Properties of the Interacting Fermi Gas in the Random Phase Approximation ($\simeq$ Bosonization)

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A Review of Bogoliubov Theory

Fock Space and Canonical Operators

State of N bosonic/fermionic particles on the torus $\mathbb{T}^3$ is described by a vector

$$\psi \in P_\pm L^2(\mathbb{T}^3)^{\otimes N} \quad \text{with} \quad \psi(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(N)}) = (\pm 1)^\pi \psi(x_1, x_2, \ldots, x_N)$$

Fock space:

$$\mathcal{F}_\pm := \bigoplus_{n=0}^{\infty} P_\pm L^2(\mathbb{T}^3)^{\otimes n}$$

$$\psi = (\psi^{(0)}, \psi^{(1)}, \psi^{(2)}, \psi^{(3)}, \ldots) , \qquad \psi^{(n)} \in P_\pm L^2(\mathbb{T}^3)^{\otimes n}$$

Creation and Annihilation Operators: for f in one-particle space $L^2(\mathbb{T}^3)$ define

$$(a^*(f)\psi)^{(n+1)} := \sqrt{n+1} P_\pm(f \otimes \psi^{(n)}) , \qquad (a(f)\psi)^{(n-1)} := \sqrt{n} \langle f, \psi^{(n)} \rangle_{L^2(\mathbb{T}^3, \mathrm{d}x_1)}$$

Zero particles state: vacuum vector $\Omega = (1, 0, 0, \ldots) \in \mathcal{F}$

$$a(f)\Omega = 0 \quad \text{for all } f \in L^2(\mathbb{T}^3)$$

For an ONB $(f_k)_k$ of the one-particle space $L^2(\mathbb{T}^3)$ we write $a_k := a(f_k)$.
Main example: plane waves $f_k(x) = (2\pi)^{-3/2} e^{ik\cdot x}$ with momenta $k \in \mathbb{Z}^3$.

Commutators and Anticommutators

Canonical Commutation Relations: on bosonic Fock space $\mathcal{F}_+$

$$[a(f), a(g)] := a(f)a(g) - a(g)a(f) = 0 = [a^*(f), a^*(g)] \ ,$$
$$[a(f), a^*(g)] = \langle f, g \rangle \ .$$

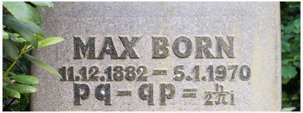
Canonical Anticommutation Relations: on fermionic Fock space $\mathcal{F}_-$

$$\{a(f), a(g)\} := a(f)a(g) + a(g)a(f) = 0 = \{a^*(f), a^*(g)\} \ ,$$
$$\{a(f), a^*(g)\} = \langle f, g \rangle \ .$$

Bogoliubov maps: linear maps from the algebra (fermionic/bosonic) of canonical operators to itself such that the CCR/CAR are preserved

Application: quadratic Hamiltonians

$$H = \sum_{j,k} D_{jk} a_j^* a_k + \frac{1}{2} \sum_{j,k} (W_{jk} a_j^* a_k^* + \overline{W_{jk}} a_k a_j)$$



19.11.1932 Germany's top industry leaders urge the president to appoint Hitler as chancellor

30.1.1933: NSDAP seizes control, governs by presidential emergency decrees

7.4.1933: "Law for the Restoration of the Professional Civil Service": Born, Courant, Noether etc. dismissed from their positions

Diagonalization of Quadratic Hamiltonians

Parametrization I: Bogoliubov maps can be written as

$$\tilde{a}_j^* = \sum_k U_{j,k} a_k^* + \sum_k V_{j,k} a_k \,, \qquad U^* U \mp V^* V = \mathbb{1} \,, \quad V^* U \mp U^* V = 0 \,.$$

Diagonalization: For reasonable matrices D and W, there exists a Bogoliubov map such that

$$H = \underbrace{\sum_j E_{j,j}\, \tilde{a}_j^* \tilde{a}_j}_{\text{excitation spectrum}} + \underbrace{\frac{1}{2} \operatorname{tr}(E - D - W)}_{\text{ground state energy}} \,, \qquad \sigma(\tilde{a}_j^* \tilde{a}_j) = \mathbb{N} \,.$$

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Parametrization II: there exists $K_{j,k}$ ("Bogoliubov kernel") such that

$U = \cosh(K)\,,\ V = \sinh(k)$ (bosonic case) / $U = \cos(k)\,,\ V = \sin(k)$ (fermionic case).

Implementation on Fock space: If $\operatorname{tr} V^* V < \infty$ we can define a unitary $R : \mathcal{F}_\pm \to \mathcal{F}_\pm$ by

$$R := \exp\Big(\sum_{j,k} K_{j,k} a_j^* a_k^* - \text{h.c.} \Big) \,,$$

which "implements" the Bogoliubov map: $R^* a_j R = \tilde{a}_j \,, \qquad R^* a_j^* R = \tilde{a}_j^* \,.$

From Fermions to Bosons

Transforming Fermions into Bosons (quasi. . .)

Pairs of fermions as bosons? Fermionic a_j^* anticommute, pairs of fermions commute:

$$\text{Let } b_{j,k}^* := a_j^* a_k^* \;, \quad \text{then} \quad [b_{j,k}^*, b_{m,n}^*] = 0 \;.$$

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The bad news... Pauli principle $\left(a_j^*\right)^2 = 0$ is difficult to escape:

$$\left(b_{j,k}^*\right)^2 = a_j^* a_k^* a_j^* a_k^* = -\left(a_j^*\right)^2 (a_k^*)^2 = 0 . \qquad \text{We cannot create more than one pair!}$$

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Solution by collective modes: pick a large index set of pairs I_μ, and define

$$b_\mu^* := |I_\mu|^{-1/2} \sum_{(j,k)\in I_\mu} a_j^* a_k^* \,.$$

then only a small fraction of terms (the diagonal "$j = \tilde{j}$") vanishes due to the Pauli principle in

$$(b_\mu^*)^2 = |I_\mu|^{-1} \sum_{(j,k)\in I_\mu} \sum_{(\tilde{j},\tilde{k})\in I_\mu} a_j^* a_k^* a_{\tilde{j}}^* a_{\tilde{k}}^* \,.$$

Quasibosonic CCR: for well-chosen index sets

$$[b_\mu^*, b_\tau^*] = 0 \,, [b_\mu^*, b_\tau^*] = \delta_{\mu,\tau} - \mathcal{E}_{\mu,\tau} \,, \text{ with "deviation operator" } \mathcal{E}_{\mu,\tau} \sim \frac{\sum_{j,k} a_j^* a_k}{|I_\mu|^{1/2} |I_\tau|^{1/2}} \,.$$

Application to the Fermi Gas

$$H_N := \sum_{i=1}^{N} (-\Delta_{x_i}) + \lambda \sum_{1 \leq i < j \leq N} V(x_i - x_j) \qquad \text{with } V : \mathbb{R}^3 \to \mathbb{R}$$

Mean–field limit: $N \to \infty$, weak interaction, mean particle distance $\ll$ interaction range

Simplest fermionic state: antisymmetric tensor product = Slater determinant

$$\psi = \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_N \qquad \Leftrightarrow \qquad \psi(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \det\left(\varphi_i(x_j)\right)_{i,j}$$

For non-interacting fermions: the ground state is a Slater determinant of plane waves.

Kinetic energy: $\sum_{k \in B_F} |k|^2 \sim N^{5/3}$

Potential energy: $\sum_{i<j} V(x_i - x_j) \sim N^2$

$$\Rightarrow \quad \textbf{Coupling constant:} \qquad \lambda := N^{-1/3}$$

Fermi momentum: $k_F = \sup_{k \in B_F} |k| \sim N^{1/3}$ $\Rightarrow$ typical velocity $\sim N^{1/3}$

Rescale time: consider typical times of order $N^{-1/3}$

$$iN^{1/3}\partial_t \psi_t = \left[\sum_{j=1}^{N} -\Delta_{x_j} + N^{-1/3} \sum_{1 \leq i < j \leq N} V(x_i - x_j) \right] \psi_t$$

Effective Planck constant: $\hbar := N^{-1/3}$, multiply both sides by $\times \hbar^2$

$\Rightarrow$ N fermions in mean-field/semiclassical scaling

$$i\hbar\partial_t \psi_t = H_N \psi_t \,, \qquad H_N = \sum_{j=1}^{N} -\hbar^2 \Delta_{x_j} + \frac{1}{N} \sum_{1 \leq i < j \leq N} V(x_i - x_j) \,, \qquad \hbar = N^{-1/3}$$

Investigate: ground state energy $E_N := \inf_{\|\psi\|=1} \langle \psi, H_N \psi \rangle$, spectrum $\sigma(H_N)$, time evolution?

Hartree–Fock Theory: optimize only the choice of orbitals φ_j (no linear combinations):

$$E_N^{\mathrm{HF}} := \left\{ \langle \psi, H_N \psi \rangle : \psi = \varphi_1 \wedge \cdots \wedge \varphi_N \text{ with } \varphi_j \in L^2(\mathbb{T}^3) \text{ normalized} \right\} .$$

In general, plane waves are only a stationary point, not a (local or global) minimizer.

Theorem: [Gontier–Hainzl–Lewin '19, Gontier–Lewin '19]

With Coulomb interaction, spin, and in the thermodynamic limit:

$$E^{\mathrm{pw}} - C_1 e^{-\rho^{1/6}} \leq E^{\mathrm{HF}} \leq E^{\mathrm{pw}} - C_2 e^{-\rho^{1/6}} , \qquad \text{for density } \rho \to \infty .$$

Theorem: [B–Nam–Porta–Schlein–Seiringer '21]

On the finite torus, without spin, for $0 \leq \hat{V} \in \ell^1(\mathbb{Z}^3)$, for $N = |\mathcal{B}_{\mathrm{F}}|$, we have

$$E_N^{\mathrm{HF}} = E_N^{\mathrm{pw}} .$$

Hartree–Fock trivially yields an upper bound:

$$E_N^{\mathrm{HF}} := \inf_{\substack{\psi \text{ Slater det.} \\ \|\psi\|=1}} \langle \psi, H_N \psi \rangle \qquad \geq E_N$$

Theorem: [Graf–Solovej '94]

For $N \to \infty$ we have

$$E_N = E_N^{\mathrm{HF}} + o(N^0) \qquad \text{for } N \to \infty \, .$$

The Hartree–Fock energy is given by the Thomas–Fermi (kinetic and direct) and the Dirac (exchange) term:

$$E_N^{\mathrm{HF}} = c_{\mathrm{TF}} N + c_{\mathrm{HF}} N^0 \, .$$

What is the next order?

Beyond Hartree–Fock Theory: Correlation Energy

Theorem: [B–Nam–Porta–Schlein–Seiringer '21]

Let $\hat{V}$ be non–negative and $\sum_{k\in\mathbb{Z}^3} \hat{V}(k)|k| < \infty$. Then

$$E_N = c_{\mathrm{TF}} N + c_{\mathrm{HF}} + c_{\mathrm{RPA}} N^{-1/3} + o(N^{-1/3}) \qquad \text{as } N \to \infty \,.$$

The leading order of the correlation energy is given by (with $\kappa_0 = (3/4\pi)^{1/3}$)

$$c_{\mathrm{RPA}} := \kappa_0 \sum_{k\in\mathbb{Z}^3} |k| \left(\frac{1}{\pi} \int_0^\infty \log\left(1 + 2\pi\kappa_0 \hat{V}(k)\Big(1 - \lambda \arctan\big(\frac{1}{\lambda}\big)\Big)\right) \mathrm{d}\lambda - \frac{\pi}{2}\kappa_0 \hat{V}(k) \right) .$$

Proof: Hamiltonian $\mathcal{H}_N$ can be generalized to an operator $\mathcal{F} \to \mathcal{F}$:

$$\mathcal{H}_N = \hbar^2 \sum_{p\in\mathbb{Z}^3} |p|^2 a_p^* a_p + \frac{1}{2N} \sum_{k,p,q\in\mathbb{Z}^3} \hat{V}(k) a_{p+k}^* a_{q-k}^* a_q a_p$$

Bosonization $\rightsquigarrow$ almost quadratic Hamiltonian $\rightsquigarrow$ approximate Bogoliubov diagonalization.

History of the Correlation Energy

- **Wigner '34, Heisenberg '47:** perturbation theory with $\hat{V}(k) = 1/|k|^2$ severely divergent already at second order
- **Macke '50:** partial resummation (log series) cures the divergence
- **Bohm–Pines '53:** relation to collective oscillations (plasmon excitation)
- **Sawada–Brueckner–Fukuda–Brout '57:** explanation through formal bosonization of pair excitations
- **Gell-Mann–Brueckner '57:** systematic partial resummation

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- **B–Nam–Porta–Schlein–Seiringer '20:** rigorous bosonization $\Rightarrow$ optimal upper bound for $\hat{V}$ non-negative and compactly supported
- **B–Nam–Porta–Schlein–Seiringer '21:** corresponding lower bound for small potential
- **B–Porta–Schlein–Seiringer '23, Christiansen–Hainzl–Nam '23:** general potentials
- **Christiansen–Hainzl–Nam '23–'24**: extension to Coulomb potential

Hartree–Fock minimizer:

$$\psi_\mathrm{F} := \bigwedge_{k \in \mathcal{B}_\mathrm{F}} f_k = \prod_{k \in \mathcal{B}_\mathrm{F}} a_k^* \Omega$$ Slater determinant of plane waves

$$\mathcal{B}_\mathrm{F} := \{k \in \mathbb{Z}^3 : |k| \leq \kappa_0 N^{1/3}\}$$ momenta in the Fermi ball

Try to find a diagonalizable (i. e,. quadratic) Hamiltonian describing only the excitations over Hartree–Fock theory.

Recall the starting point: fermionic Hamiltonian

$$\mathcal{H}_N = \hbar^2 \sum_{p \in \mathbb{Z}^3} |p|^2 a_p^* a_p + \frac{1}{2N} \sum_{k,p,q \in \mathbb{Z}^3} \hat{V}(k) a_{p+k}^* a_{q-k}^* a_q a_p$$

Particle-hole transformation $R : \mathcal{F} \to \mathcal{F}$ unitarily acting by

$$R\, a_k^*\, R^* := \begin{cases} a_k^* & \text{for } |k| > cN^{1/3} \\ a_k & \text{for } |k| \leq cN^{1/3}\,, \end{cases} \qquad R\psi_{\mathrm{F}} := \Omega\,.$$

Expand $RH_N R^*$ **and normal-order:**

$$RH_N R^* = E_N^{\mathrm{HF}} + \underbrace{\hbar^2 \sum_{p \in \mathcal{B}_{\mathrm{F}}^c} p^2 a_p^* a_p - \hbar^2 \sum_{h \in \mathcal{B}_{\mathrm{F}}} h^2 a_h^* a_h}_{=:\ H^{\mathrm{kin}}} + \underbrace{Q}_{\substack{\text{interaction, quartic in} \\ \text{operators } a^* \text{ and } a}}$$

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Define collective pair–creation operators

$$b_k^* := \sum_{\substack{p \in \mathcal{B}_{\mathrm{F}}^c \\ h \in \mathcal{B}_{\mathrm{F}}}} \delta_{p-h,k}\, a_p^*\, a_h^*$$

p “particle” outside the Fermi ball

h “hole” inside the Fermi ball

Bosonizable terms of the interaction:

$$Q = \frac{1}{N} \sum_{k \in \mathbb{Z}^3} \hat{V}(k) \Big(2 b_k^* b_k + b_k^* b_{-k}^* + b_{-k} b_k \Big) + \text{non–bosonizable terms} \,.$$

This is convenient because the b_k^* and b_k have **approximately bosonic (CCR) commutators**:

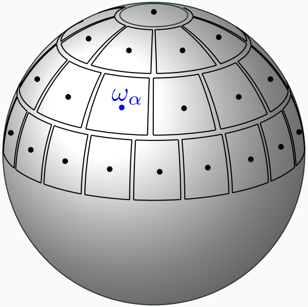
$$b_k^* b_\ell^* = b_\ell^* b_k^* \,, \qquad [b_\ell, b_k^*] = b_\ell b_k^* - b_k^* b_\ell = n_k^2 \left(\delta_{k,\ell} + \mathcal{E}_{k,\ell} \right) \,.$$

How to express $H^{\text{kin}} \sim a^* a$ as a quadratic form in pair operators?

Use a discrete partition of unity of the surface of the Fermi ball

$$b_k^* = \sum_{\alpha=1}^{M} n_{\alpha,k} b_{\alpha,k}^* + \text{negligible} \,.$$

Fermi ball $\mathcal{B}_\mathrm{F}$



[Benfatto–Gallavotti '90]
[Haldane '94]
[Fröhlich et al. '95]

Pair creation operators, localized to patches:

$$b^*_{\alpha,k} := \frac{1}{n_{\alpha,k}} \sum_{\substack{p \in \mathcal{B}_\mathrm{F}^c \cap B_\alpha \\ h \in \mathcal{B}_\mathrm{F} \cap B_\alpha}} \delta_{p-h,k}\, a^*_p a^*_h \,.$$

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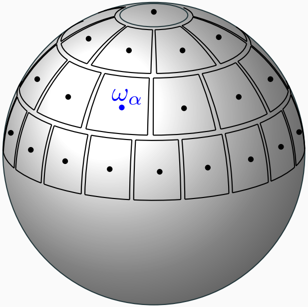

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Linearize kinetic energy around patch center ω_α:

$$[H^\mathrm{kin}, b^*_{\alpha,k}] \simeq 2\hbar \, |k \cdot \hat{\omega}_\alpha| \, b^*_{\alpha,k} \,.$$

$\Rightarrow$ **in commutators:**

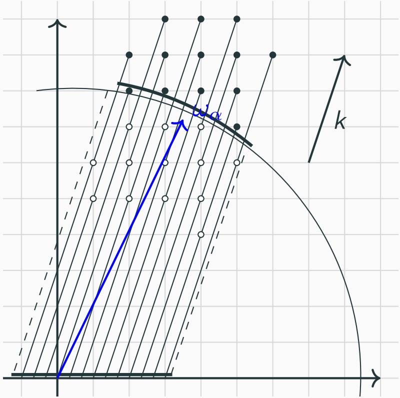
$$H^\mathrm{kin} \simeq \sum_{k \in \mathbb{Z}^3} \sum_{\alpha=1}^{M} 2\hbar \, |k \cdot \hat{\omega}_\alpha| \, b^*_{\alpha,k} b_{\alpha,k} \,.$$

Normalization:

To get explicitly boson-like commutators, choose $n_{\alpha,k}$ such that $\|b^*_{\alpha,k}\Omega\| = 1$.

$$n_{\alpha,k}^2 = \sum_{\substack{p \in \mathcal{B}_F^c \cap B_\alpha \\ h \in \mathcal{B}_F \cap B_\alpha}} \delta_{p-h,k} \simeq \frac{4\pi N^{2/3}}{M} |k \cdot \hat{\omega}_\alpha| \,.$$

Closing gap of kinetic energy balanced by excitation energies vanishing at the same rate.



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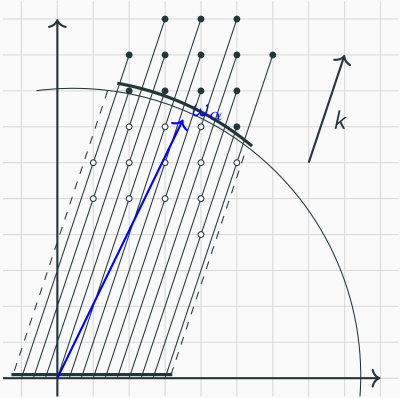


Effective Quadratic Bosonic Hamiltonian

$$H^{\text{eff}} = \hbar \sum_{k \in \mathbb{Z}^3} \left[\sum_\alpha u_\alpha^2 b^*_{\alpha,k} b_{\alpha,k} + \frac{\hat{V}(k)}{M} \sum_{\alpha,\beta} \left(u_\alpha u_\beta b^*_{\alpha,k} b_{\beta,k} + u_\alpha u_\beta b^*_{\alpha,k} b^*_{\beta,-k} + \text{h.c.} \right) \right]$$

Can be diagonalized by a Bogoliubov transformation, i. e., a unitary $T : \mathcal{F} \to \mathcal{F}$ preserving the canonical commutation relations of bosonic creation and annihilation operators.

$$T := \exp\left(\sum_{k \in \mathbb{Z}^3} \sum_{\alpha,\beta=1}^{M} K(k)_{\alpha,\beta} b^*_{\alpha,k} b^*_{\beta,-k} - \text{h.c.} \right)$$

$$T b_{\alpha,k} T^* \simeq \sum_{\beta} \cosh(K(k))_{\alpha,\beta} b_{\beta,k} + \sum_{\beta} \sinh(K(k))_{\alpha,\beta} b^*_{\beta,-k}$$

Optimize the matrix $K(k)$ such that $b^* b^*$– and bb–terms vanish from $T^* H^{\text{eff}} T$:

$$T H^{\text{eff}} T^* \simeq E_N^{\text{RPA}} + \hbar \sum_{k \in \mathbb{Z}^3} \sum_{\alpha,\beta=1}^{M} E(k)_{\alpha,\beta} b^*_{\alpha,k} b_{\beta,k} \quad \geq E_N^{\text{RPA}} .$$

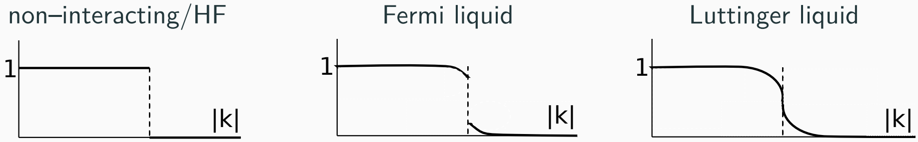
Non–bosonizable terms and deviation from bosonic CCR controlled through all the analysis. ■

Phases of Matter and Universality in Bosonization

Universality Conjectures:

- **Fermi liquid** [Landau '56]:
  in $d \geq 2$, in absence of superconductivity, almost non–interacting effective description
- **Luttinger liquid** [Haldane '81]:
  in $d = 1$, behavior similar to [Lieb–Mattis '65]'s exact solution of the Luttinger model

Signatures: number of fermions with momentum k in the ground state



Renormalization Group Analysis: [Benfatto–Gallavotti–Mastropietro '92] $d = 1$ is Luttinger liquid; [Feldmann–Knörrer–Trubowitz '00–'04] $d = 2$ for asymmetric Fermi surface (to suppress Kohn–Luttinger superconductivity) is Fermi liquid.

Bosonization is sufficient to identify the Fermi liquid phase

Theorem: [B–Lill '23] Let $\psi := T\Omega$ the trial state for the optimal upper bound. Then for any $\epsilon > 0$ and all momenta $q \in \mathbb{Z}^3$ such that $\nexists k \in B_R(0) : \frac{|k\cdot q|}{|k||q|} \in (0,\epsilon)$ we have

$$0 \leq \langle \psi, a_k^* a_k \psi \rangle \leq N^{-\frac{2}{3}} \sum_{k \in \mathcal{C}^q \cap \mathbb{Z}^3} \frac{\hat{V}(k)}{2\pi\kappa_0 |k|} \int_0^\infty \frac{(\mu^2 - \lambda_{q,k}^2)(\mu^2 + \lambda_{q,k}^2)^{-2}}{1 + Q_k(\mu)} \, \mathrm{d}\mu + \mathcal{E} \, ,$$

where

$$\lambda_{q,k} := \frac{|k \cdot q|}{|k||q|} \, , \qquad Q_k(\mu) := 2\pi\kappa_0 \hat{V}(k) \left(1 - \mu \arctan\left(\frac{1}{\mu}\right)\right)$$

and the error term is $|\mathcal{E}| \leq C\epsilon^{-1} N^{-\frac{2}{3} - \frac{1}{12}}$.

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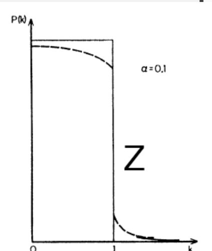
Corollary: The momentum distribution of the trial state has a jump of height $Z \geq 1 - CN^{-\frac{2}{3}+\frac{1}{12}}$ at the non-interacting Fermi momentum.



Graph: [Daniel–Vosko '60]