Gran Sasso Science Institute
Mathematics Area
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Introduction to the mathematical analysis of incompressible Euler and Navier-Stokes equations
Sheet 5 - 2025, January 23

Exercise 18. Consider the operator $\mathbb{A} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2;\mathbb{R}^2)$ defined as Fourier multiplier by

$$\mathbb{A}(\vartheta) := \mathcal{F}^{-1}\left(\frac{\mathrm{i}\xi^\perp}{|\xi|}\mathcal{F}(\vartheta)\right),$$

where $\xi^\perp := (-\xi_2, \xi_1)$.

1. Show that $\mathbb{A}$ commutes with derivatives, i.e.

$$\mathbb{A}(\partial^\alpha \vartheta) = \partial^\alpha \mathbb{A}(\vartheta)$$

for every $\vartheta \in \mathcal{S}(\mathbb{R}^2)$ and every multiindex $\alpha \in \mathbb{N}^2$.

2. For all $s \geq 0$, use Point 1. and the density of $\mathcal{S}(\mathbb{R}^2)$ in H^s to show that $\mathbb{A} : H^s \to H^s$ is linear and bounded.

3. Show that $\mathrm{div}\mathbb{A}(\vartheta) = 0$ (in the sense of distributions) for all $\vartheta \in L^2$.

Exercise 19. Consider the surface quasigeostrophic equations (SQG)[a] on $\mathbb{R}^2$

$$\begin{cases} \partial_t \vartheta + u \cdot \nabla \vartheta = 0, \\ u = -\mathbb{A}(\vartheta), \\ \vartheta|_{t=0} = \vartheta_0, \end{cases} \tag{11}$$

where

- $\mathbb{A}$ is the operator defined in Exercise 18,
- $\vartheta : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is the unknown,
- $u : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field, which is obtained from ϑ by applying the operator $\mathbb{A}$ at each time $t \in [0,T]$, i.e. $u(t,\cdot) = \mathbb{A}(\vartheta(t,\cdot))$.

Let $s > 2$. In analogy to the local well posedness theory for the Euler equations in H^s:

1. give a reasonable definition of *strong solution* to (11), using the integral form of the equation;

2. show that any strong solution (as defined in Point 1.) is continuously differentiable in space and time and it solves the equation in differential form;

[a]The SQG equations arise in the study of atmospheric dynamics and it is also considered an interesting model to study an easier global regularity problem closely related to the 3D Euler regularity problem.

3. show that, for all $\vartheta_0 \in H^s$, there exists a unique strong solution $\vartheta \in C([-T,T];H^s)$ to (11) where

$$T := \frac{1}{C_s \|\vartheta_0\|_{H^s}}$$

and C_s is a constant depending only on s.

Exercise 20. Show that for $s > 0$, $\xi, \eta \in \mathbb{R}^d$,

$$|\langle \xi \rangle^s - \langle \eta \rangle^s| \leq C_s \left(\langle \eta \rangle^{s-1} + \langle \xi - \eta \rangle^{s-1} \right) \langle \xi - \eta \rangle.$$

Exercise 21. Let $f \in \mathcal{S}(\mathbb{R}^d)$. Let $m \in \mathbb{N}$. Show that

$$\|\nabla^i f\|_{L^{\frac{2m}{i}}} \leq C_{i,m} \|f\|_{L^\infty}^{1-\frac{i}{m}} \|\nabla^m f\|_{L^2}^{\frac{i}{m}}$$

for all $0 \leq i \leq m$.

Deduce that the following three norms on H^m are equivalent

$$\begin{aligned} \|u\|_{H^m} &:= \|\langle \xi \rangle^m \hat{u}\|_{L^2} \\ \|u\|_{H^m} &:= \sum_{i=0}^{m} \|\nabla^i u\|_{L^2} \\ \|u\|_{H^m} &:= \|u\|_{L^2} + \|\nabla^m u\|_{L^2}. \end{aligned}$$

Exercise 22. Show rigorously that the strong solutions to the Euler equations constructed in the lecture are indeed strongly continuous as maps $(-T_0, T_0) \to H^s$.