^2(\mathbb{S}^2),\ R \in SO(3),\ \hat{\boldsymbol{x}} \in \mathbb{R}^3 :\ |\hat{\boldsymbol{x}}| = 1.$$

One can always decompose a rotation in $\mathbb{R}^3$ as a combination of three consecutive (counterclockwise) rotations around the axes identified by the canonical basis $\{\boldsymbol{e}_j\}_{j=1}^3$

$$R(t_1,t_2,t_3) = M_3(t_3)M_2(t_2)M_1(t_1), \qquad t_i \in [0, 2\pi),$$

$$M_1(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{bmatrix}, \qquad M_2(t) = \begin{bmatrix} \cos(t) & 0 & \sin(t) \\ 0 & 1 & 0 \\ -\sin(t) & 0 & \cos(t) \end{bmatrix},$$

$$M_3(t) = \begin{bmatrix} \cos(t) & -\sin(t) & 0 \\ \sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, a rotation by a given angle θ around the axis $\boldsymbol{e}_j$ is $\rho_{M_j(\theta)} : \psi(\hat{\boldsymbol{x}}) \longmapsto \psi(M_j^{-1}(\theta)\,\hat{\boldsymbol{x}})$ which turns out to be a strongly-continuous, one-parameter, unitary group with generator

$$L_j = -i\sum_{k,\ell=1}^{3} \epsilon_{jk\ell}\, x_k\, \tfrac{\partial}{\partial x_\ell}$$

called **angular momentum** (here ϵ_{ijk} is the *Levi-Civita symbol*). In other words, $\boldsymbol{L} = \boldsymbol{Q}\wedge\boldsymbol{P}$ and

$$[L^2, L_i] = 0, \qquad [L_j, X_k] = i\sum_{\ell=1}^{3}\epsilon_{jk\ell}X_\ell, \qquad X_k \in \{L_k, Q_k, P_k\},$$

with $L^2 = L_1^2 + L_2^2 + L_3^2$. Concerning the spectrum, one has that L^2 and one single L_j (*e.g.* without loss of generality L_3) have a common orthonormal basis of eigenfunctions $\psi_{\ell,m}$ (known as *spherical harmonics*) with

$$L^2\psi_{\ell,m} = \ell(\ell+1)\psi_{\ell,m}, \qquad L_3\psi_{\ell,m} = m\psi_{\ell,m}, \qquad \ell \in \mathbb{N}_0,\ m \in \mathbb{Z} :\ |m| \le \ell.$$

$$\sigma(L^2) = \sigma_{\text{disc}}(L^2) = \{\ell(\ell+1) \mid \ell \in \mathbb{N}_0\}, \qquad \sigma(L_3) = \sigma_{\text{disc}}(L_3) = \mathbb{Z}.$$

Our last example will be given by the action of the matrix group $SU(2,\mathbb{C})$ in the Hilbert space $\mathbb{C}^2$, where

$$SU(2,\mathbb{C}) := \left\{ \begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix} \,\middle|\, \alpha,\beta \in \mathbb{C} : \ |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

In particular, given the **Pauli matrices** $\{\sigma_i\}_{i=1}^3$

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

one has for any $\boldsymbol{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$

$$U_j(a_j) := \cos(a_j)\,\mathbb{1} - i\sin(a_j)\,\sigma_j \in SU(2,\mathbb{C}), \qquad \forall\, j \in \{1,2,3\},$$

since $\alpha_j = \cos(a_j) - i\,\delta_{j,3}\sin(a_j)$ and $\beta_j = -i^j\sin(a_j)(1-\delta_{j,3})$. Our representation of $SU(2,\mathbb{C})$ in $\mathbb{C}^2$ shall be given by

$$\rho_{\mathcal{U}(\boldsymbol{t})} = \mathcal{U}(\boldsymbol{t}) := U_3(\tfrac{t_3}{2})\,U_2(\tfrac{t_2}{2})\,U_1(\tfrac{t_1}{2}) \in SU(2,\mathbb{C}), \ \boldsymbol{t} = (t_1,t_2,t_3) \in \mathbb{R}^3.$$

One can verify that $\{U_i(\frac{t_i}{2})\}_{i=1}^3$ are three strongly-continuous, one-parameter unitary groups with generators $S_i = \frac{1}{2}\sigma_i$, which are known as **$^1\!/_2$ – spin operators**. In this situation the spectrum is simply $\sigma(S_i) = \sigma_{\text{disc}}(S_i) = \{\pm\frac{1}{2}\}$ and once again

$$[S^2, S_i] = 0, \qquad [S_j, S_k] = i\sum_{\ell=1}^3 \epsilon_{jk\ell}\, S_\ell,$$

where $S^2 = S_1^2 + S_2^2 + S_3^2$. This is due to the fact that the *Lie algebras* associated with the *Lie groups* $SO(3)$ and $SU(2,\mathbb{C})$ are isomorphic.

This construction can be generalized in order to obtain a unitary representation of $SU(2,\mathbb{C})$ in $\mathbb{C}^{2s+1}$, with $s \in \frac{1}{2}\mathbb{N}$, obtaining the s – spin operators satisfying

$$S^2 = s(s+1)\,\mathbb{1}, \qquad [S_j, S_k] = i\sum_{\ell=1}^3 \epsilon_{jk\ell}\, S_\ell.$$

In this case one has a common orthonormal basis of eigenvectors $\psi_{s,m} \in \mathbb{C}^{2s+1}$ for both S^2 and S_3, provided by $\psi_{s,m} = \boldsymbol{e}_{s-m+1}$ with $\{\boldsymbol{e}_i\}_{i=1}^{2s+1}$ the canonical basis of $\mathbb{R}^{2s+1}$ and

$$S^2\psi_{s,m} = s(s+1)\,\psi_{s,m}, \qquad S_3\,\psi_{s,m} = m\,\psi_{s,m}, \qquad m \in \{-s, -s+1, \dots, s\}.$$

More precisely, one has the explicit entries of such matrices (let $j,k \in \{1,\dots,2s+1\}$)

$$(S_1)_{jk} = \tfrac{\delta_{j,k+1}+\delta_{j+1,k}}{2}\sqrt{(s+1)(j+k-1)-jk},$$
$$(S_2)_{jk} = \tfrac{\delta_{j,k+1}-\delta_{j+1,k}}{2}\, i\sqrt{(s+1)(j+k-1)-jk},$$
$$(S_3)_{jk} = (s+1-j)\,\delta_{j,k}.$$

For instance, in case $s = 1$

$$S_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad S_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \qquad S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Proposition 3.4. *Given $\boldsymbol{S}$ a spin operator acting on $\mathbb{C}^{2s+1}$ and $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ one has*

$$[\boldsymbol{a}\cdot\boldsymbol{S}, \boldsymbol{b}\cdot\boldsymbol{S}] = i\,(\boldsymbol{a}\wedge\boldsymbol{b})\cdot\boldsymbol{S},$$
$$R_{\boldsymbol{a}}\boldsymbol{b} = \tfrac{1}{2}\operatorname{Tr}\left[e^{-i\boldsymbol{a}\cdot\boldsymbol{S}}(\boldsymbol{b}\cdot\boldsymbol{S})\,e^{i\boldsymbol{a}\cdot\boldsymbol{S}}\boldsymbol{S}\right],$$

where $R_{\boldsymbol{a}} \in SO(3)$ is a rotation by an angle $|\boldsymbol{a}|$ around the axis generated by $\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$.

Remark 3.2. *From the previous proposition we find that a generic element of $SU(2,\mathbb{C})$ denoted by $U(\boldsymbol{a})$ (in our case $U(\boldsymbol{a}) = e^{-i\boldsymbol{a}\cdot\boldsymbol{S}}$) is mapped into an element of $SO(3)$. This map is surjective, but not injective. Indeed, in case $s = 1/2$, one can exploit the fact that $(\boldsymbol{a}\cdot\boldsymbol{\sigma})^2 = |\boldsymbol{a}|^2\,\mathbb{1}$ in order to explicitly compute*

$$e^{-i\boldsymbol{a}\cdot\boldsymbol{S}} = \cos\left(\tfrac{|\boldsymbol{a}|}{2}\right)\mathbb{1} - i\,\tfrac{\boldsymbol{a}\cdot\boldsymbol{\sigma}}{|\boldsymbol{a}|}\sin\left(\tfrac{|\boldsymbol{a}|}{2}\right)$$

which is in $SU(2,\mathbb{C})$ since $\alpha = \cos\left(\frac{|\boldsymbol{a}|}{2}\right) - i\,\frac{a_3}{|\boldsymbol{a}|}\sin\left(\frac{|\boldsymbol{a}|}{2}\right)$ and $\beta = \frac{a_2 - i a_1}{|\boldsymbol{a}|}\sin\left(\frac{|\boldsymbol{a}|}{2}\right)$. Here one can consider the couple of elements

$$U(\boldsymbol{a}) = \cos\left(\tfrac{|\boldsymbol{a}|}{2}\right)\mathbb{1} - i\,\hat{\boldsymbol{a}}\cdot\boldsymbol{\sigma}\sin\left(\tfrac{|\boldsymbol{a}|}{2}\right), \qquad U(2\pi\hat{\boldsymbol{a}} - \boldsymbol{a}) = -\cos\left(\tfrac{|\boldsymbol{a}|}{2}\right)\mathbb{1} - i\,\hat{\boldsymbol{a}}\cdot\boldsymbol{\sigma}\sin\left(\tfrac{|\boldsymbol{a}|}{2}\right),$$

where we took the angle $|\boldsymbol{a}| \in [0, 2\pi)$. By the statement of the previous proposition, both elements of $SU(2,\mathbb{C})$ are associated with the same rotation. Actually, one has the following isomorphism

$$SO(3) \cong {}^{SU(2,\mathbb{C})}\!\big/_{\{\pm\mathbb{1}\}}\,.$$

In general, in the one-particle Hilbert space $L^2(\mathbb{R}^3)\otimes\mathbb{C}^{2s+1} \cong L^2(\mathbb{R}_+, r^2dr)\otimes L^2(\mathbb{S}^2)\otimes\mathbb{C}^{2s+1}$ one can define the **total angular momentum** given by $\boldsymbol{J} = \mathbb{1}\otimes\boldsymbol{L}\otimes\mathbb{1} + \mathbb{1}\otimes\mathbb{1}\otimes\boldsymbol{S}$.

In conclusion we want to convince the reader that in case the quantum system exhibits a *symmetry*, we know, by the means of the Noether's theorem, that there shall be a conserved quantity.
More precisely, given a group $(G,\cdot)$ and a Hamiltonian $\mathcal{H}, \mathfrak{D}(\mathcal{H}) \in \mathscr{L}(\mathfrak{H})$ we say that G is a group of symmetry for $\mathcal{H}, \mathfrak{D}(\mathcal{H})$ if there exists a unitary representation $\rho\colon G \longrightarrow \mathscr{B}(\mathfrak{H})$ such that for all $g \in G$ one has ρ_g and $\mathcal{H}, \mathfrak{D}(\mathcal{H})$ commute, namely[1] $\rho_g\mathfrak{D}(\mathcal{H}) \subseteq \mathfrak{D}(\mathcal{H})$ and $\rho_g\mathcal{H}\rho_g^* = \mathcal{H}$ on $\mathfrak{D}(\mathcal{H})$.
Notice that in case there are two equivalent representations ρ and $\tilde{\rho}$ of the group G, namely there exists a unitary operator $U \in \mathscr{B}(\mathfrak{H})$ such that $\tilde{\rho}_g U = U\rho_g$ for all $g \in G$, one has

$$\rho_g\mathcal{H} \subseteq \mathcal{H}\rho_g \qquad \Longleftrightarrow \qquad \tilde{\rho}_g\mathcal{H}' \subseteq \mathcal{H}'\tilde{\rho}_g,$$

where $\mathcal{H}' = U\mathcal{H}U^*$ and $\mathfrak{D}(H') = U\mathfrak{D}(\mathcal{H})$. However, in case a quantum system whose energy is described by the Hamiltonian $\mathcal{H}, \mathfrak{D}(\mathcal{H})$ has a group of symmetry, we still need to understand which is the conserved quantity, since we know that the unitary representation of the group commutes with the Hamiltonian, but it is not in general an observable (*i.e.* a self-adjoint operator). Therefore, suppose the group of symmetry for a Hamiltonian $\mathcal{H}, \mathfrak{D}(\mathcal{H}) \in \mathscr{L}(\mathfrak{H})$ is a topological

[1] Actually, $\rho_g\mathfrak{D}(\mathcal{H}) \subseteq \mathfrak{D}(\mathcal{H})$ for any $g \in G$ implies $\mathfrak{D}(\mathcal{H}) \subseteq \rho_{g^{-1}}\mathfrak{D}(\mathcal{H})$, hence $\rho_g\mathfrak{D}(\mathcal{H}) = \mathfrak{D}(\mathcal{H})$.

group $(G,\cdot)$ and suppose there exists a *group homomorphism*[2] $\jmath\colon(\mathbb{R},+)\longrightarrow(G,\cdot)$, *i.e.* a continuous map satisfying

$$\jmath(t+s)=\jmath(t)\cdot\jmath(s),\qquad\forall\, t,s\in\mathbb{R},$$

from which one also obtains that $\jmath(0)$ is the neutral element of G and $\jmath(-t)=\jmath(t)^{-1}$. We stress that this means the image of $\jmath$ is in general an *abelian subgroup* of G (surjectivity is not required). In this situation, since ρ is a unitary representation, one has

$$\rho_{\jmath(0)}=\mathbb{1}_{\mathfrak{H}},\qquad \rho_{\jmath(s+t)}=\rho_{\jmath(s)\cdot\jmath(t)}=\rho_{\jmath(s)}\rho_{\jmath(t)},\qquad \rho_{\jmath(t)}\xrightarrow[t\to t_0]{s}\rho_{\jmath(t_0)}.$$

Now it should be clear that $\{\rho_{\jmath(t)}\}_{t\in\mathbb{R}}$ is a strongly-continuous, one-parameter, unitary group, hence, by Stone's theorem we know there exists a unique self-adjoint operator $\mathcal{G},\mathfrak{D}(\mathcal{G})\in\mathscr{L}(\mathfrak{H})$ such that $\rho_{\jmath(t)}=e^{-it\mathcal{G}}$ and since ρ_g commutes with $\mathcal{H},\mathfrak{D}(H)$ for any $g\in G$ one also has that $\mathcal{G},\mathfrak{D}(\mathcal{G})$ is compatible with the Hamiltonian and it is a conserved quantity of the system, by Noether's theorem.

[2] Notice that there is always the trivial case for which $\jmath$ maps all real numbers in the neutral element of the group, but in this situation the generator of the corresponding strongly-continuous, one-parameter, unitary group is the zero operator, which is not interesting.

References

[1] B. C. Hall, *Quantum Theory for Mathematicians*. Springer New York, Graduate Texts in Mathematics **267** (2013).

[2] G. Teschl, *Mathematical Methods in Quantum Mechanics - with Applications to Schrödinger operators* 2^{nd} edition. AMS, Graduate Studies in Mathematics **157** (2014).

[3] A. Teta, *A Mathematical Primer on Quantum Mechanics*. Springer Cham, UNITEXT for Physics (2018).