Gran Sasso Science Institute
Mathematics Area
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Introduction to the mathematical analysis of incompressible Euler and Navier-Stokes equations
Sheet 3 - 2024, November 28

In this exercise sheet we prove that the Hilbert transform is a bounded operator $L^p(\mathbb{R}) \to L^p(\mathbb{R})$ for $p \in (1,2)$.

Exercise 9 (The Hilbert transform is well defined)**.** For $f \in \mathcal{S}(\mathbb{R})$, consider the Hilbert transform

$$\mathcal{H}(f)(x) := \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy. \tag{4}$$

Show that the limit in (4) exists for all x and it is uniform in x. Deduce from this that $\mathcal{H}f$ is continuous and bounded.

Exercise 10 (Strong L^2 estimate)**.** Show that

$$\|\mathcal{H}f\|_{L^2(\mathbb{R})} \le C\|f\|_{L^2(\mathbb{R})}.$$

In particular $\mathcal{H}$ defined a linear bounded operator $L^2 \to L^2$.
Hint: Follow the steps:

1. *Show that the equality*
$$\mathcal{H}f = T * f \tag{5}$$
holds in $\mathcal{S}'(\mathbb{R})$ (space of tempered distributions). Here T is the tempered distribution associated to the map $1/t$, as defined in Exercise 8.
2. *By taking the Fourier transform in (5), deduce that $\mathcal{H}f \in L^2(\mathbb{R})$ and*
$$\|\mathcal{H}f\|_{L^2(\mathbb{R})} \le C\|f\|_{L^2(\mathbb{R})}.$$
In particular, $\mathcal{H}$ extends to a bounded linear operator $L^2 \to L^2$.

Exercise 11 (Calderon-Zygmund decomposition)**.** Let $f : \mathbb{R} \to [0,\infty)$ be in $L^1(\mathbb{R})$. Let $\lambda > 0$. Show that there is a (finite or countable) family $\{I_k\}_{k\in\mathbb{N}}$ of closed intervals with pairwise disjoint interiors such that

(a) $f(x) \le \lambda$ for a.e. $x \in \mathbb{R} \setminus \bigcup_k I_k$,

(b) $|\bigcup_k I_k| \le \frac{C}{\lambda}\|f\|_{L^1(\mathbb{R})}$,

(c) $\lambda \le \fint_{I_k} f < 2\lambda$ for all k.

Hint: Follow the steps:

1. *Find first $a > 0$ such that $\fint_{[na,(n+1)a]} f < 2\lambda$ for all $n \in \mathbb{Z}$.*
2. *Select all intervals $[na,(n+1)a]$ where it also holds $\lambda \le \fint_{[na,(n+1)a]} f$.*
3. *Split the remaining ones in two and proceed by induction.*

Exercise 12 (Weak L^1-estimate)**.** Show that, for all $f \in \mathcal{S}(\mathbb{R})$,

$$|\{x \in \mathbb{R} : \mathcal{H}f(x) > \lambda\}| \leq \frac{C}{\lambda}\|f\|_{L^1(\mathbb{R})}. \tag{6}$$

Follow the steps:

1. *Assume that* $f \in L^1(\mathbb{R})$ *with* $\operatorname{supp} f \subseteq [-a, a]$ *and* $⨍_{[-a,a]} f = 0$. *Show that*

$$\|\mathcal{H}f\|_{L^1(\mathbb{R}\setminus[-2a,2a])} \leq C\|f\|_{L^1([-a,a])}.$$

2. *More generally, assume that* $f = \sum_{k\in\mathbb{N}} f_k$, *where each* $f_k \in L^1(\mathbb{R})$, $\operatorname{supp} f_k \subseteq I_k$, $⨍_{I_k} f_k = 0$, *and the* I_k*'s are closed intervals with pairwise disjoint interior. Similarly to the previous point, show that*

$$\|\mathcal{H}f\|_{L^1(\mathbb{R}\setminus\bigcup_k 2I_k)} \leq C\|f\|_{L^1(\mathbb{R})},$$

   where $2I_k$ *denotes the interval having the same center as* I_k, *but double length.*

3. *Prove the weak* L^1*-estimate* (6) *holds when* $\lambda \geq \|f\|_{L^\infty}$.
4. *Prove now* (6) *for any* $f \geq 0$. *To do this, combine the Calderon-Zygmund decomposition of Exercise 11 with the previous two points.*
5. *For general* f, *write* $f = f^+ - f^-$, *where* $f^\pm$ *are the positive/negative part of* f.

Exercise 13 (Strong L^p estimate, $p \in (1, 2)$)**.** Show that, for all $p \in (1, 2)$,

$$\|\mathcal{H}f\|_{L^p(\mathbb{R})} \leq C_p\|f\|_{L^p(\mathbb{R})}.$$

In particular, $\mathcal{H}$ defines a linear bounded operator $L^p \to L^p$, $p \in (1, 2)$.

Hint: Use an interpolation *argument similar to the one used for the maximal function in Exercise 6.*

Exercise 14 (Strong L^p estimate, $p \in (2, \infty)$)**.** Show that, for all $p \in (2, \infty)$,

$$\|\mathcal{H}f\|_{L^p(\mathbb{R})} \leq C_p\|f\|_{L^p(\mathbb{R})}.$$

In particular, $\mathcal{H}$ defines a linear bounded operator $L^p \to L^p$, $p \in (2, \infty)$.

Hint: Follow the steps:

1. *Prove that* $\mathcal{H}$ *is skew adjoint, i.e.*

$$\int \mathcal{H}f g dx = -\int f\mathcal{H}g, \qquad f, g \in \mathcal{S}(\mathbb{R}).$$

2. *Use a* duality *argument based on the dual characterization of the* L^p *norm*

$$\|g\|_{L^p} = \sup_{\substack{\varphi\in\mathcal{S}(\mathbb{R}) \\ \|\varphi\|_{L^{p'}}\leq 1}} \int g\varphi dx.$$