Gran Sasso Science Institute
Mathematics Area
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**Introduction to the mathematical analysis of incompressible Euler and Navier-Stokes equations**
Sheet 2 - November 14, 2024

**Exercise 5** (Vitali type covering)**.** Let $E \subseteq \mathbb{R}^d$ be measurable and let $\mathcal{F}$ be a (possibly uncountable) family of balls such that

$$E \subseteq \bigcup \mathcal{F}.$$

Then there is a (finite or countable) subfamily

$$\mathcal{F}' = \{B_1, B_2, \cdots\} \subseteq \mathcal{F}$$

such that:

(a) balls in $\mathcal{F}'$ are pairwise disjoint,

(b) $|E| \leq C_d \sum_k |B_k|$, for some constant $C_d$ depending only on $d$.

**Exercise 6** (Maximal function)**.** For $f \in L^1_{\text{loc}}(\mathbb{R}^d)$, recall the definition of the *maximal function*

$$Mf(x) := \sup_{r>0} \fint_{B_r(x)} |f(y)|dy.$$

Show that

$$\|Mf\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p} \tag{2}$$

for any $p \in (1, \infty]$.
*Follow the steps:*

1. *Prove first the trivial case $p = \infty$.*
2. *For a measurable map $\varphi$ on $\mathbb{R}^d$, denote by*

$$D_\varphi(\alpha) := |\{x : |\varphi(x)| > \alpha\}|.$$

*Using Exercise 5, show that*

$$D_{Mf}(\alpha) \leq \frac{C}{\alpha} \|f\|_{L^1}.$$

*This is called* weak-$L^1$ estimate.

3. *For fixed $\alpha \in (0, \infty)$, split $f$ as the sum of an $L^1$ and an $L^\infty$ map:*

$$f = g + h = f\chi_{\{|f|>\alpha\}} + f\chi_{\{|f|\leq\alpha\}} \leq f\chi_{\{|f|>\alpha\}} + \alpha.$$

*and observe that*

$$D_{Mf}(2\alpha) \leq D_{Mg}(\alpha) \leq \frac{C}{\alpha}\|g\|_{L^1(\mathbb{R}^d)} = \frac{C}{\alpha}\int_{\{x\,:\,|f(x)|>\alpha\}} |f(x)|dx.$$

4. *Show that, for any measurable map $\varphi$ and any $p \in [1, \infty)$,*

$$\int_{\mathbb{R}^d} |\varphi(x)|^p dx = p\int_0^\infty \alpha^{p-1} D_\varphi(\alpha) d\alpha.$$

5. *Combine all previous points to estimate* $\|Mf\|_{L^p}$. *This is a typical example of the more general* Marcinkiewicz interpolation Theorem.

**Exercise 7.** Prove that (2) does not hold for $p = 1$.

**Exercise 8.** Consider the map $T : \mathcal{S}(\mathbb{R}) \to \mathbb{R}$

$$f \mapsto T(f) := \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{f(x)}{x}. \tag{3}$$

(i) Show that $T$ is well defined.

(ii) Show that $T$ is a tempered distribution.

(iii) Show that $\hat{T}$ (the Fourier transform of $T$) is represented by a function $m(\xi) = c\,\mathrm{sign}(\xi)$, for some constant $c \in \mathbb{C}$.

(iv) Compute $c$.