Gran Sasso Science Institute
Mathematics Area
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**Introduction to the mathematical analysis**
**of incompressible Euler and Navier-Stokes equations**
Sheet 1 - November 7, 2024

**Exercise 1.** Let $u : I \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuously differentiable vector field satisfying $\sup_t \|\nabla u(t, \cdot)\|_{L^\infty} < \infty$. Let $X, A : I \times \mathbb{R}^d \to \mathbb{R}^d$ be the associated flow map and back-to-labels map respectively and $J(t, a) := \det \nabla_a X(t, a)$ the associated Jacobian. Show that

(i) $X$ satisfies

$$e^{-\int_0^t \|\nabla u(s)\|_{L^\infty} ds} \leq \frac{|X(t, a) - X(t, b)|}{|a - b|} \leq e^{\int_0^t \|\nabla u(s)\|_{L^\infty} ds},$$

(ii) $A$ satisfies $\partial_t A + u \cdot \nabla A = 0$, $A|_{t=0} = \mathrm{id}$,

(iii) $J$ satisfies $\partial_t J(t, a) = J(t, a)(\mathrm{div} u)(t, X(t, a))$, $J(0, a) \equiv 1$.

**Exercise 2.** Let $f \in W^{1,1}(\mathbb{R}^d)$. Show that

$$\int_{\mathbb{R}^d} \partial_j f = 0 \text{ for all } j \in 1, \dots, d.$$

**Exercise 3** (Laplace equation)**.** Let $\Omega \subseteq \mathbb{R}^d$. Let $u \in C^2(\Omega)$. Show that

(i) $\Delta u = 0$ if and only if

$$u(x) = \fint_{\partial B_r(x)} u(y) dS(y) = \fint_{B_r(x)} u(y) dy \text{ for all } B_r(x) \subset\subset \Omega.$$

Such $u$'s are called *harmonic maps* (on $\Omega$).

(ii) Let $u$ be harmonic on $\Omega$. Prove that $u \in C^\infty$.

(iii) (Liouville's Theorem) Let $u$ be harmonic on $\mathbb{R}^d$. Show that $u$ bounded implies $u$ constant.

**Exercise 4** (Poisson equation)**.**

(i) Find all radial maps $u$ satisfying $\Delta u = 0$ on $\Omega = \mathbb{R}^d \setminus \{0\}$.

*$u$ being* radial *means that there is $g : (0, \infty) \to \mathbb{R}$ such that $u(x) = g(|x|)$.*

(ii) Show that all $u$'s from the previous point are in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. What is $\Delta u$ in $\mathcal{D}'(\mathbb{R}^d)$?

(iii) Define, for $x \neq 0$, the *Newton potential*

$$N(x) := \begin{cases} -\frac{1}{2\pi} \log |x|, & d = 2, \\ \frac{1}{d(d-2)\alpha_d} \frac{1}{|x|^{d-2}}, & d \geq 3, \end{cases}$$

where $\alpha_d$ is the volume of the unit ball in $\mathbb{R}^d$. Show that for all $f \in C_c^\infty(\mathbb{R}^d)$, the map $u := N * f$ solves

$$-\Delta u = f. \tag{1}$$

*Here $N * f$ denotes the convolution between $N$ and $f$. Note that this convolution is well defined because $N \in L^1_{\text{loc}}(\mathbb{R}^d)$ and $f$ has compact support.*

(iv) Let $d \geq 3$. Show that $N * f$ is the unique *bounded* solution solution (up to additive constant) to (1).