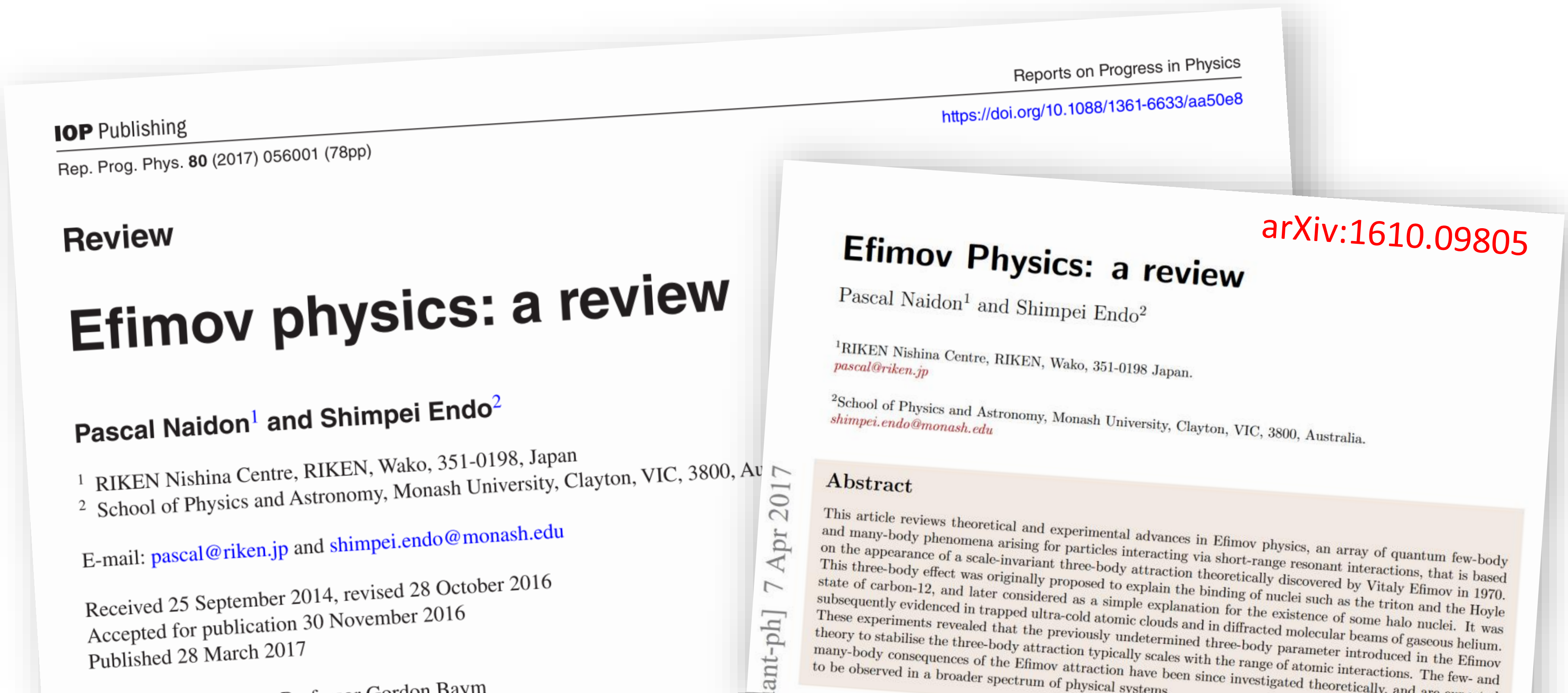
A brief overview of Efimov physics

and related mathematical problems

Pascal Naidon, RIKEN (Wako, Japan)


IOP Publishing
Reports on Progress in Physics
Rep. Prog. Phys. 80 (2017) 056001 (78pp)
https://doi.org/10.1088/1361-6633/aa50e8

Review

Efimov physics: a review

Pascal Naidon[1] and Shimpei Endo[2]

[1] RIKEN Nishina Centre, RIKEN, Wako, 351-0198, Japan
[2] School of Physics and Astronomy, Monash University, Clayton, VIC, 3800, Au

E-mail: pascal@riken.jp and shimpei.endo@monash.edu

Received 25 September 2014, revised 28 October 2016
Accepted for publication 30 November 2016
Published 28 March 2017

arXiv:1610.09805

Efimov Physics: a review

Pascal Naidon[1] and Shimpei Endo[2]

[1]RIKEN Nishina Centre, RIKEN, Wako, 351-0198 Japan.
pascal@riken.jp

[2]School of Physics and Astronomy, Monash University, Clayton, VIC, 3800, Australia.
shimpei.endo@monash.edu

ant-ph] 7 Apr 2017

Abstract

This article reviews theoretical and experimental advances in Efimov physics, an array of quantum few-body and many-body phenomena arising for particles interacting via short-range resonant interactions, that is based on the appearance of a scale-invariant three-body attraction theoretically discovered by Vitaly Efimov in 1970. This three-body effect was originally proposed to explain the binding of nuclei such as the triton and the Hoyle state of carbon-12, and later considered as a simple explanation for the existence of some halo nuclei. It was subsequently evidenced in trapped ultra-cold atomic clouds and in diffracted molecular beams of gaseous helium. These experiments revealed that the previously undetermined three-body parameter introduced in the Efimov theory to stabilise the three-body attraction typically scales with the range of atomic interactions. The few- and many-body consequences of the Efimov attraction have been since investigated theoretically, and are ... to be observed in a broader spectrum of physical systems

Plan

1. The Efimov effect for three particles
   1. A physical description
   2. Physical examples (experiments)
   3. A mathematical description
2. More than three particles
   1. Universal tetramers (4 identical bosons)
   2. The four-body Efimov effect (3 fermions + 1)

Physical description

What is the Efimov effect?

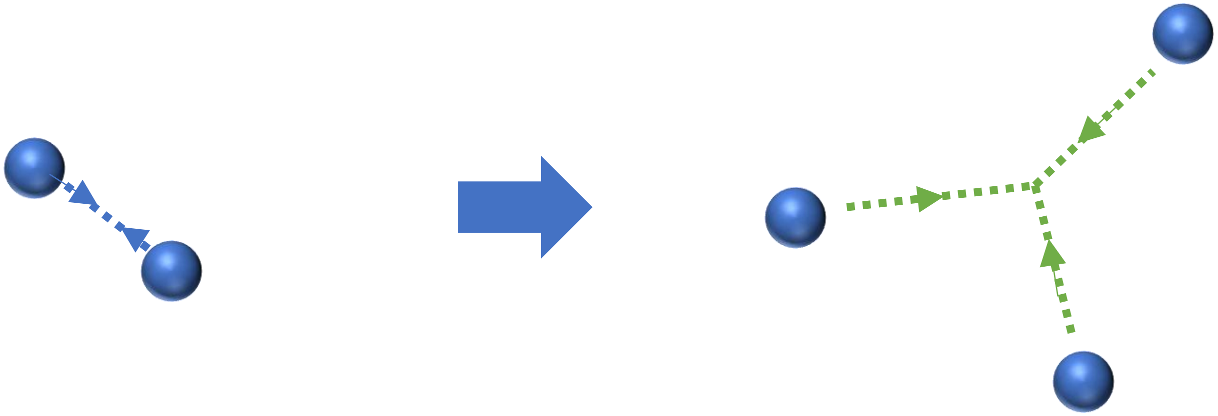
Vitaly Efimov in 1977

V. Efimov, “Weakly-bound states of three resonantly-interacting particles,” **Yad. Fiz., 12, 1080–1091**, November 1970, [**Sov. J. Nucl. Phys. 12, 589-595 (1971)**].

V. Efimov, “Energy levels arising from resonant two-body forces in a three-body system.” **Physics Letters B, 33, 563 – 564, 1970**.

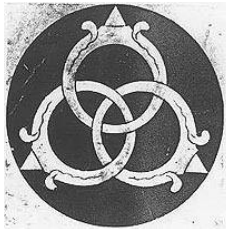
What is the Efimov effect?

The appearance of an effective **long-range three-body attractive force** between three particles interacting via **short-range two-body attractive forces**

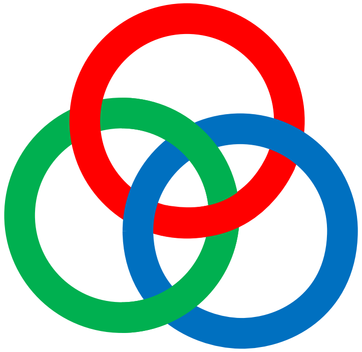


This leads to remarkable properties:

Existence of **an infinite number of Borromean states**, i.e. three-body bound states in the absence of two-body bound state.

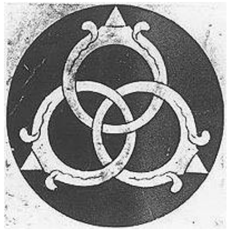


Borromeo family emblem

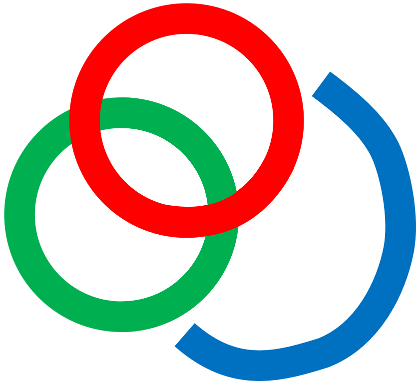


This leads to remarkable properties:

Existence of **an infinite number of Borromean states**, i.e. three-body bound states in the absence of two-body bound state.

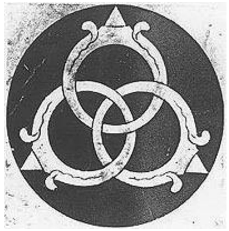


Borromeo family emblem

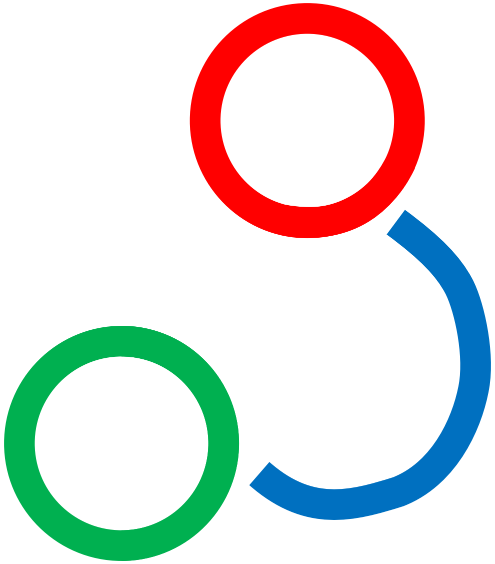


This leads to remarkable properties:

Existence of **an infinite number of Borromean states**, i.e. three-body bound states in the absence of two-body bound state.

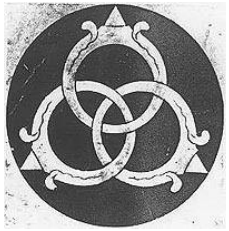


Borromeo family emblem



This leads to remarkable properties:

Existence of an infinite number **of Borromean states**, i.e. three-body bound states in the absence of two-body bound state.



Borromeo family emblem

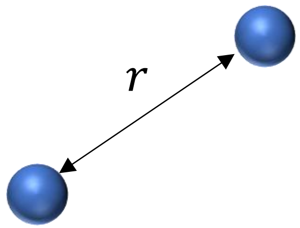
Discrete scale invariance of the spectrum: each three-body state can be obtained from another by a scale transformation

Matryoshka (Russian nesting dolls)

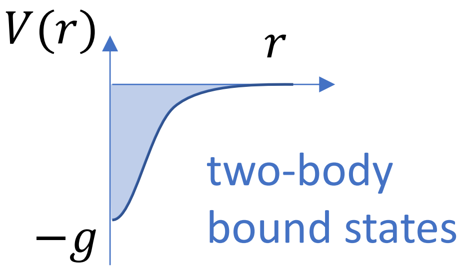
Universality of the physical properties: they do not depend on the details of the two-body force, but **only on two parameters**.

What is the origin of the Efimov effect?

- Is there an Efimov effect in classical physics?

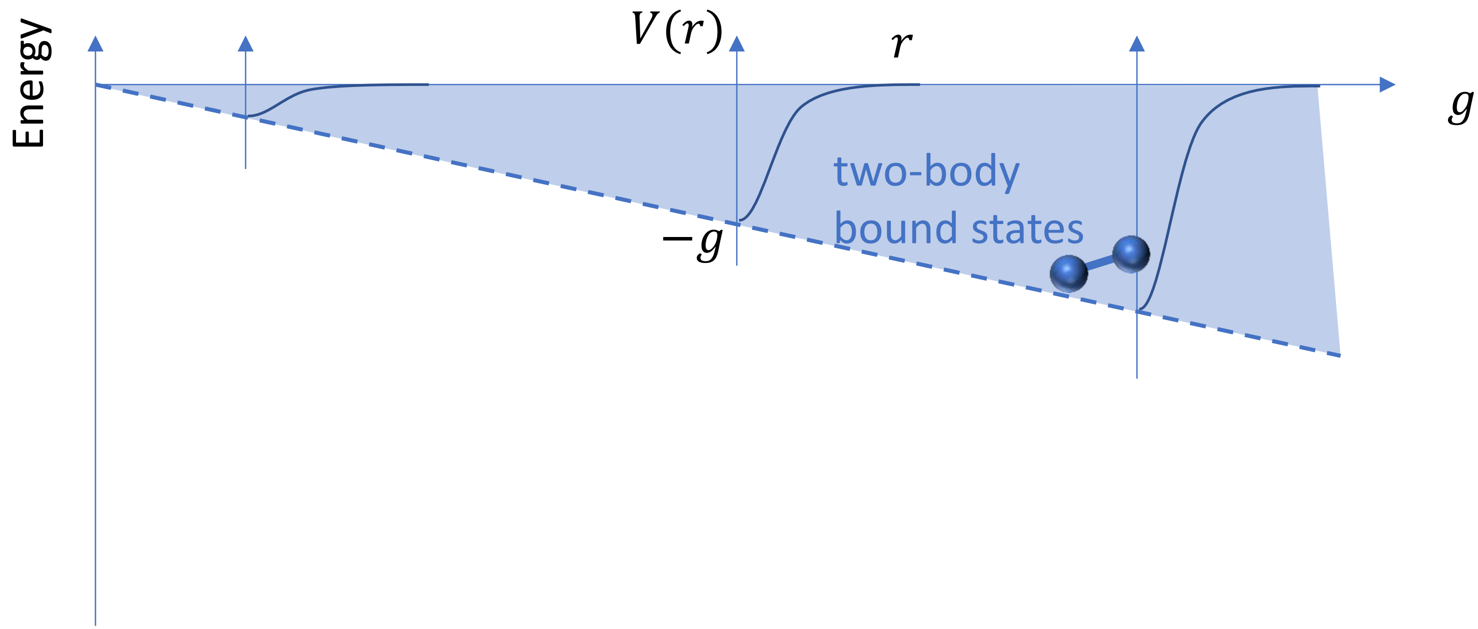


Short-range attractive two-body interaction potential



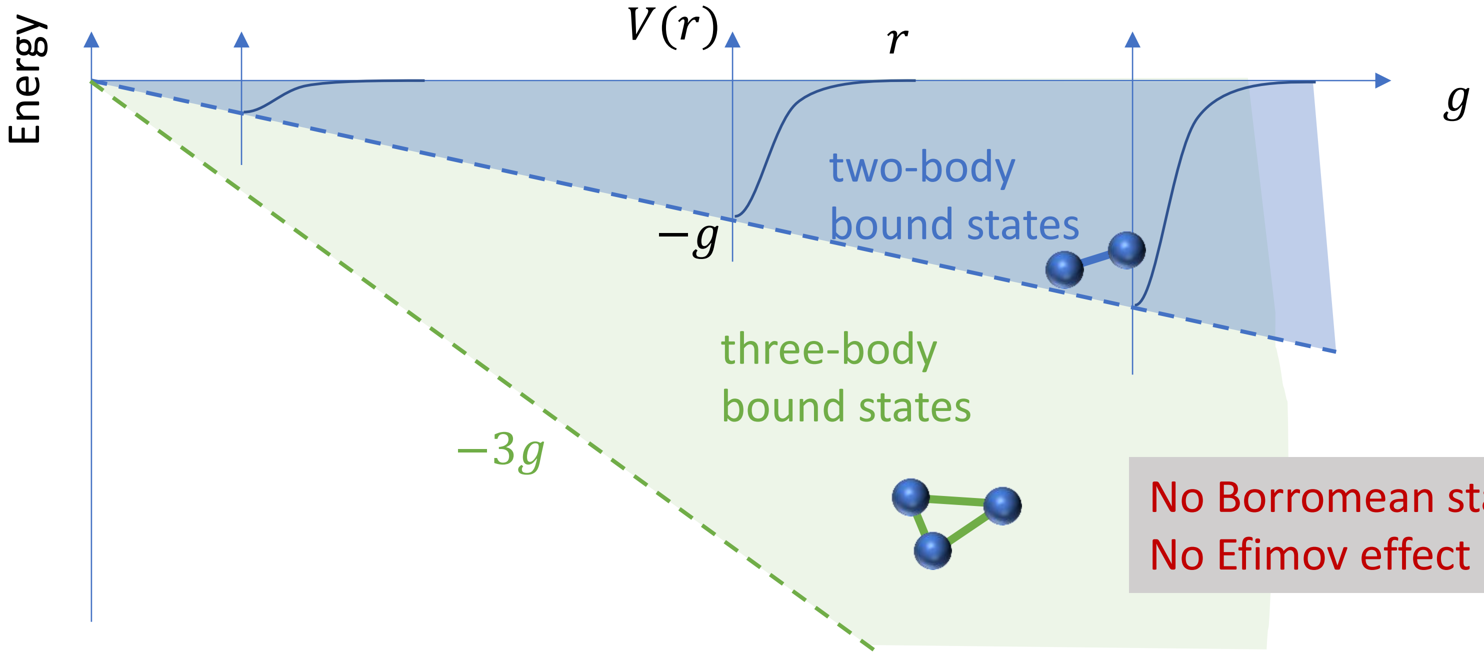
What is the origin of the Efimov effect?

- Is there an Efimov effect in classical physics?



What is the origin of the Efimov effect?

- Is there an Efimov effect in classical physics?

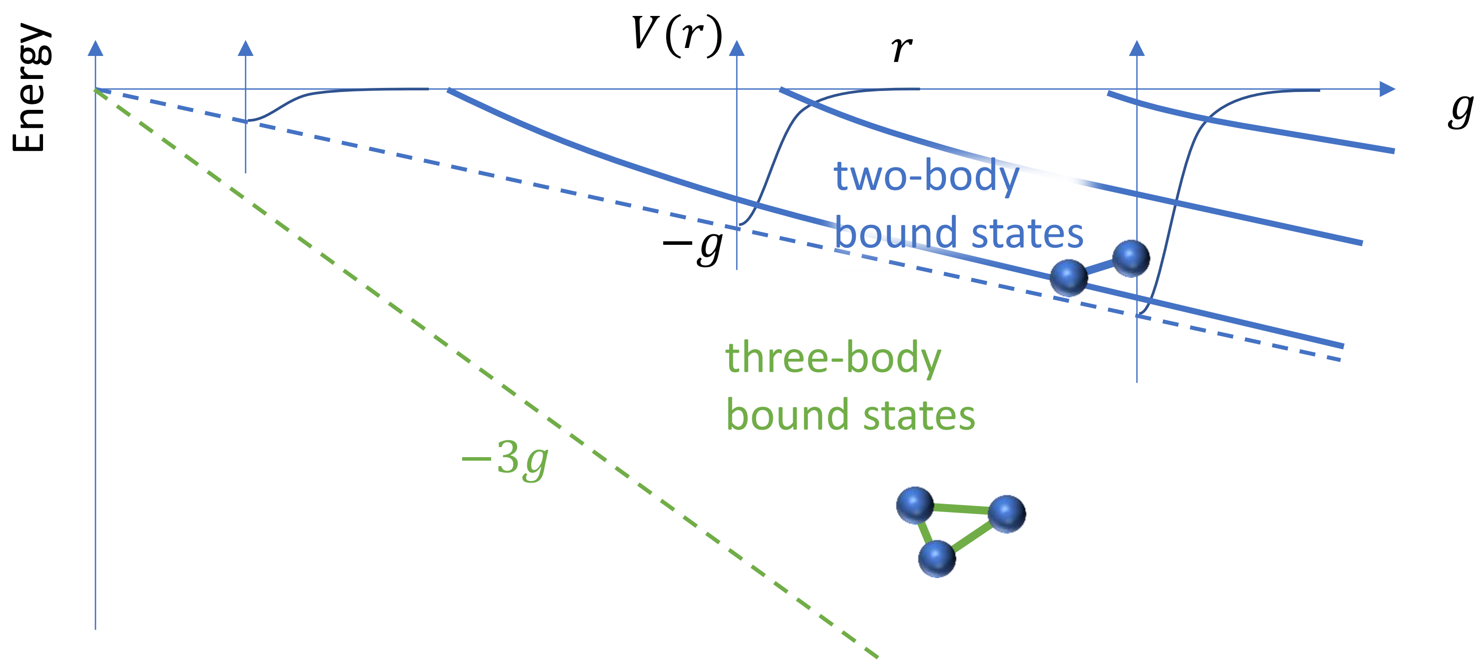


No Borromean state
No Efimov effect

What is the origin of the Efimov effect?

- The quantum origin of the Efimov effect

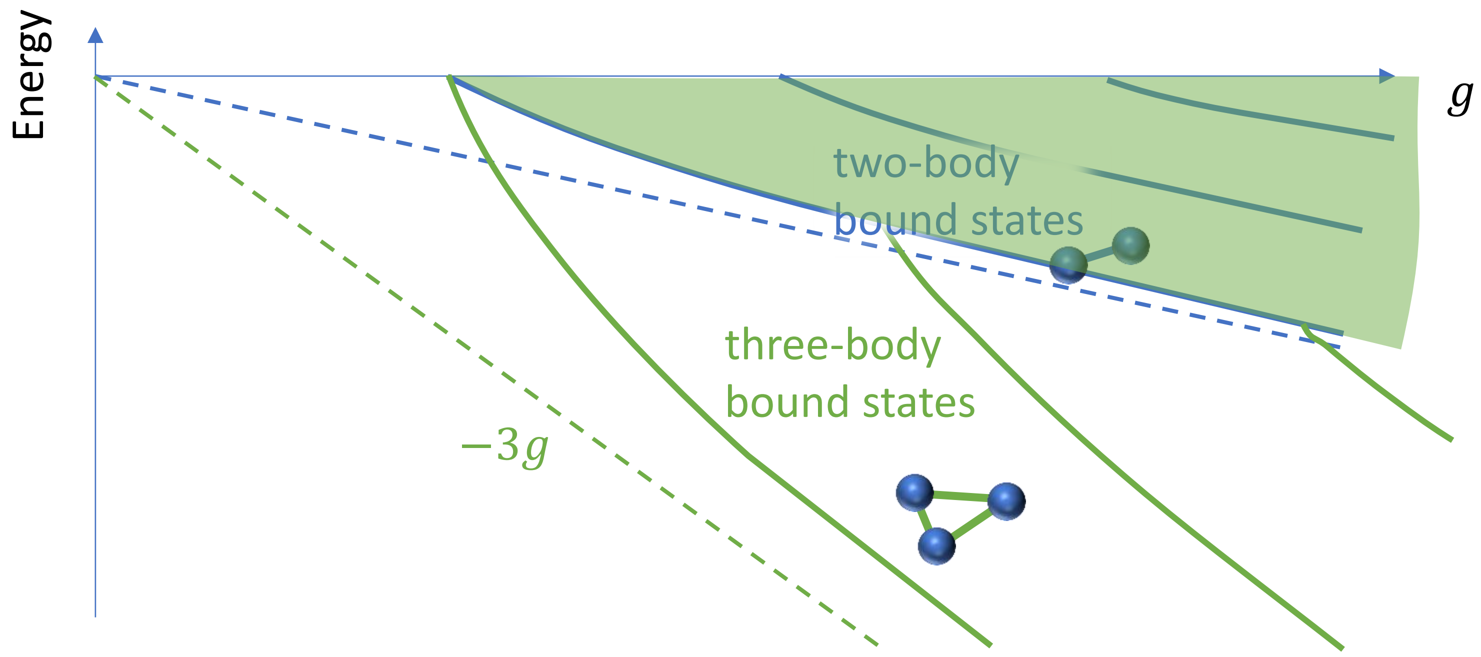
Quantum fluctuations + interactions $\Longrightarrow$ critical strength g (zero-point) and quantisation



What is the origin of the Efimov effect?

- The quantum origin of the Efimov effect

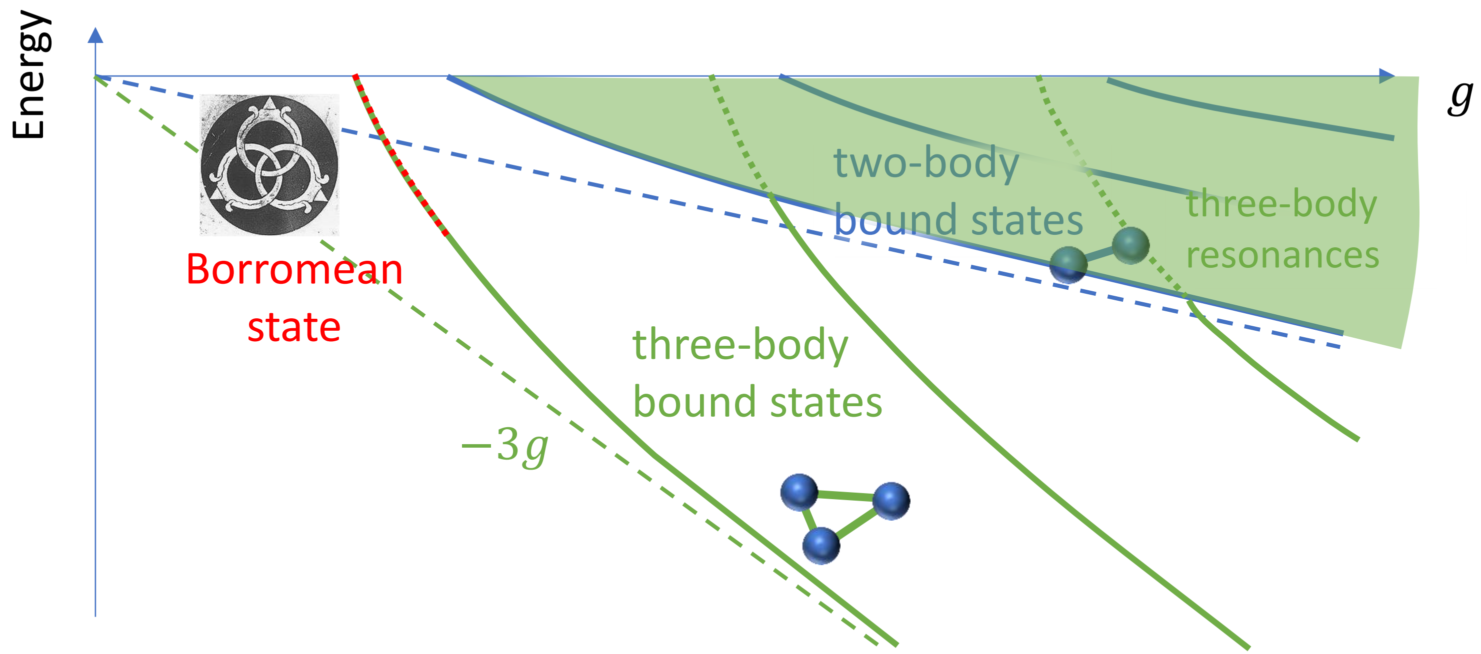
Quantum fluctuations + interactions $\Longrightarrow$ critical strength g (zero-point) and quantisation



What is the origin of the Efimov effect?

- The quantum origin of the Efimov effect

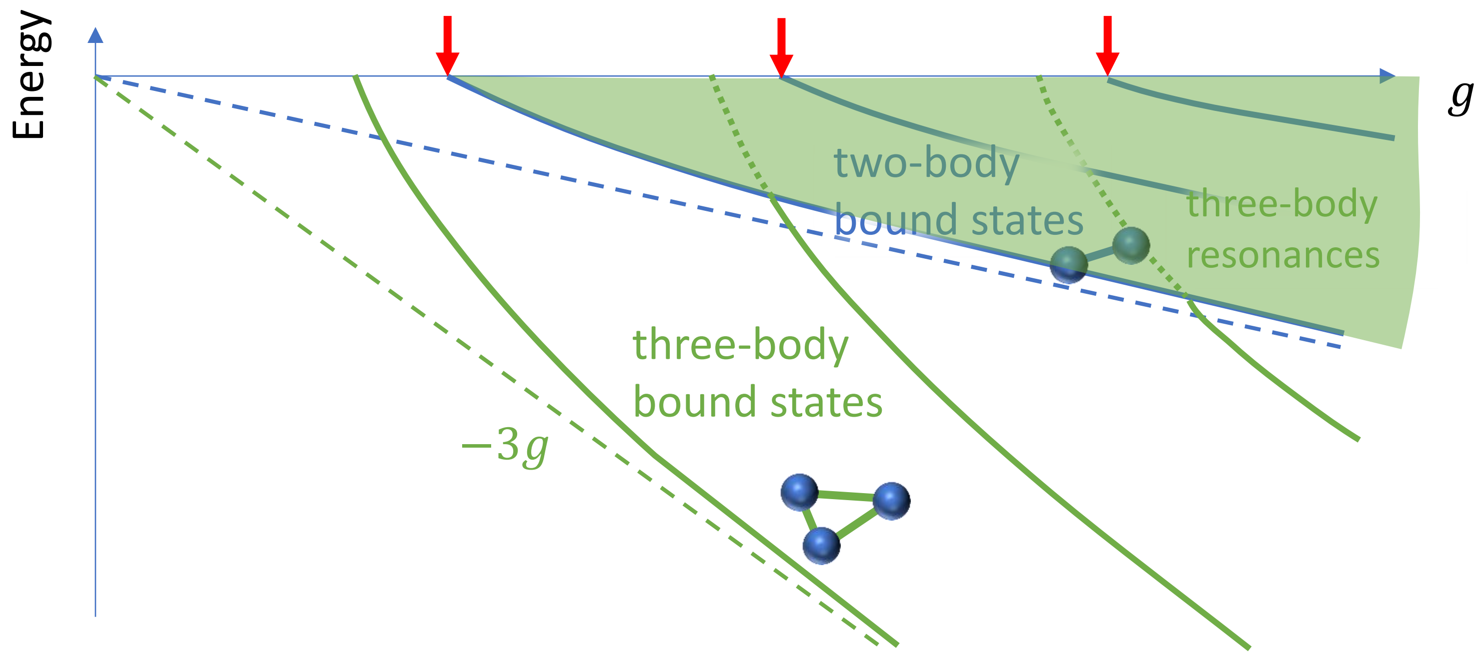
Quantum fluctuations + interactions $\Longrightarrow$ critical strength g (zero-point) and quantisation



What is the origin of the Efimov effect?

- The quantum origin of the Efimov effect

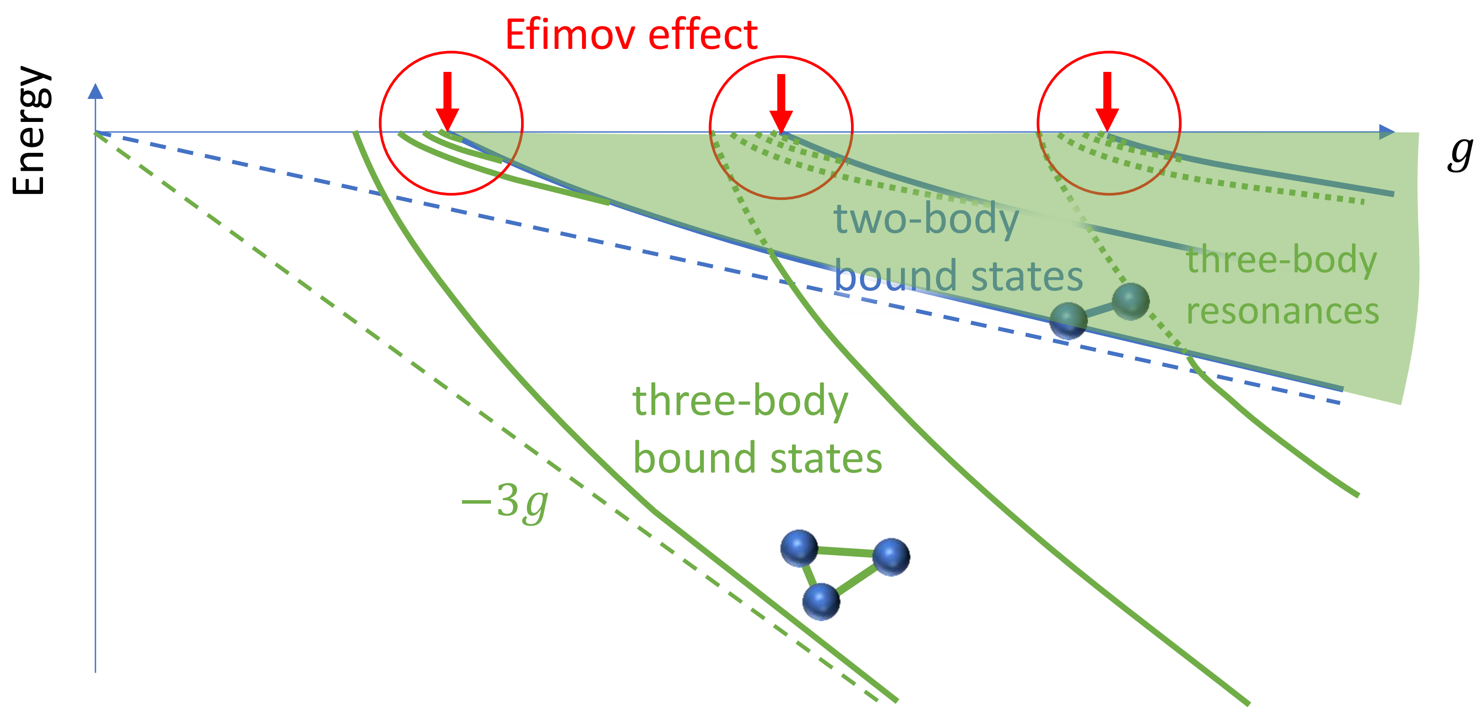
Critical strengths g for binding two particles $\Longrightarrow$ "unitarity"



What is the origin of the Efimov effect?

- The quantum origin of the Efimov effect

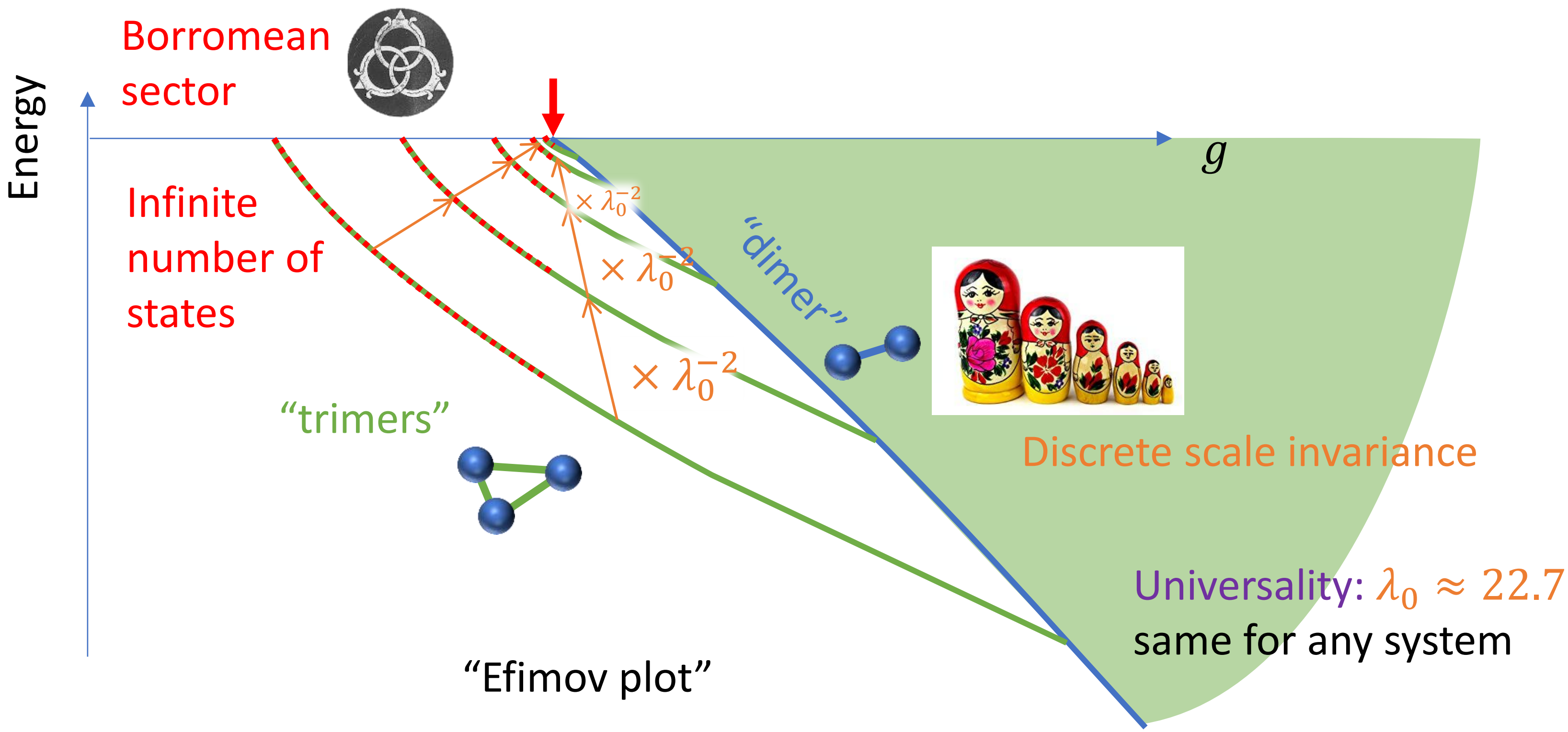
Critical strengths g for binding two particles $\Longrightarrow$ "unitarity"



What is the origin of the Efimov effect?

- The quantum origin of the Efimov effect

Critical strengths g for binding two particles $\Longrightarrow$ "unitarity"



"Efimov plot"

What is the origin of the Efimov effect?

- The quantum origin of the Efimov effect

Why universality and scale invariance?

At the two-body level:

Short-range interaction → At low-energy → At "unitarity" ($a \to \infty$)

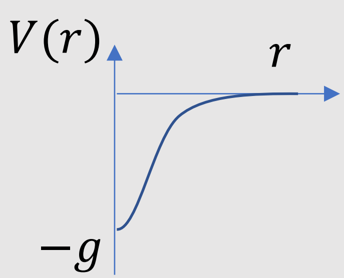
Single physical scale:
"Scattering length" a

No more length scale!
(Scale invariance)

↕

"Low-energy universality"
"Short-range universality"
"Zero-range universality"

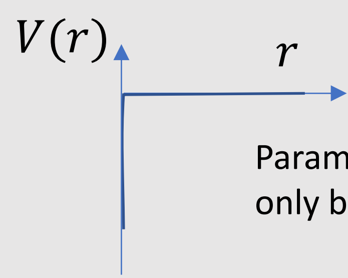
Original interaction



The wave function is much larger than the range of interactions

can be replaced by →

"Zero-range" interaction



Parametrised only by a

What is the origin of the Efimov effect?

- The quantum origin of the Efimov effect

Why infinitely many states?

At the three-body level:

Effective three-body potential $V(R) \propto -\frac{\hbar^2}{m}\frac{1}{R^2}$ ⟺ Scales like the kinetic energy $K \propto \frac{\hbar^2}{m}\frac{d^2}{dR^2}$

⇒

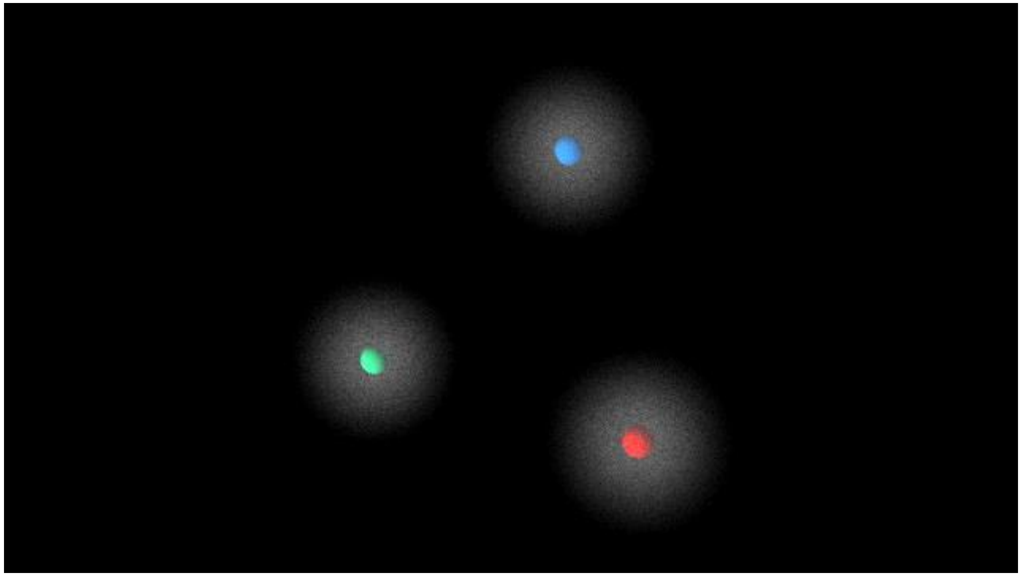
- Scale invariant equation
- Long-range attractive potential, may support **infinitely many states**

What is the origin of the Efimov effect?

- The quantum origin of the Efimov effect

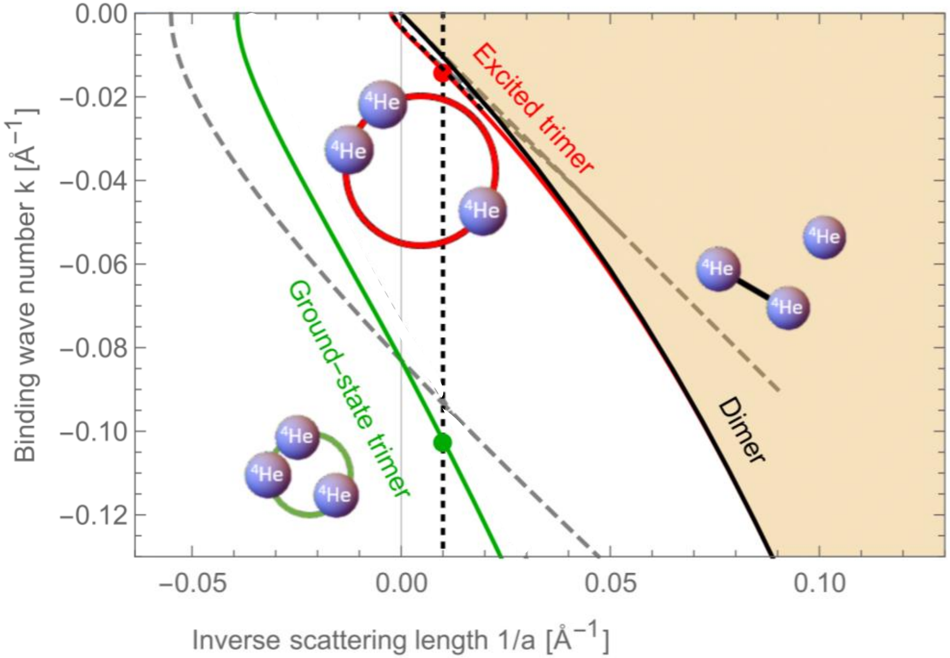
Why long range ? in spite of short-range two-body interactions?

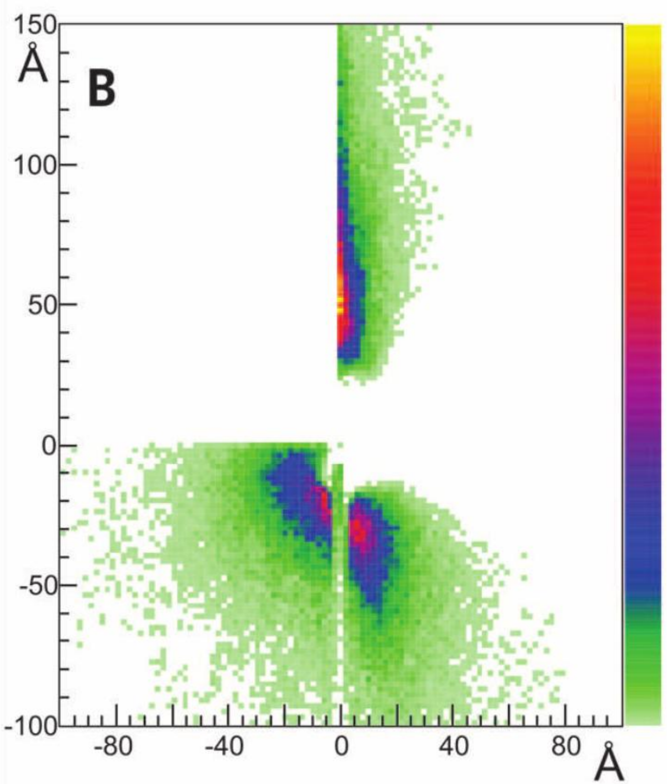
The Efimov attraction may be viewed as an interaction between two particles mediated by a third particle



Physical examples

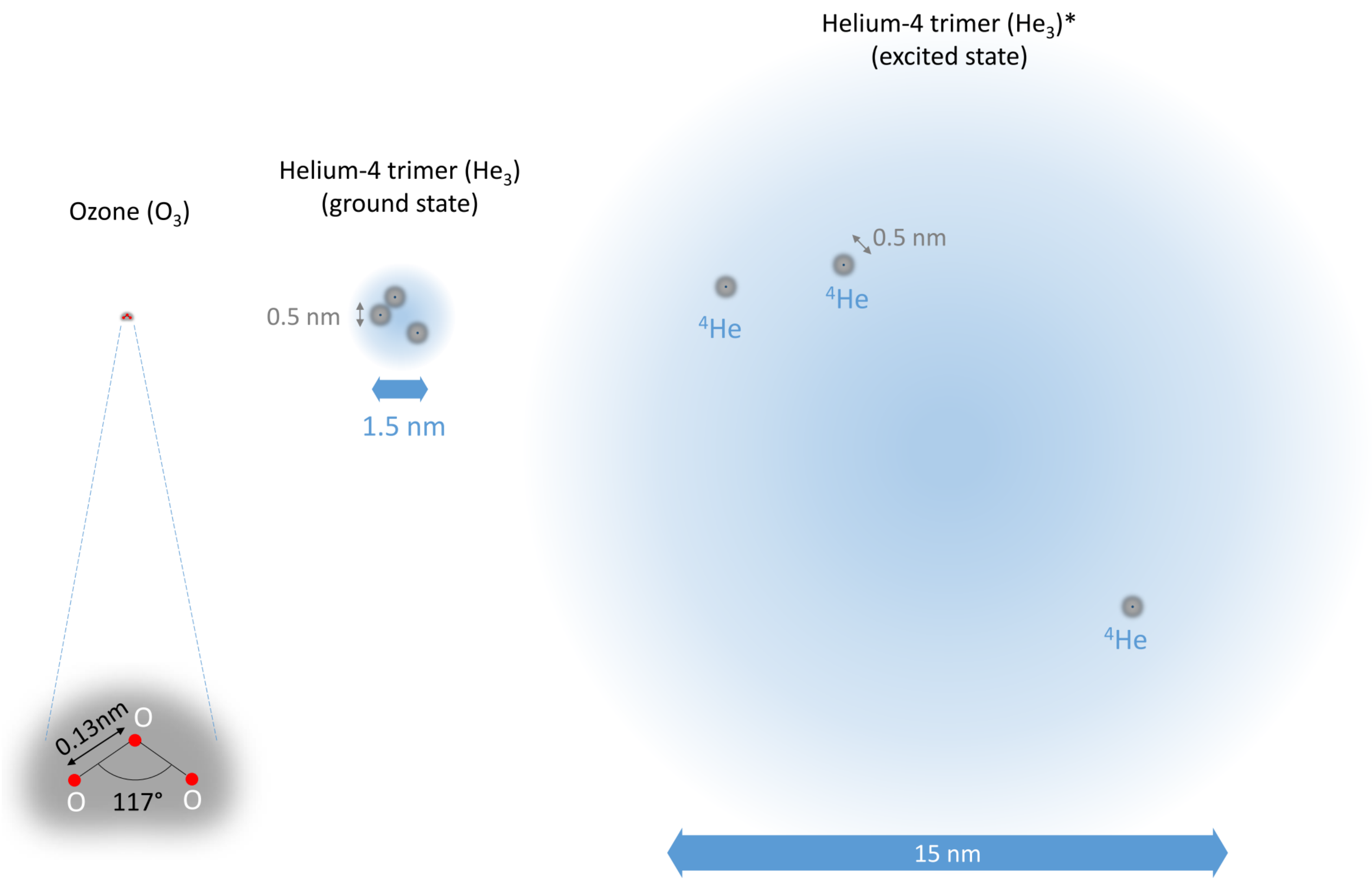
The helium triatomic molecules $^4He_3$



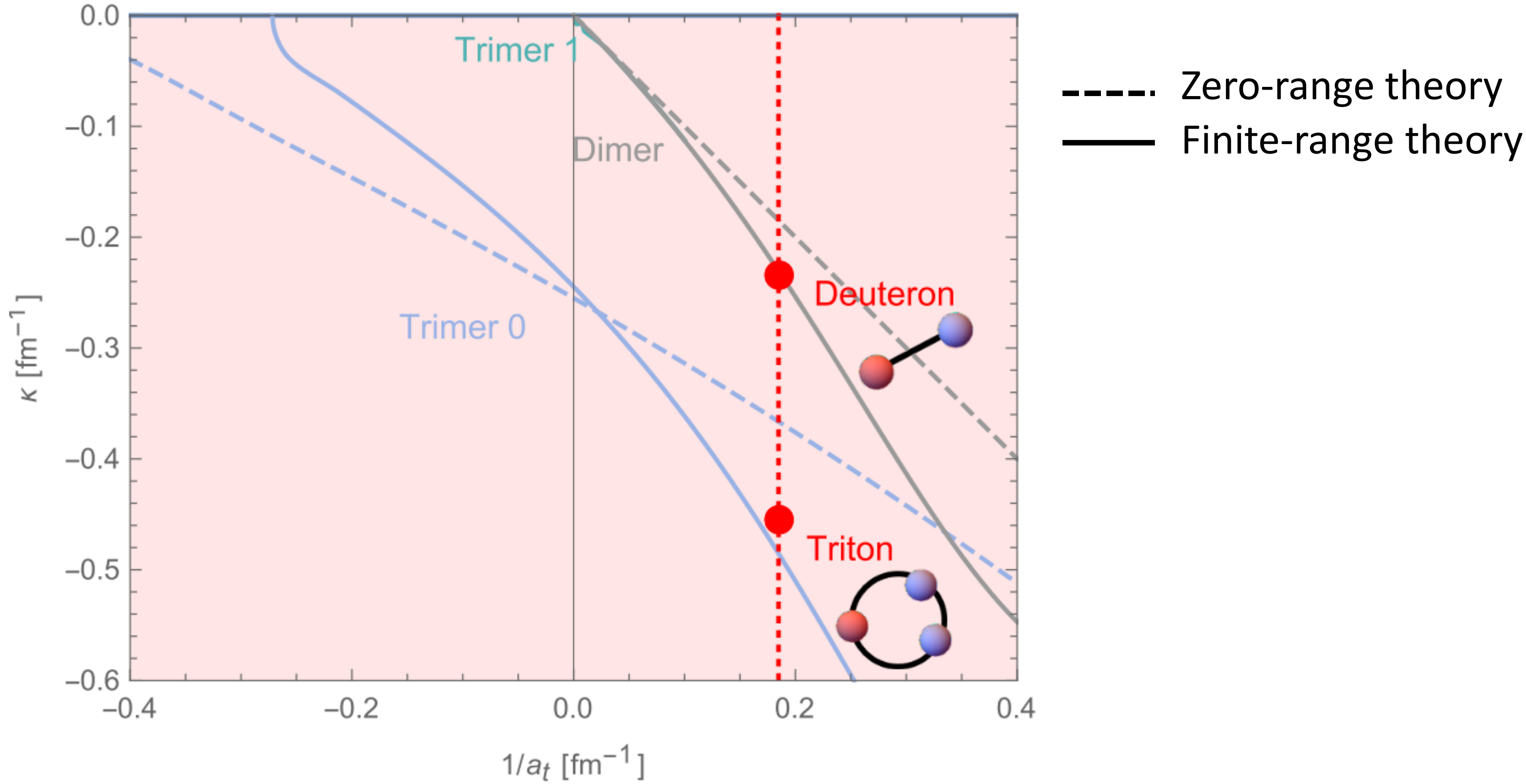


Kunitski, Science 348, 551 (2015)

The helium triatomic molecules $^{4}He_{3}$



The triton (2 neutrons + 1 proton)

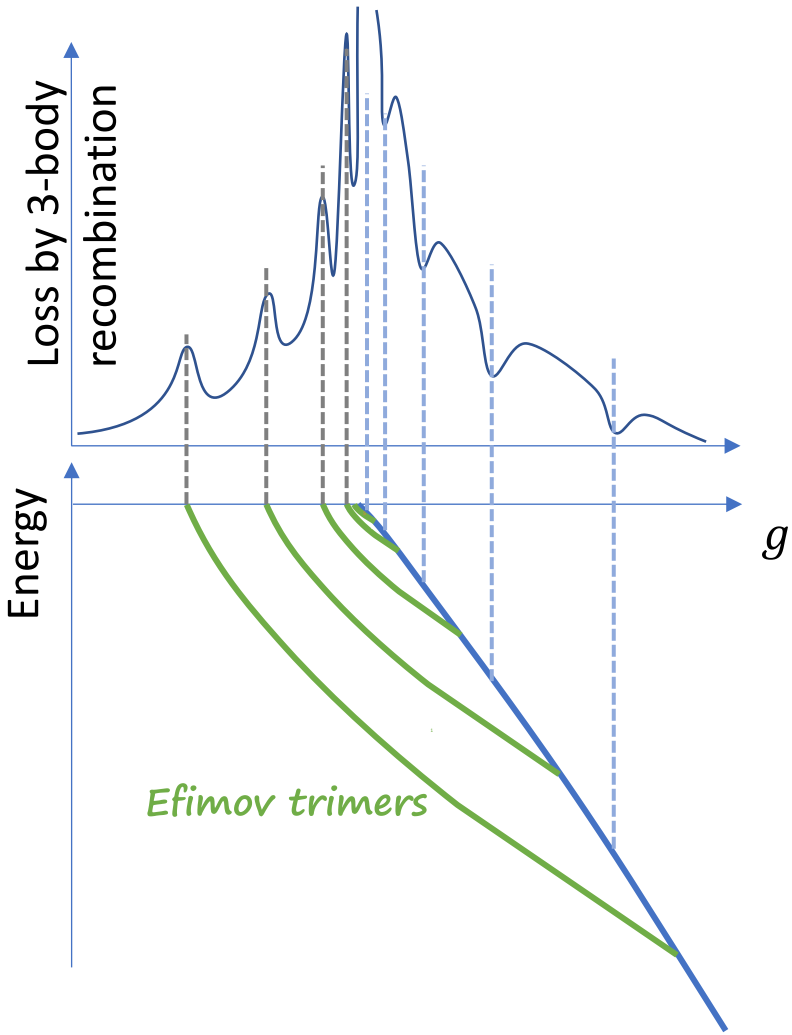


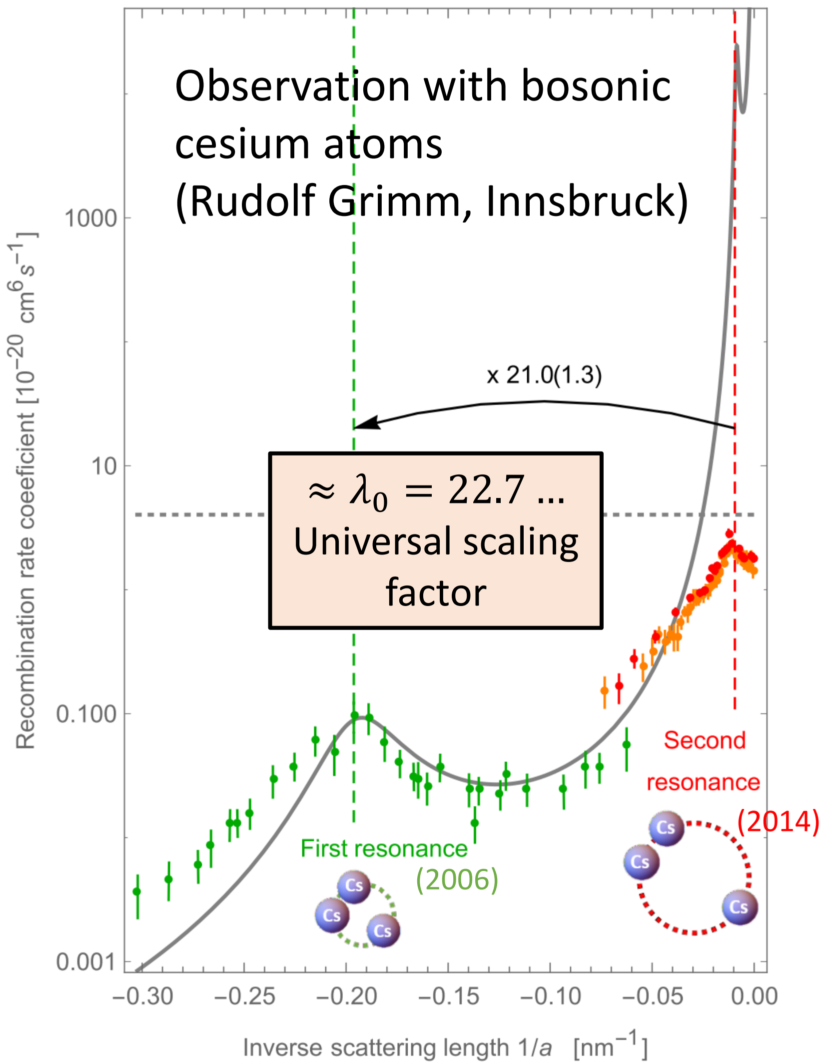
Observations in ultra-cold atomic gases

Ultra-cold atoms are gases of atoms cooled to extremely low temperatures (< μK) to reach the quantum degeneracy regime.

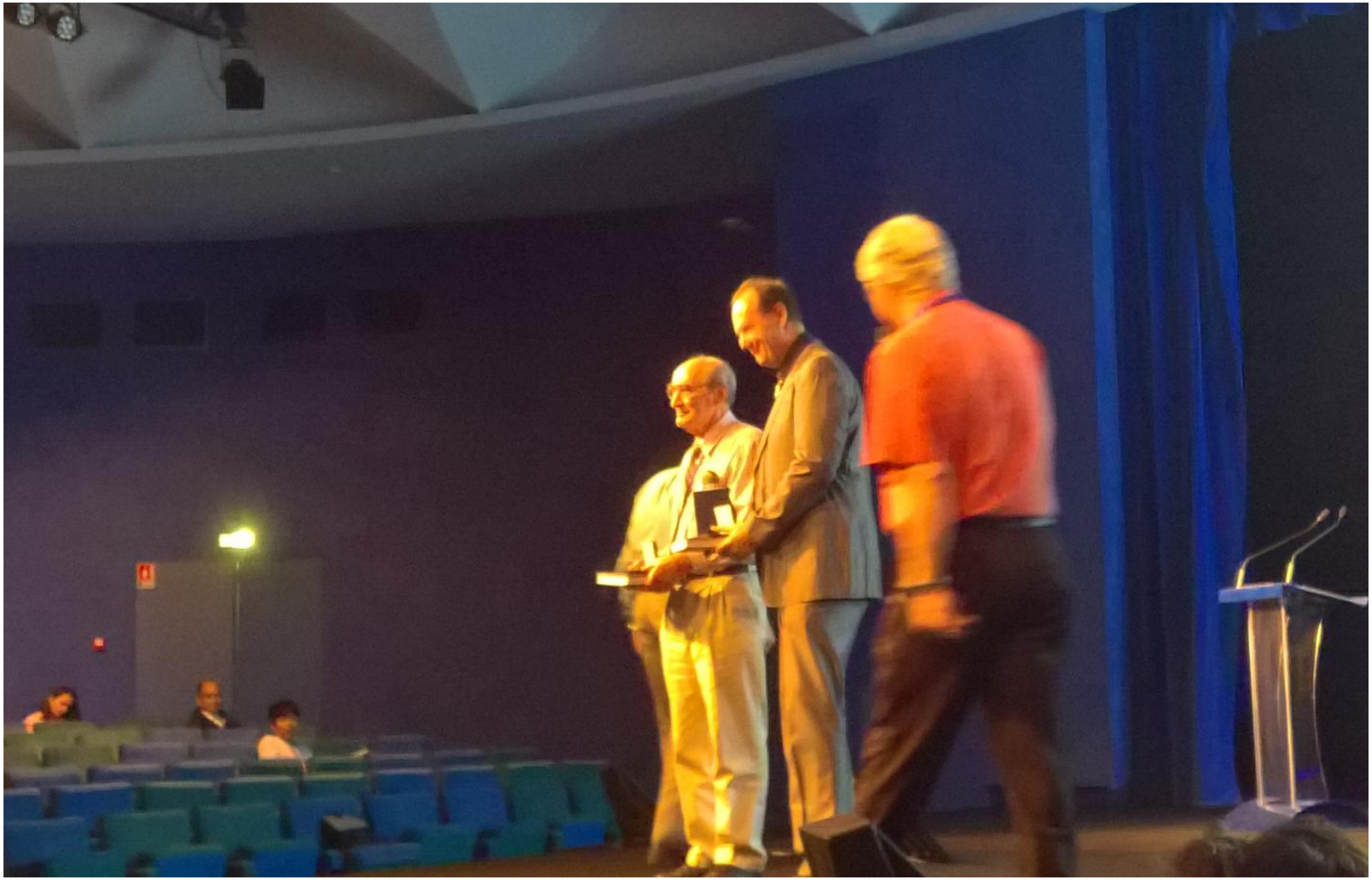
Magic tool: Feshbach resonances
By applying a magnetic field, it is possible to change the strength of interatomic interactions!

Observations in ultra-cold atomic gases

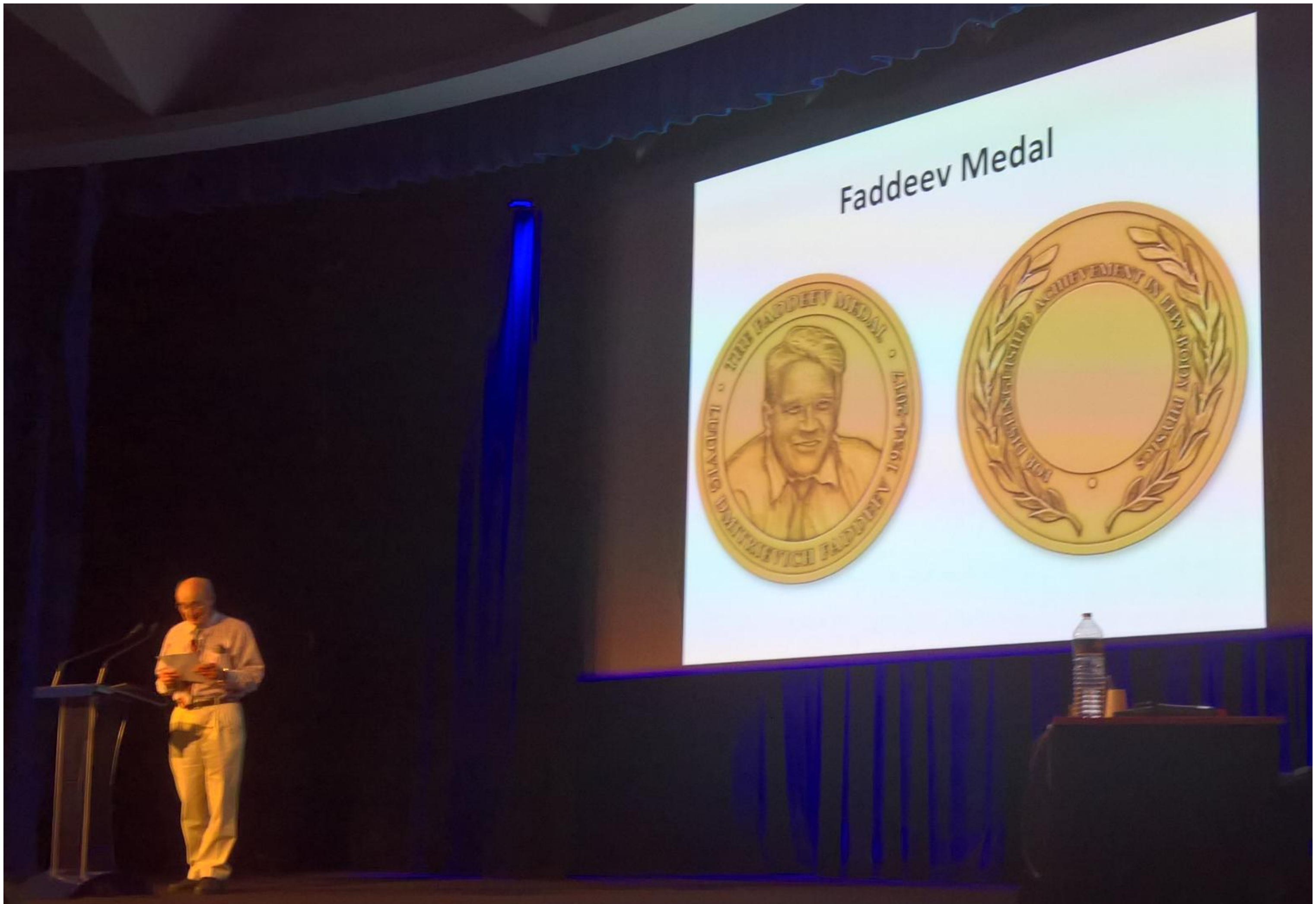




Vitaly Efimov and Rudolf Grimm receive the first Faddeev medal in Caen (July 11, 2018)

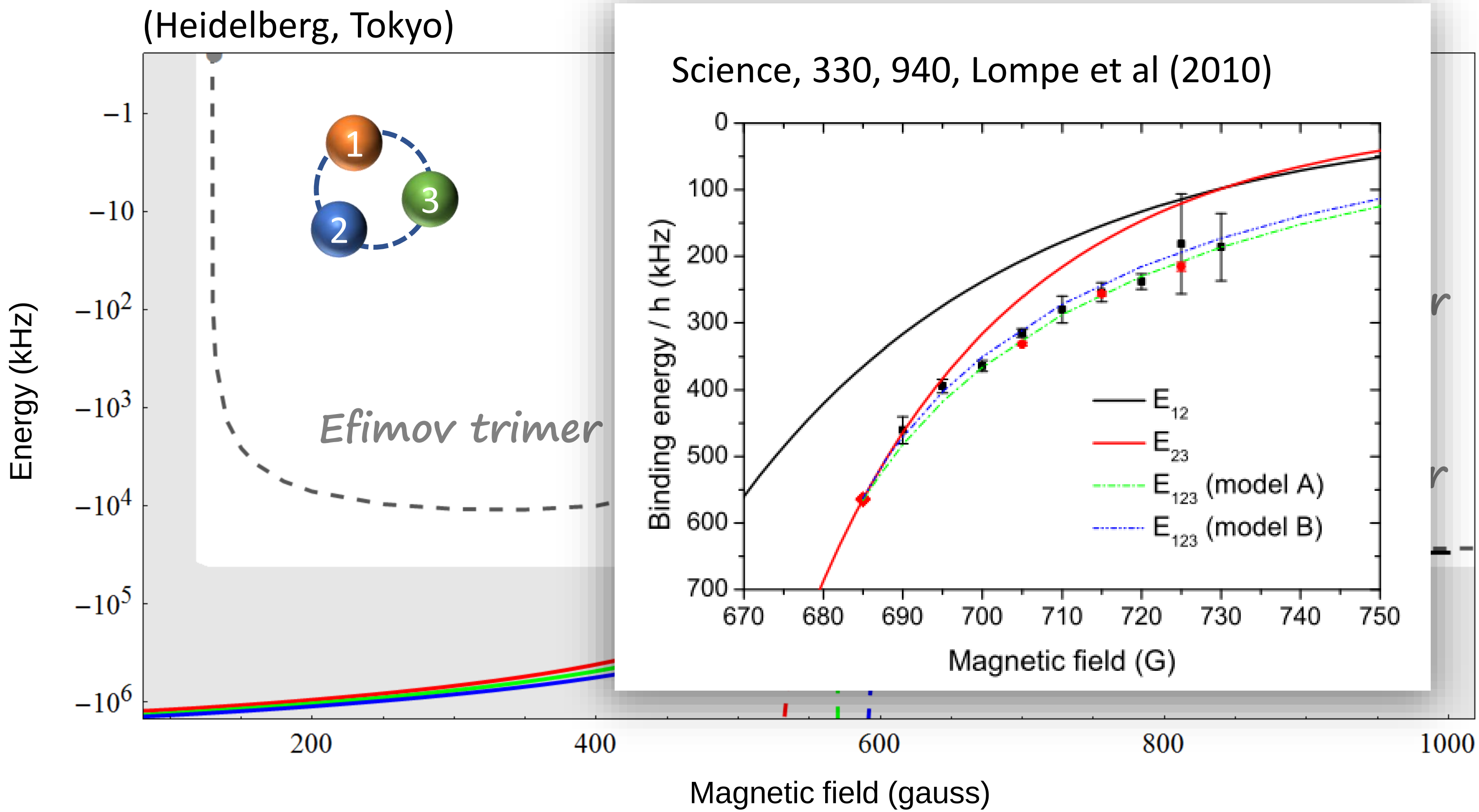


Vitaly Efimov’s speech after receiving the prize



Observations in ultra-cold atomic gases

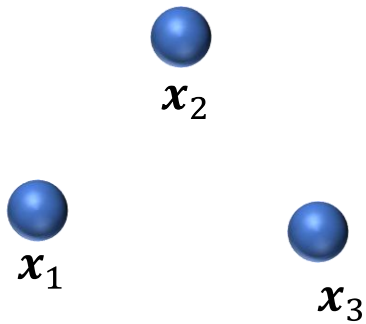
Observation with three distinguishable states of lithium atoms (Heidelberg, Tokyo)



Mathematical description

Three-body equation

3 identical bosons interacting through the two-body potential $V(\boldsymbol{r})$



Bosonic symmetry:
$\psi(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = \psi(\boldsymbol{x}_i, \boldsymbol{x}_j, \boldsymbol{x}_k)$

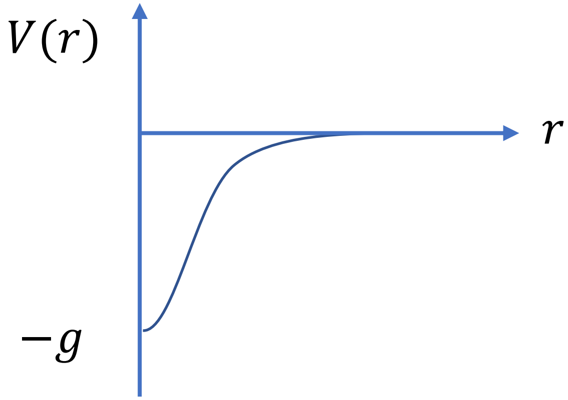
Schrödinger equation: $(\hat{H} - E)\psi(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = 0$

Hamiltonian: $\hat{H} = -\frac{\hbar^2}{2m}(\Delta_\mathbf{1} + \Delta_\mathbf{2} + \Delta_\mathbf{3}) + V(\boldsymbol{r}_{12}) + V(\boldsymbol{r}_{23}) + V(\boldsymbol{r}_{31})$

With $\boldsymbol{r}_{ij} = \boldsymbol{x}_i - \boldsymbol{x}_j$

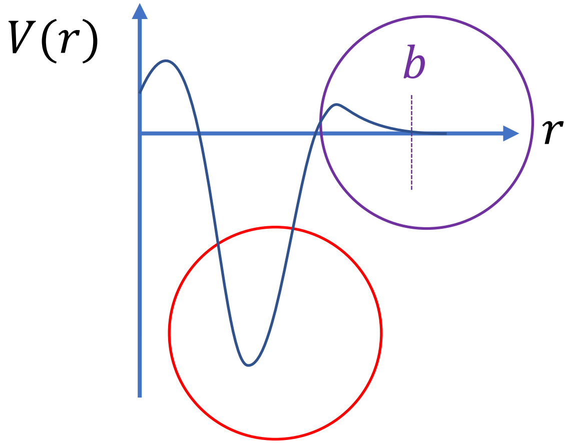
Conditions on the two-body interaction potentials $V(r)$

What is a short-range resonant interaction?



Conditions on the two-body interaction potentials $V(r)$

What is a short-range resonant interaction?



Should decay faster than $1/r^3$ to qualify as "**short-range**" and admit:

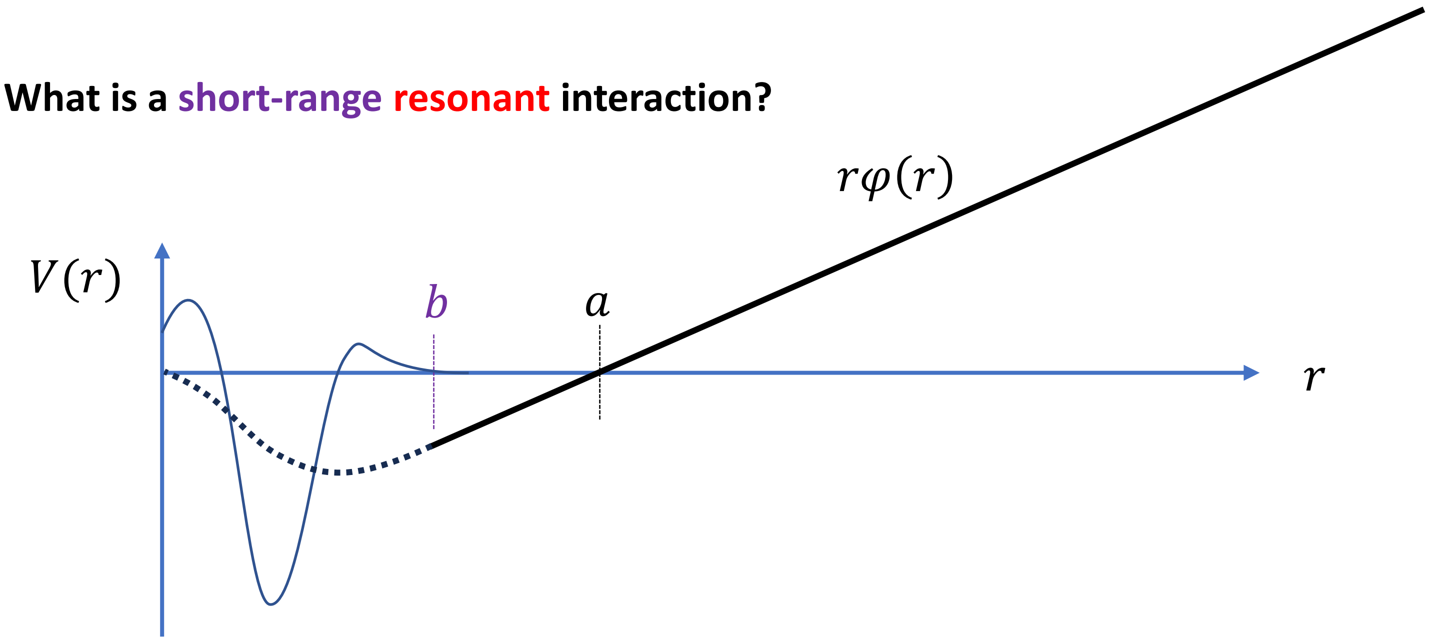
- **typical range b**: $V(r) \approx 0$ for $r \gg b$
- **scattering length a**

Should feature enough **attractive** part to approach the appearance of a two-body bound state.

- $|a| \gg b$

Conditions on the two-body interaction potentials $V(r)$

What is a short-range resonant interaction?



Two-body equation at zero energy:

$$\left(-\frac{1}{2}\Delta_1 - \frac{1}{2}\Delta_2 + V(r_{12}) - 0\right)\varphi(\boldsymbol{x}_1, \boldsymbol{x}_2) = 0$$

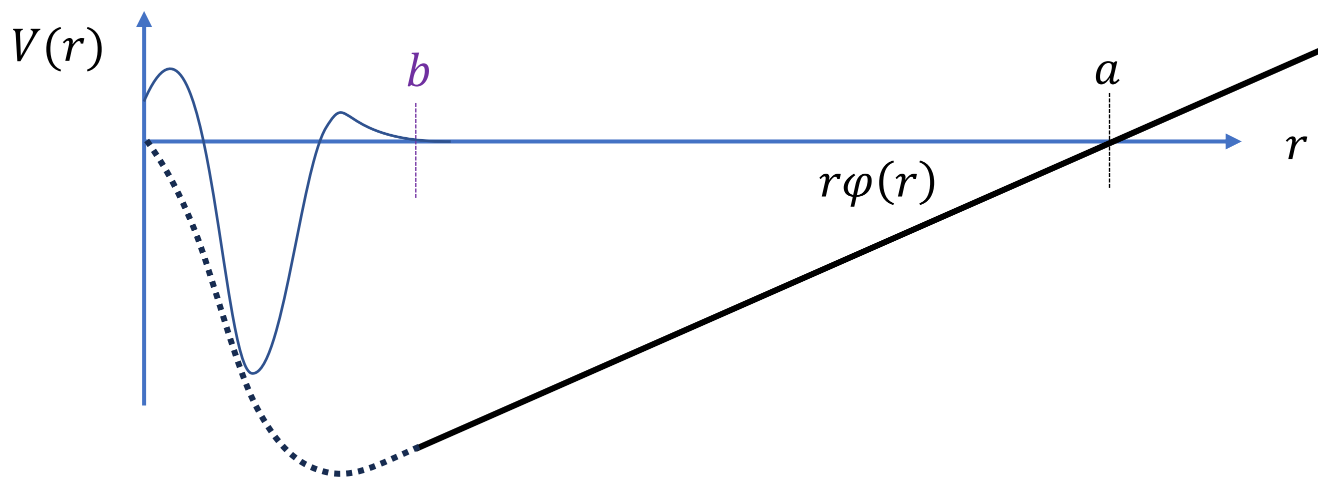
Elimination of the centre of mass and s-wave:

$$\left(-\frac{d^2}{dr^2} + V(r)\right)[r\varphi(r)] = 0$$

For $r \gg b$, $r\varphi(r) \propto r - a$

Conditions on the two-body interaction potentials $V(r)$

What is a short-range resonant interaction?



Two-body equation at zero energy:

$$\left(-\frac{1}{2}\Delta_1 - \frac{1}{2}\Delta_2 + V(r_{12}) - 0\right)\varphi(\boldsymbol{x}_1, \boldsymbol{x}_2) = 0$$

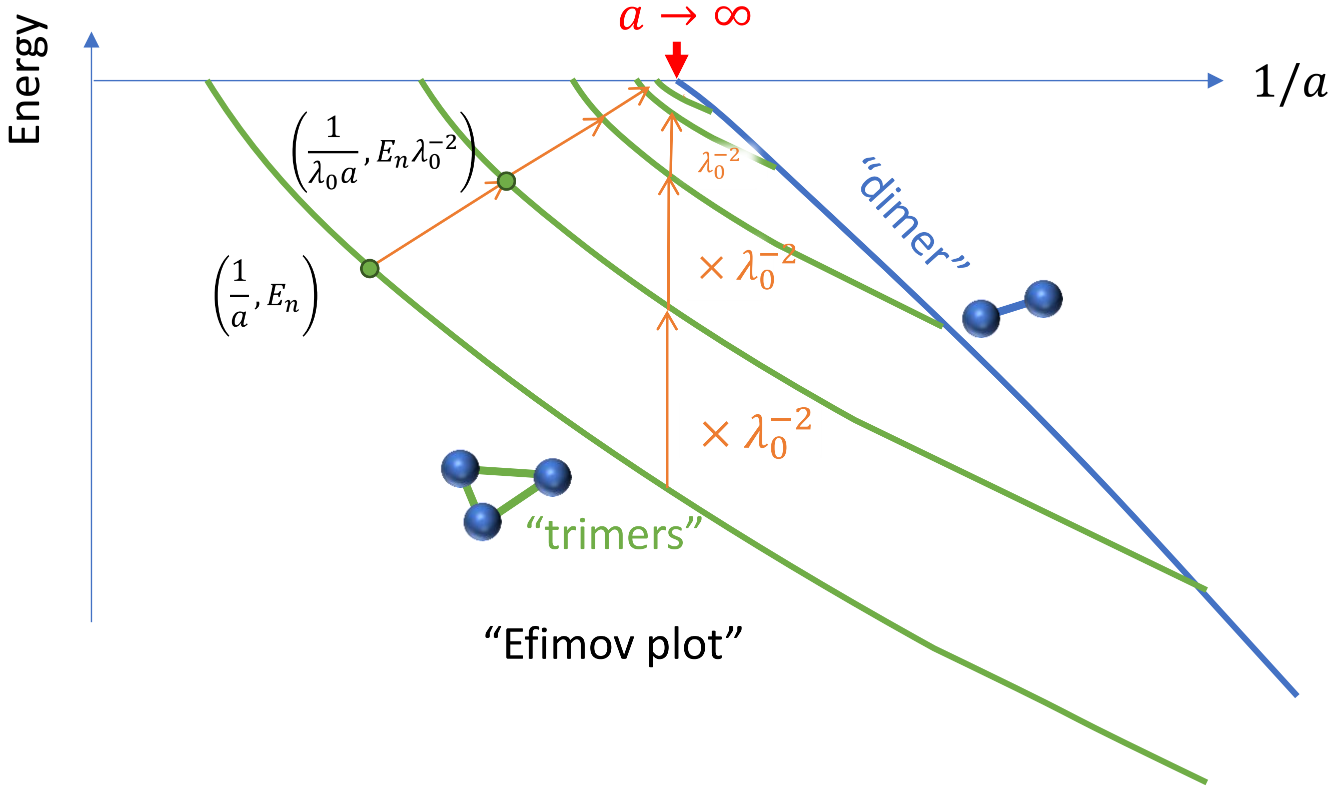
Elimination of the centre of mass and s-wave:

$$\left(-\frac{d}{dr} + V(r)\right)[r\varphi(r)] = 0$$

For $r \gg b$, $r\varphi(r) \propto r - a$

Characterisation of the Efimov effect

- For $a \to \infty$, infinite sequence of discrete eigenvalues: $E_n \xrightarrow[n\to\infty]{} E_0 \lambda_0^{-2n}$
- **Discrete scale invariance**: for large n, if $\psi_n(r)$ is an eigenvector of $\hat{H}(a)$ with eigenvalue E_n, then $\psi_n(r/\lambda_0)$ is (almost) an eigenvector of $\hat{H}(a\,\lambda_0)$ with eigenvalue $E_n\,\lambda_0^{-2}$



“Efimov plot”

How did Efimov “demonstrate” the effect?

Vitaly Efimov in 1977

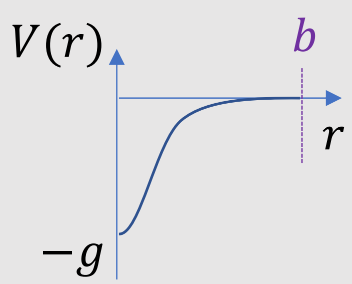
V. Efimov, “Weakly-bound states of three resonantly-interacting particles,” **Yad. Fiz., 12, 1080–1091**, November 1970, [**Sov. J. Nucl. Phys. 12, 589-595 (1971)**].

V. Efimov, “Energy levels arising from resonant two-body forces in a three-body system.” **Physics Letters B, 33, 563 – 564, 1970**.

How did Efimov “demonstrate” the effect?

(1) Zero-range approximation:

Original interaction

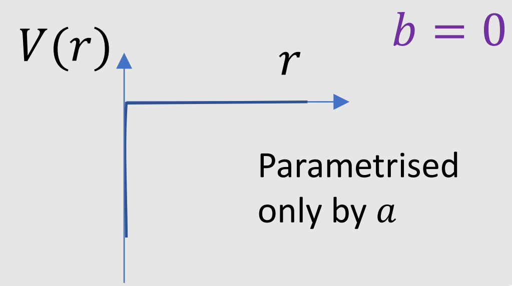


The wave function is much larger than the range of interactions

can be replaced by

Universality and scale invariance become exact!

“Zero-range” interaction



Parametrised only by a

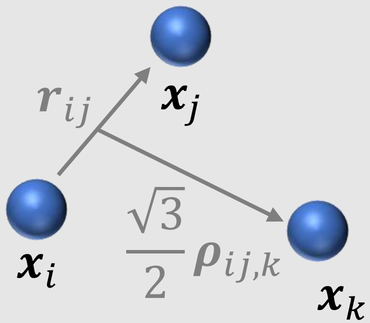
There are many ways to implement this:

- “Delta function” $V(r) \to g\delta^3(\boldsymbol{r})$ regularised by a cutoff (like renormalisation in quantum field theory)
- Zero-range boundary condition: $\psi \xrightarrow[r\to 0]{} \propto \frac{1}{r} - \frac{1}{a} \quad \Leftrightarrow \quad \left[\frac{d}{dr}\ln(r\psi)\right]_{\ell=0} \xrightarrow[r\to 0]{} -\frac{1}{a}$
- Pseudo-potential $V(r) = g\delta^3(\boldsymbol{r})\frac{d}{dr}(r\,\cdot)$

How did Efimov "demonstrate" the effect?

However, the 3-boson problem with such zero-range interaction is not well posed: divergence of energies with large cutoff ("Thomas collapse")

(2) Hyper-spherical adiabatic expansion (crucial insight!)



Change to hyper-spherical coordinates

$$\psi(\boldsymbol{r}_{ij}, \boldsymbol{\rho}_{ij,k}) = \psi(\rho, \overbrace{\alpha, \hat{r}_{ij}, \hat{\rho}_{ij,k}}^{\Omega})$$

Hyper-radius $\rho = \sqrt{r_{ij}^2 + \rho_{ij,k}^2}$

Hyper-angle $\alpha = \arctan\frac{r_{ij}}{\rho_{ij,k}}$

$$\rho^{-\frac{5}{2}}\left(-\frac{d^2}{d\rho^2} - \frac{1/4}{\rho^2} - \frac{\hat{\Lambda}_\Omega}{\rho^2} + \sum_{ij} V(\rho \sin \alpha_{ij}) - E\right)(\rho^{5/2}\psi) = 0$$

How did Efimov "demonstrate" the effect?

$$\rho^{-\frac{5}{2}}\left(-\frac{d^2}{d\rho^2}-\frac{1/4}{\rho^2}-\frac{\hat{\Lambda}_\Omega}{\rho^2}+\sum_{ij}V(\rho\sin\alpha_{ij})-E\right)(\rho^{5/2}\psi)=0$$

Hyper-radial Hyper-angular

Expansion: $\psi(\rho,\Omega)=\frac{1}{\rho^{5/2}}\sum_{n=1}^{\infty} f_n(\rho)\ \Phi_n(\Omega;\rho)$

Hyper-angular equation at fixed ρ: $$\left(\hat{\Lambda}_\Omega-\rho^2\sum_{ij}V(\rho\sin\alpha_{ij})+s_n^2(\rho)\right)\Phi_n(\Omega;\rho)=0$$

Coupled hyper-radial equations: $$\left(-\frac{d^2}{d\rho^2}+\frac{s_n^2(\rho)-1/4}{\rho^2}-E\right)f_n-\sum_{p}^{\infty}\left(2P_{np}\frac{df_p}{d\rho}+Q_{np}f_p\right)=0$$

For zero-range interactions and $a\to\infty$, the equations decouple!

How did Efimov "demonstrate" the effect?

$$\left(\hat{\Lambda}_\Omega - \rho^2 \sum_{ij} V(\rho \sin \alpha_{ij}) + s_n^2(\not{\rho})\right)\Phi_n(\Omega;\rho) = 0 \quad \Longrightarrow \quad s_n \cos\left(\frac{s_n \pi}{2}\right) + \frac{8}{\sqrt{3}} \sin\left(\frac{s_n \pi}{6}\right) = 0$$

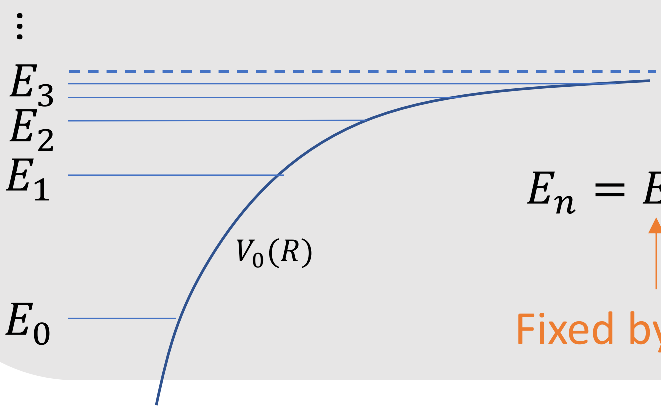
$$\left(-\frac{d^2}{d\rho^2} + \frac{s_n^2(\not{\rho}) - 1/4}{\rho^2} - E\right) f_n = 0$$

All s_n are real, except one:
$s_0 = \pm i1.00624$

For $n = 0$, one gets the Efimov attractive potential $V_0(\rho) = -\frac{|s_0|^2 + \frac{1}{4}}{\rho^2}$

For small ρ, $f_0(\rho) = \alpha \rho^{i|s_0|} + \beta \rho^{-i|s_0|} \propto \cos(|s_0| \ln \Lambda \rho)$

→ Three-body parameter set by a short-range 3-body boundary condition



$$E_n = E_0 \left(\exp \pi / |s_0|\right)^{-2n}$$

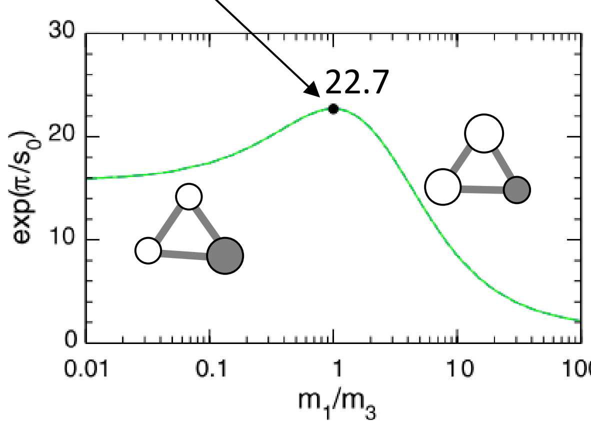
Fixed by Λ

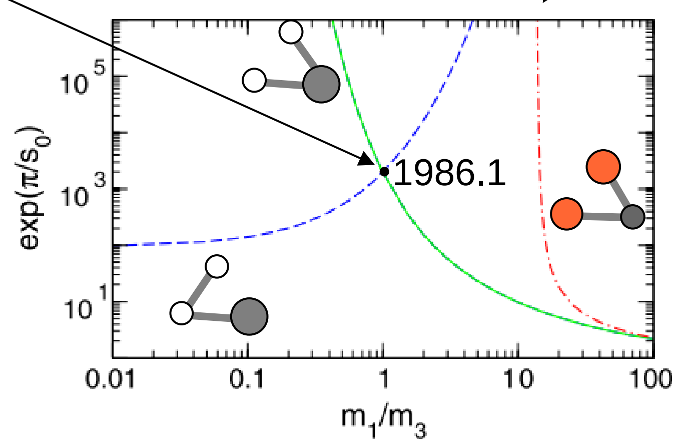
$\lambda_0 \approx 22.7$ (scaling factor)

	3 Bosons	2 Bosons + 1 particle	3 Distin-guishable	3 Fermions	2 Fermions + 1 particle
Efimov effect	Yes	Yes	No	No	Yes For mass ratio > **13.607**
Scaling factor e^{π/s_0} for equal masses	22.7	1986.1			

Number of resonant pairs

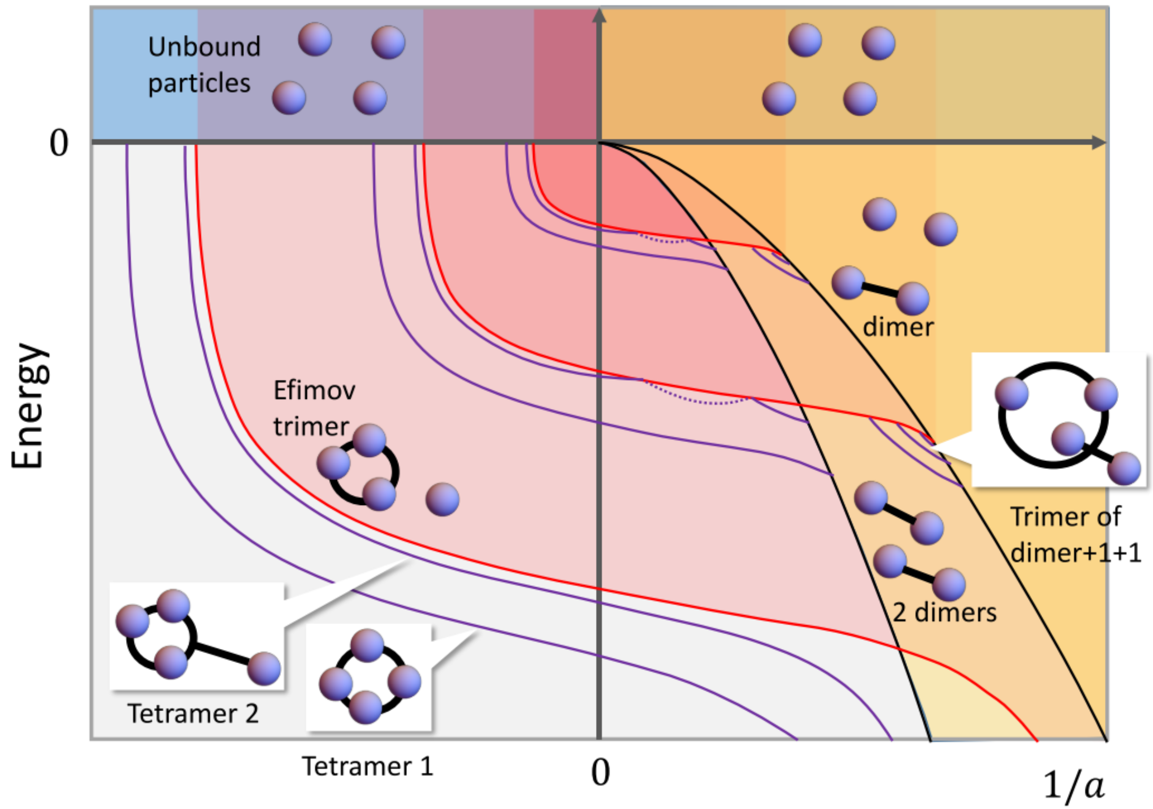
Scaling factor e^{π/s_0} for unequal masses





More than three particles

Tetramers of four identical bosons



No four-body Efimov effect

Two “universal tetramers” attached to each Efimov trimer

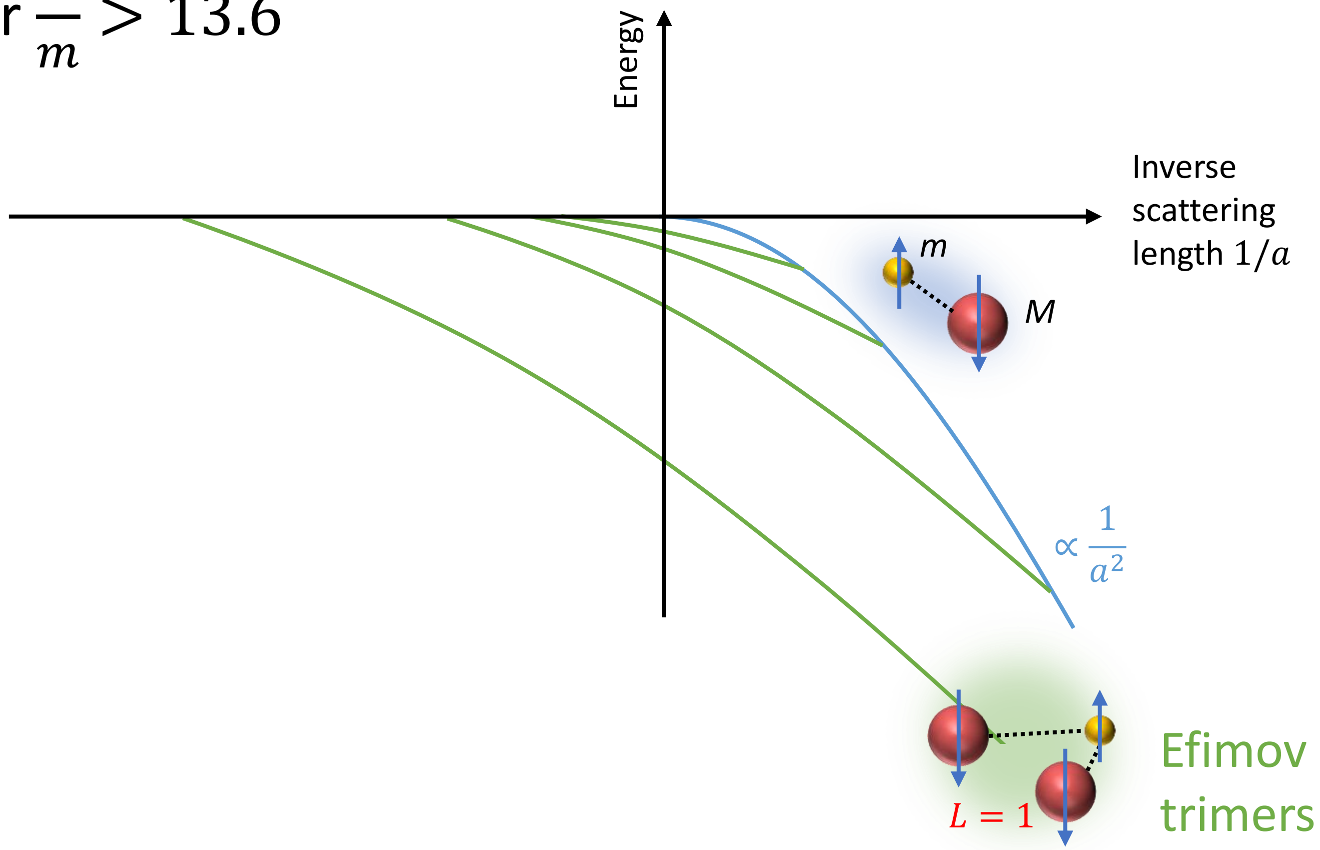
Controversy:

1) There is in general a need for a 4-body parameter
2) The universal states do not require any 4-body parameter

J. von Stecher, J. P. D’Incao, and C. H. Greene, Nature Physics, 5, 417–421, 2009.
A. Deltuva, Europhysics Letters, 95, 43002, 2011.

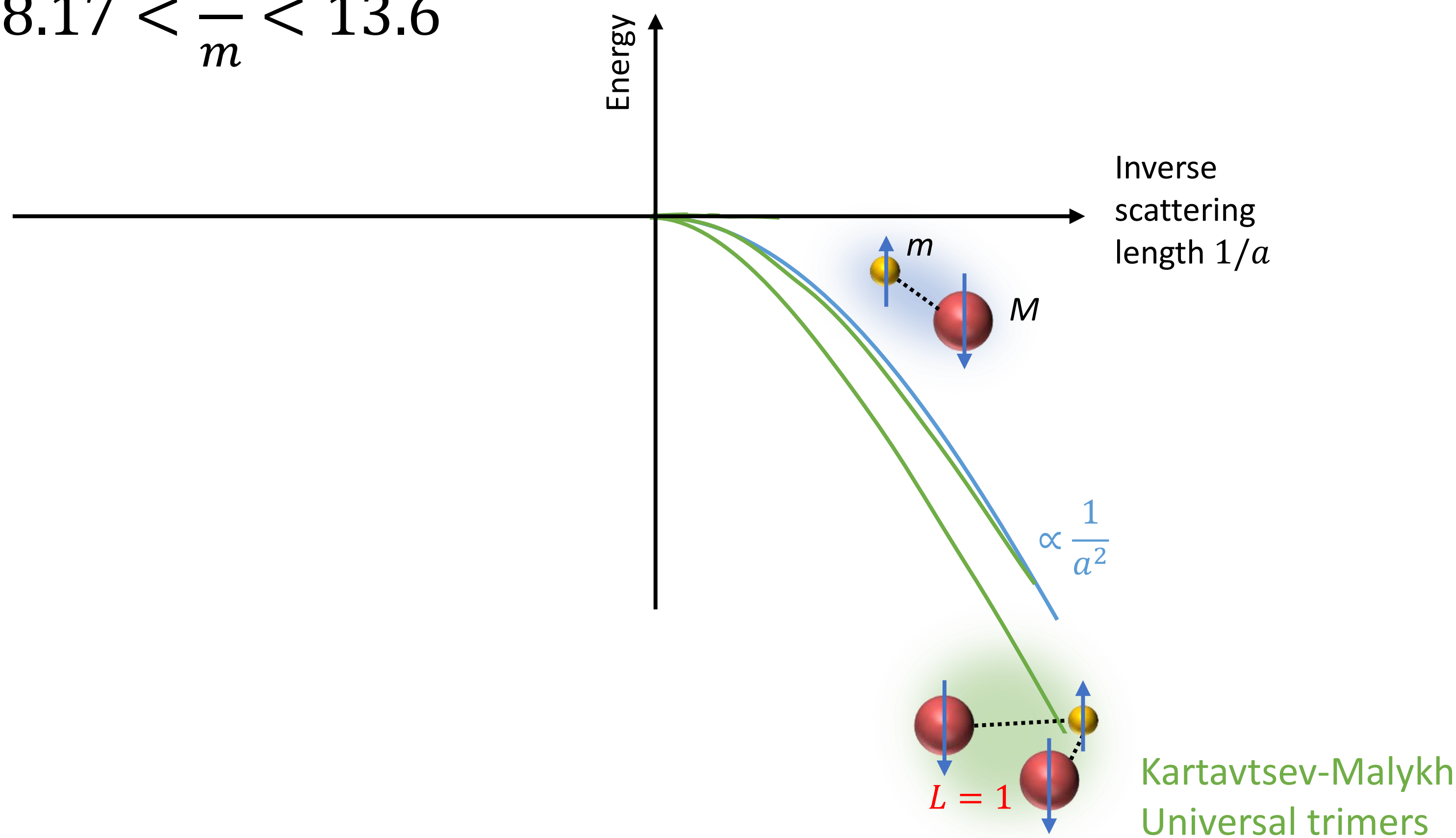
Tetramers of 3+1 fermions

For $\frac{M}{m} > 13.6$



Tetramers of 3+1 fermions

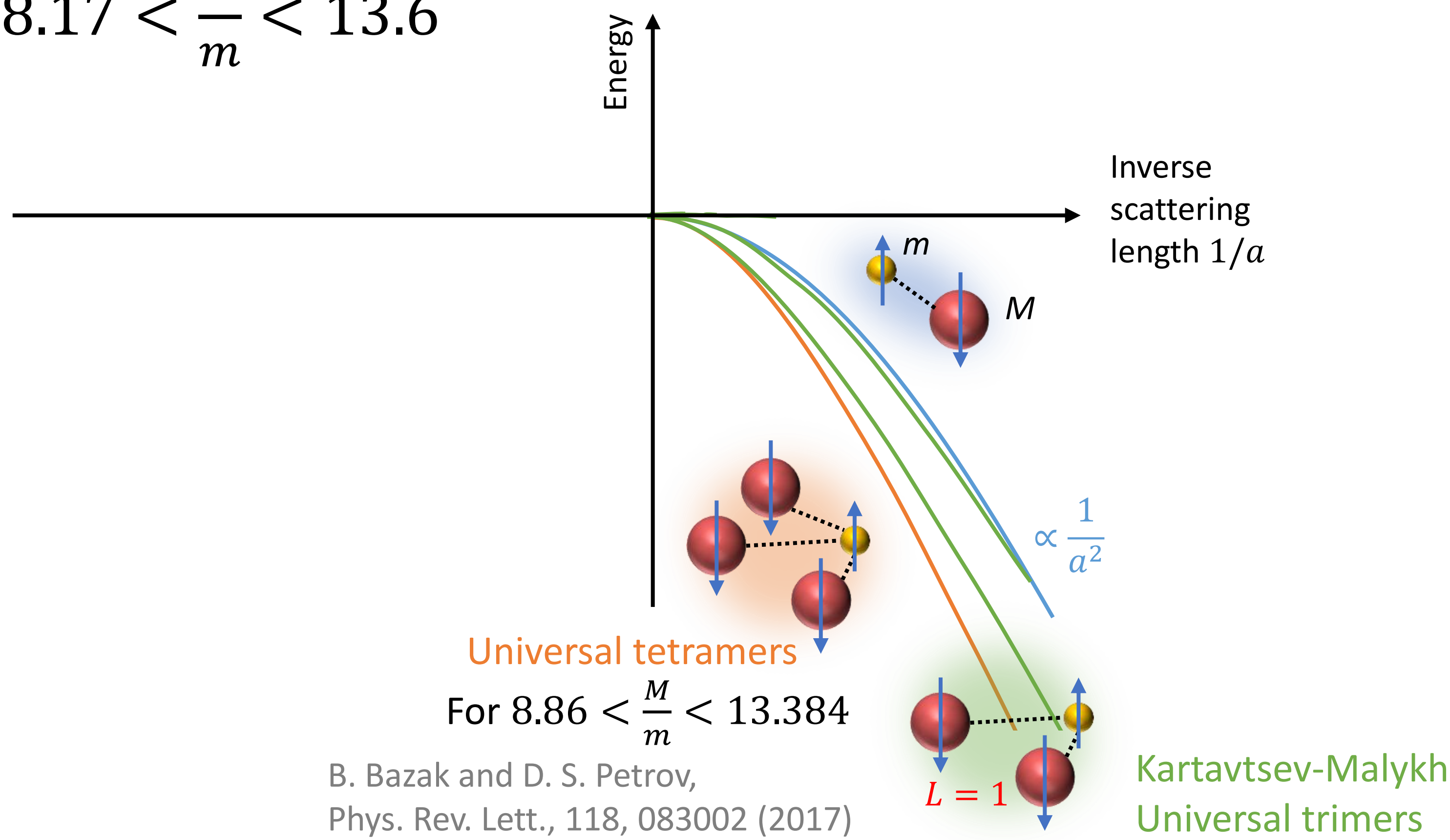
For $8.17 < \frac{M}{m} < 13.6$



Kartavtsev+Malykh
J Phys B 40, 1429 (2007)

Tetramers of 3+1 fermions

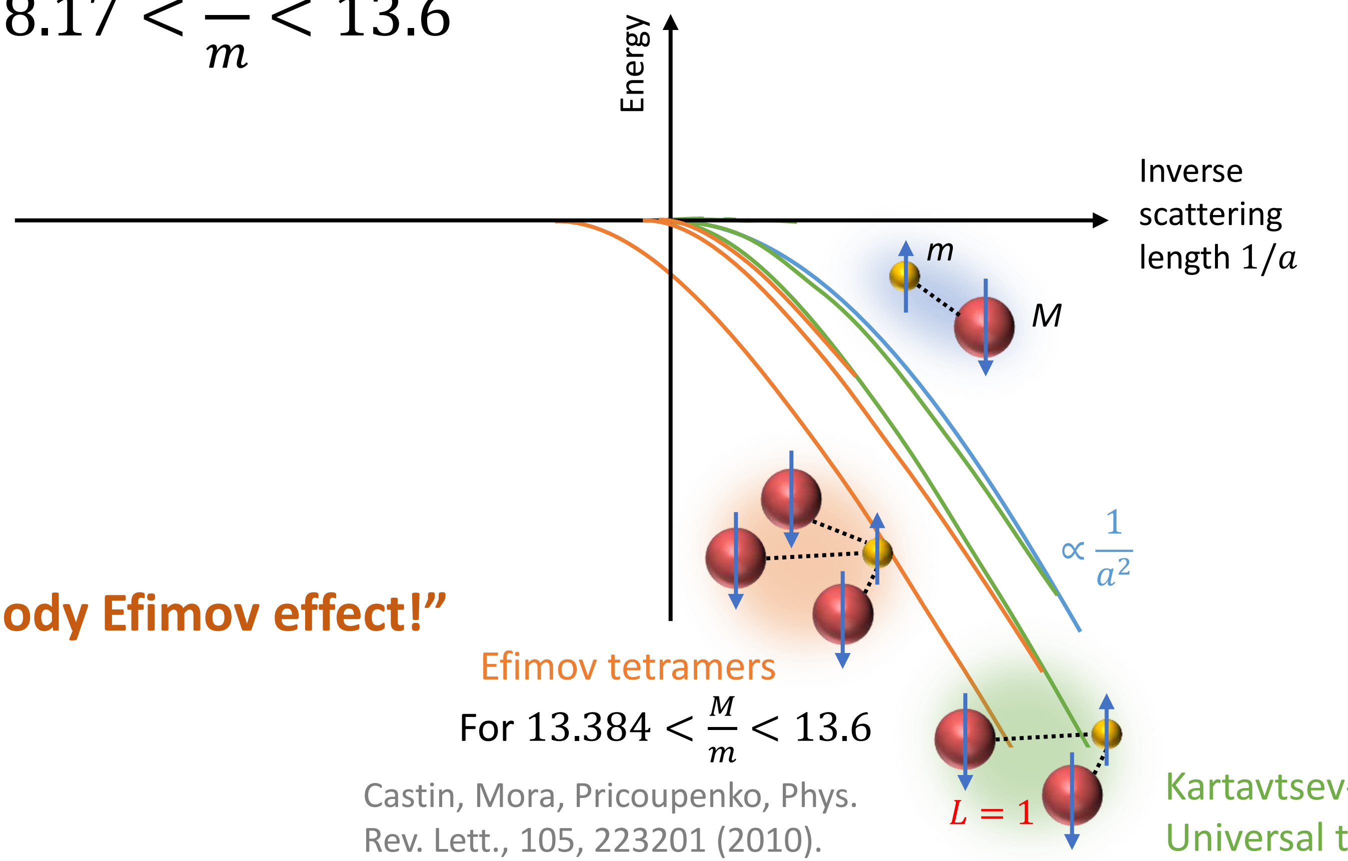
For $8.17 < \frac{M}{m} < 13.6$



Kartavtsev+Malykh
J Phys B 40, 1429 (2007)

Tetramers of 3+1 fermions

For $8.17 < \frac{M}{m} < 13.6$



"4-body Efimov effect!"

For $13.384 < \frac{M}{m} < 13.6$

Castin, Mora, Pricoupenko, Phys. Rev. Lett., 105, 223201 (2010).

Kartavtsev+Malykh
J Phys B 40, 1429 (2007)

Conclusion

Efimov physics has been a developing field of quantum physics, both theoretical and experimental, unveiling a whole collection of universal few-body states with remarkable mathematical properties and challenges.