Gran Sasso Science Institute
Mathematics Area
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Introduction to the mathematical analysis
of incompressible Euler and Navier-Stokes equations
Sheet 4 - 2024, December 9

Exercise 15 (Infinite time blow up of the vorticity)**.** For 3D Euler equations it is not known whether the vorticity can blow up in finite time. Show that, at least, blow up in *infinite* time is possible. More precisely:

1. Construct an explicit, smooth, global solution to 3D Euler on $\mathbb{R}^3$ such that
$$\lim_{t\to\infty} \|\omega(t,\cdot)\|_{L^\infty} = \infty.$$
Hint (by Yudovich): Use a 2.5 dimensional flows, whose "purely 2d" part is a shear flow.

2. At what rate does $\|\omega(t,\cdot)\|_{L^\infty}$ blow up in the previous point?

3. Is it possible to obtain exponential blow up? I.e.
$$\|\omega(t,\cdot)\|_{L^\infty} \geq ce^t, \text{ for some } c > 0.$$
Hint: Substitute the shear flow in Point 1 with $V(y,x) = (-y,z)$.

4. (*) Observe that the stationary vector field V (and thus the whole 2.5 dimensional velocity field) used in Point 3 is not bounded. Modify the example to show exponential blow up of the vorticity on a periodic domain (thus, in particular, with a bounded velocity field).
Hint: Substitute the stationary vector field V in Point 3 with a cellular flow.

Exercise 16 (Biot-Savart on $\mathbb{R}^3$)**.** Consider the $\operatorname{div} - \operatorname{curl}$ system on $\mathbb{R}^3$

$$\begin{cases} \operatorname{curl} u = \omega, \\ \operatorname{div} u = 0 \end{cases} \tag{7}$$

for a given $\omega \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$ with $\operatorname{div}\omega = 0$. We showed in the lecture that one solution to this system is provided by the 3D Biot-Savart law

$$u = \operatorname{curl}\psi, \qquad \psi := (-\Delta)^{-1}\omega, \tag{8}$$

and $(-\Delta)^{-1}$ is meant in the sense of convolution with the Newton kernel.

1. Show that if $\tilde{\Psi}$ is an arbitrary solution to $-\Delta\tilde{\psi} = \omega$, then it is not true in general that $u = \operatorname{curl}\tilde{\psi}$ solves (7).

2. Show that solutions to (7) are not unique in general.

3. Show that the solution u given by the Biot-Savart law (8) can be written in convolution form as

$$u(x) = \int_{\mathbb{R}^3} K(x-y) \times \omega(y) dy$$

for some kernel $K \in L^1_{\text{loc}}(\mathbb{R}^3; \mathbb{R}^3)$, and compute explicitly K.

4. Consider the mapping $\omega \mapsto \nabla u$, where u is given by the Biot-Savart law (8). Show that in Fourier variables it holds

$$\widehat{(\nabla u)}(\xi) = -\left(\frac{\xi}{|\xi|} \times \hat{\omega}(\xi)\right) \otimes \frac{\xi}{|\xi|}$$

Exercise 17 (Biot-Savart on $\mathbb{R}^2$)**.** Consider the $\text{div} - \text{curl}$ system on $\mathbb{R}^2$

$$\begin{cases} \text{curl}\, u = \omega, \\ \text{div} u = 0 \end{cases} \tag{9}$$

for a given $\omega \in C_c^\infty(\mathbb{R}^2)$.

1. Show that one solution to (9) is given by the 2D Biot-Savart law

$$u = \nabla^T \psi, \qquad \psi := \Delta^{-1}\omega, \tag{10}$$

and Δ^{-1} is meant in the sense of convolution with the Newton kernel. Notice that, differently from the 3D case, here both ω and ψ are scalar functions.

2. Differently from the 3D case, show that if $\tilde{\Psi}$ is an arbitrary solution to $\Delta\tilde{\psi} = \omega$, then still $u = \nabla^T \tilde{\psi}$ solves (9).

3. Deduce from the previous point that, as in the 3D case, solutions to (9) are not unique in general.

4. Show that the solution u given by the Biot-Savart law (10) can be written in convolution form as

$$u(x) = \int_{\mathbb{R}^3} K(x-y)\omega(y) dy$$

for some kernel $K \in L^1_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2)$, and compute explicitly K.

5. Consider the mapping $\omega \mapsto \nabla u$, where u is given by the Biot-Savart law (10). Show that in Fourier variables it holds

$$\widehat{(\nabla u)}(\xi) = -\left(\frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|}\right) \hat{\omega}(\xi).$$