RAGE and the Machine

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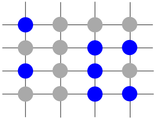
GSSI February 2025

The Anderson Model

P. W. Anderson 1958 :
"Absence of diffusion in certain random lattices", Phys. Rev. (Nobel 1977)

Consider many possible realizations of the potential ω, where ω is in a probability space $(\Omega, \mathbb{P})$.



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$$V_\omega \varphi(x) = \omega_x \varphi(x),$$

is a random potential generated by $\omega = (\omega_x)_{x \in \mathbb{Z}^d}$ bounded, iid random variables.

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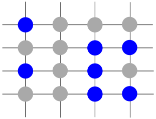

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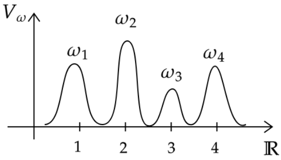

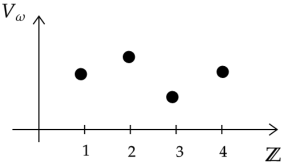

Propagation of quantum waves in disordered media

$$\begin{cases} i\partial_t \psi(x,t) = H_\omega \psi(x,t), \quad \psi(\cdot,t) \in \ell^2(\mathbb{Z}^d), \\ \psi(x,0) = \psi_0, \end{cases}$$

Therefore

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We say that H_ω exhibits *dynamical localization* (DL) in I if there exist constants $C < \infty$ and $c > 0$ such that for all $x, y \in \mathbb{Z}^d$,

$$(DL) \qquad \mathbb{E}\left(\sup_{t\in\mathbb{R}} |\langle \delta_y, e^{-itH_\omega}\chi_I(H_\omega)\delta_x\rangle|\right) \le Ce^{-c|x-y|}$$

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In particular, for any $\varphi \in \ell^2(\mathbb{Z}^d)$ with compact support we have

$$\sup_t \left\| |X|^p e^{-itH_\omega}\chi_I(H_\omega)\psi \right\| < \infty$$

for every $p \ge 0$, with probability one.

R.A.G.E.

$\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_{pp}$, the Lebesgue decomposition of a Hilbert space according to continuous spectrum and pure point spectrum.

Theorem (Ruelle '69, Amrein-Georgescu '73/74, Enss'78)

Let H be a s.a. operator on $\ell^2(\mathbb{Z}^d)$, let P_c and P_{pp} be the orthogonal projections onto $\mathcal{H}_c$ and $\mathcal{H}_{pp}$, resp. Let Λ_L be a cube of side L around the origin. Then, for any $\varphi \in \ell^2(\mathbb{Z}^d)$,

$$\|P_c\varphi\|^2 = \lim_{L\to\infty} \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\sum_{x \notin \Lambda_L} |e^{-itH}\varphi(x)|^2 \right) dt$$

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Theorem

If (DL) in I then $P_c\chi_I(H) = 0$, so pp spectrum in I.

If $\sup_t \left\| |X| e^{-itH_\omega}\chi_I(H_\omega)\psi \right\| < \infty$ a.s., then pp spectrum in I.

The expected metal-insulator transition

Conjecture and what is known for $H_\omega = -\Delta + V_\omega$

- In $d = 1$ H_ω exhibits localization everywhere in the spectrum. ✓
- In $d = 2$, localization as in $d = 1$. (Localization at spectral band edges)
- In $d \geq 3$ there is a transition between localized and delocalized states. (Localization at spectral band edges)

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Proofs for $d \geq 1$ rely on two alternative methods :

- the Multiscale Analysis (Fröhlich-Spencer'83).
- the Fractional Moment Method (Aizenman-Molchanov'93)

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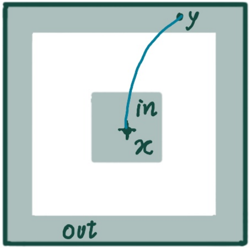
The Multiscale Analysis - The Machine ! !

$\Lambda_L(x) := [-\frac{L}{2}, \frac{L}{2}]^d + x \subset \mathbb{Z}^d$, with boundary $\partial\Lambda_L$. Write $H_\Lambda = \chi_{\Lambda_L} H_\omega \chi_{\Lambda_L}$

$\Lambda_L(x) \subset \mathbb{Z}^d$ is (E,m)-**good** if $E \notin \sigma(H_\omega)$ and

$$\left|\langle \delta_y, (H_\Lambda - E)^{-1}\delta_x \rangle\right| \leq e^{-mL/2}, \quad \forall y \in \partial\Lambda_L$$



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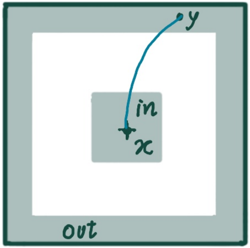

The box $\Lambda_L(x)$ is **non resonant** if $\left\|(H_\Lambda - E)^{-1}\right\| \leq L^\theta$ for some $\theta > 0$.

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Theorem (Fröhlich-Spencer'83, von Dreifus-Klein'89)

Suppose that for E_0 in the a.s. spectrum, there is L_0 and $m_0(L_0)$ such that

$$\mathbb{P}(\Lambda_{L_0}(0) \textit{ is } (E, m_0) - \textit{good}) \geq 1 - \frac{1}{L_0^p},$$

then, for $L_{k+1} = [L_k^\alpha]$, $k = 0, 1, 2, \ldots$ with $\alpha \in (1, 2)$ and $p' < p$

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Moreover, there exists $\delta > 0$ such that for $I = (E_0 - \delta, E_0 + \delta)$, for all k,

$$\mathbb{P}(for\ all\ E \in I,\ either\ \Lambda_{L_k}(x)\ or\ \Lambda_{L_k}(y)\ is\ (E, \frac{m_0}{2}) - good) \geq 1 - \frac{1}{L_k^{2p'}}$$

for all x, y with $\|x - y\| > L_k$.

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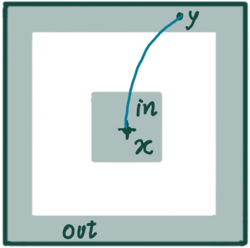
for all x, y with $\|x - y\| > L_k$. Then... dynamical localization in I.

Sketch of proof

The box Λ_{L_0} is good (initial estimate)
Iteration step $L_k \Rightarrow L_{k+1}$.
Assume $\Lambda_{L_{k+1}}$ non resonant.



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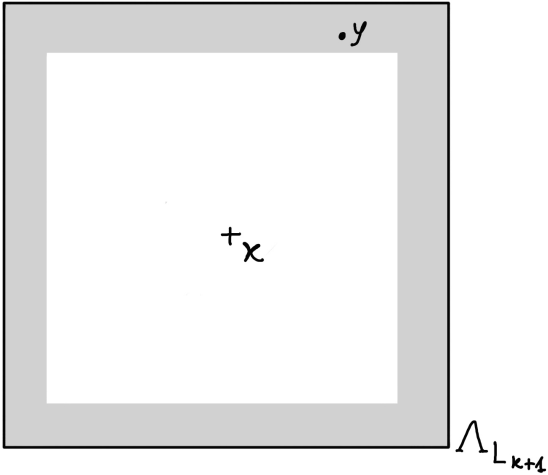

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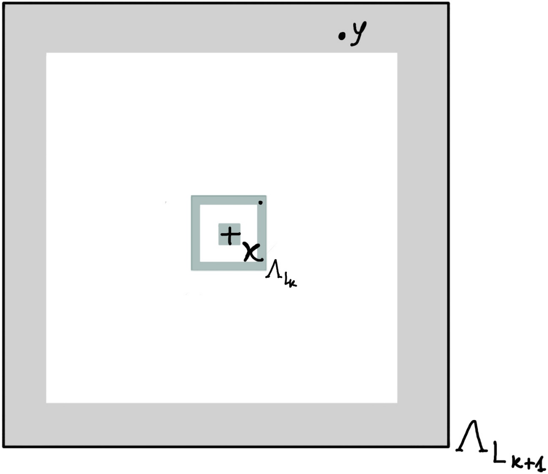

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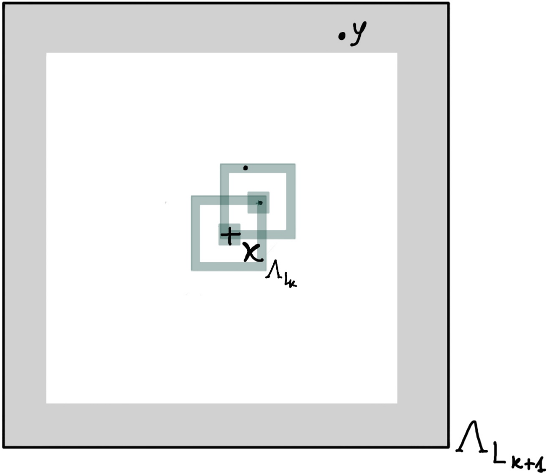

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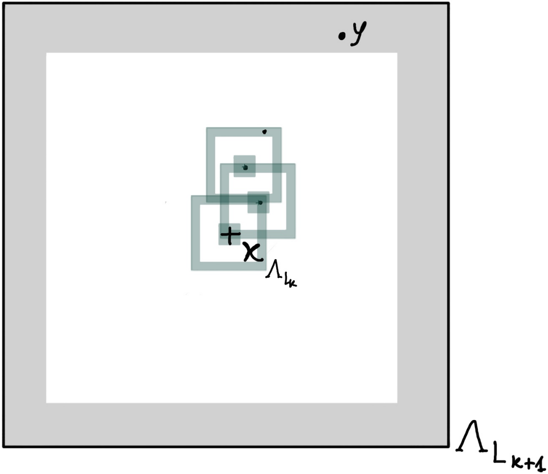

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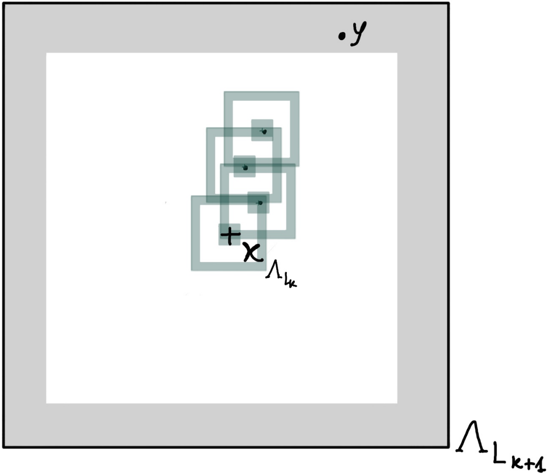

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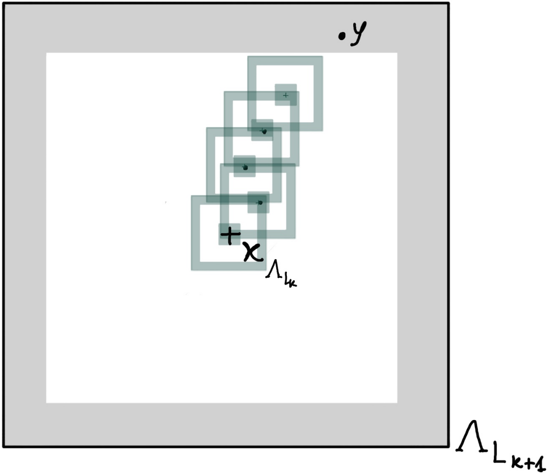

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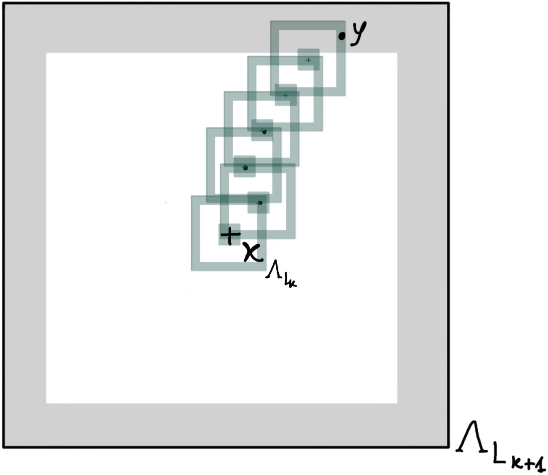

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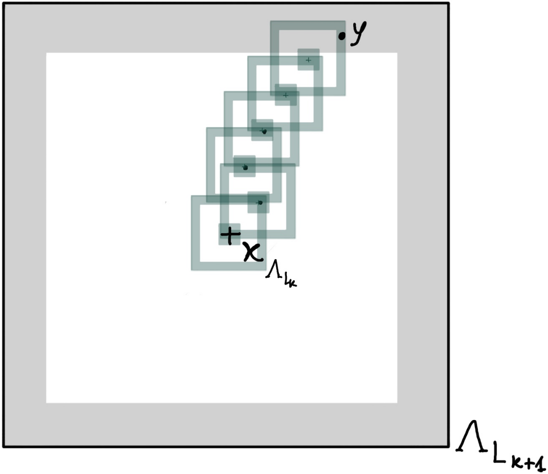
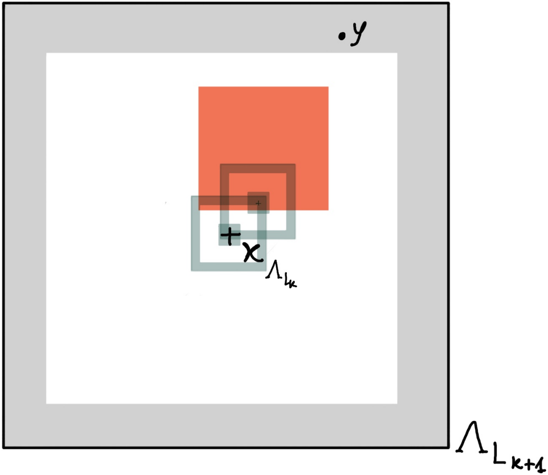
If all boxes Λ_{L_k} are good, then $\Lambda_{L_{k+1}}$ is good.



Sketch of proof

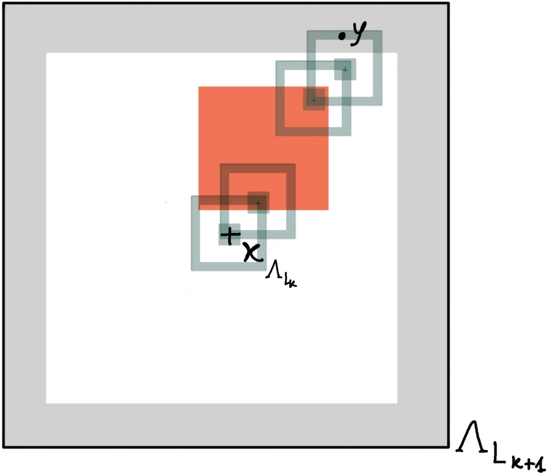
Bad things can happen $\rightarrow$ existence of bad regions



Sketch of proof

Bad things will happen !

Assume the bad region can be covered by a non-resonant box and move along !



Sketch of proof

Deterministic part

If

(A) All except some boxes of size L_k are good

(B) The bad boxes can be covered by a box of size slightly larger than L_k that is non-resonant.

(C) The box of size L_{k+1} is non-resonant.

Then, (D) the box $\Lambda_{L_{k+1}}$ is good.

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Probabilistic part

$$A \cap B \cap C \subset D,$$

so

$$\mathbb{P}(D^c) \leq \mathbb{P}(A^c) + \mathbb{P}(B^c) + \mathbb{P}(C^c)$$

$\mathbb{P}(A^c)$ is given by the hypothesis and independence of the random variables.

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$\mathbb{P}(B^c), \mathbb{P}(C^c)$ bounded using the

Wegner estimate : for $\eta > 0, L \in \mathbb{N}$

$$\mathbb{P}(\mathrm{dist}(\sigma(H_{\omega,L}), E) \geq \eta) \leq C_E \eta L^d$$

□

The Multiscale Analysis

Theorem

Suppose that for E_0 in the a.s. spectrum, there is L_0 and $m_0(L_0)$ such that

$$(\textit{initial step}) \qquad \mathbb{P}(\Lambda_{L_0}(0) \textit{ is } (E, m_0) - \textit{good}) \geq 1 - \frac{1}{L_0^p},$$

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- Usually, the Wegner estimate holds for all energies in the spectrum.
- The initial step holds for either strong disorder ($\|V_\omega\|_\infty >> 1$) or at spectral band edges (E near a spectral gap).

The Machine

"I gave a seminar at Caltech in 1988, I think, on my work with Dreifus on the **multiscale analysis**, which we had just finished. The title of the talk was "Localization without tears" (...).

At the end of the seminar, Barry (Simon) told me that my title was misleading. The Fröhlich-Spencer multiscale analysis had left the feeling of a root canal, but after my talk he still felt like he had been to the dentist."

Abel Klein on Barry Simon's 60th birthday

A challenge for the machine

The Fractional Anderson Model on $\ell^2(\mathbb{Z}^d)$

$$H_{\alpha,\omega} = (-\Delta)^\alpha + V_\omega, \quad \alpha \in (0,1)$$

- The fractional Laplacian is defined via the Spectral Theorem as a function of $-\Delta$, where $-\Delta\varphi(n) = -\sum_{m\sim n}(\varphi(m) - \varphi(n))$.

  [Ciaurri et al. '17, Gebert-RM'20] There exist $c_{d,\alpha}, C_{d,\alpha} > 0$ such that

$$\frac{c_{d,\alpha}}{|n-m|^{d+2\alpha}} \leq |(-\Delta)^\alpha(n,m)| \leq \frac{C_{d,\alpha}}{|n-m|^{d+2\alpha}}, \; n \neq m$$

- $V_\omega\varphi(n) = \omega_n\varphi(n)$, with $\omega := (\omega_n)_{n\in\mathbb{Z}^d}$ iid random variables supported in $[0,M]$, $M > 0$.

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The fractional Anderson model is a *long-range random operator*.

The Fractional Anderson Model on $\ell^2(\mathbb{Z}^d)$

$$H_{\alpha,\omega} = (-\Delta)^\alpha + V_\omega, \quad \alpha \in (0,1)$$

Theorem (Disertori-Maturana-Escobar-RM'23)

For $d \geq 1$ and $\alpha \in (0,1)$, and M large enough (i.e. $\|V_\omega\|_\infty >> 1$), $H_{\omega,\alpha}$ exhibits pp spectrum everywhere in its a.s. spectrum, with polynomially decaying e.f.
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- Conjecture : The fAM does not exhibit dynamical localization.

Discussion : the definition of dynamical localization

Recent work on localization and MSA for long-range operators : Shi '21, Jian-Sun'23, Shi-Wen'23, Sun-Wang '24.

Theorem

Jian-Sun'23 : power-law dynamical localization
$H_\omega = H_0 + V_\omega$, *with*

$$|H_0(x,y)| \leq \frac{1}{|x-y|^\gamma}, \quad \gamma > 1800d$$

There exists p such that

$$\sup_t \left\| |X|^p e^{-itH_\omega} \varphi \right\| < \infty.$$

Why use a strong notion of dynamical localization

• [Germinet-Klein'07] Characterization of localization and delocalization energies using transport exponents.

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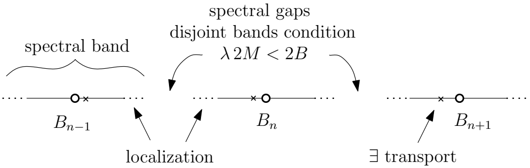

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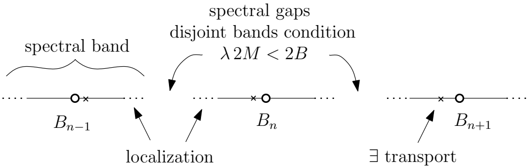

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Question : standard mechanism for metal-insulator transition in generic topological insulators in 2d ?

Thank you for your attention !

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Workshop RAD - Recent Advanced in Disordered systems

April 7-9, 2025

CY Advanced Studies, Campus Neuville, CY Cergy Paris University.

Speakers : Margherita Disertori (Bonn), Alain Joye (Grenoble), Benoit Duçot (Paris), Mostafa Sabri (Abu Dhabi), Houssam Abdul-Rahman (Abu Dhabi), Rudolph Römer (Warwick), Cristopher Cedzich (Düsseldorf), Xiaolin Zeng (Strasbourg).

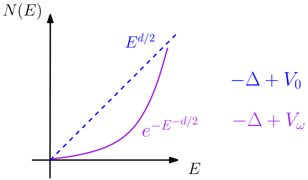
Organizers : Hakim Boumaza and CRM

Info : crojasmo -at- cyu.fr

Lifshitz Tails

The Integrated density of States (IDS) is defined as follows (when it exists)
$H_\omega|_\Lambda$ restriction of H_ω to a cube Λ.

$$\text{(IDS)} \qquad N(E) := \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{\sharp\{\text{eigenvalues of } H_\omega|_\Lambda \leq E\}}{\text{vol}(\Lambda)}$$



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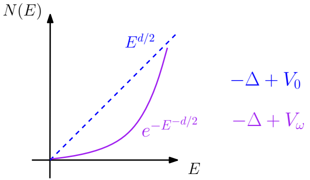

- For the Anderson model, the IDS exists and it is deterministic.
- Asymptotics near infimum of spectrum E_0 : large deviation principle.

$$\text{Lifshitz Tails} \qquad N(E) \sim e^{-E^{-d/2}} \quad E \searrow E_0$$

- This is an ingredient in proofs of localization at spectral band edges.

Just a few names : Pastur, Klopp, Kirsch, Simon, Veselić, Raikov...

Fractional Lifshitz tails : the continuous case

The fractional Anderson model : $H_{\alpha,\omega} = (-\Delta)^{\alpha} + V_{\omega}$

The fractional Laplacian on $L^2(\mathbb{R}^d)$:

$$(-\Delta)^{\alpha} u(x) := C_{d,\alpha} \lim_{r \to 0^+} \int_{\mathbb{R}^d \setminus B_r(x)} \frac{u(x) - u(y)}{|x-y|^{d+2\alpha}} dy$$

Ten equivalent definitions of the fractional Laplacian, M. Kwasnicki'17

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Fractional Anderson model on $L^2(\mathbb{R}^d)$: *fractional Lifshitz tails*

$$N(E) \sim e^{-E^{-d/2\alpha}}, \quad E \searrow E_0$$

- Okura'77 : Case of Poissonian random potential.
- Pietruska-Paluba, Kaleta'19 : Case of Anderson potentials.

Probabilistic proofs based on path integrals in terms of Lévy processes.

Related results for deterministic operators : Carmona- Chen–Masters -Simon'89

Fractional Lifshitz tails : the discrete case, $d \geq 1$

Theorem (Gebert-RM'20)

Fractional Lifshitz Tails : the IDS N exists and

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Proof : Let χ_L be the characteristic function of the cube $\Lambda = [-L, L]^d \cap \mathbb{Z}^d$.
The finite-volume restriction of $H_{\alpha,\omega}$ given by $H_{\alpha,L} = \chi_L H_{\alpha,\omega} \chi_L$. Then

$$N(E) = \lim_{\Lambda} \frac{1}{|\Lambda|} \mathbb{E}(\mathrm{tr}\, \chi_{(-\infty,E]}(H_{\alpha,L}))$$

$x \mapsto x^s$ is operator monotone for $s \in (0,1]$, i.e. A, B s.a. operators with $0 \leq A \leq B$, then $0 \leq A^s \leq B^s$ for $s \in (0,1]$. We obtain the Dirichlet-Neumann bracketing :

$$(-\Delta_\Lambda^N)^\alpha \leq \chi_L (-\Delta)^\alpha \chi_L \leq (-\Delta_\Lambda^D)^\alpha$$

Obtain upper and lower bounds for $N(E)$ as in the case of the Anderson model, we get all relevant bounds with exponents α. □

Corollary (Disertori-Maturana Escobar-RM'24)

For $\alpha \in (0,1)$, let $H_{\alpha,\omega} = (-\Delta)^\alpha + \lambda V_\omega$ on $\ell^2(\mathbb{Z}^d)$, with $(\omega_n)_{n\in\mathbb{Z}^d}$ iid random variables with bounded density. Then, for $\lambda > 0$ large enough,

i. *$H_{\alpha,\omega}$ exhibits pure point spectrum, with polynomially decaying eigenfunctions.*

i.. *If $\alpha \in (\frac{1}{2}, 1)$, for any $\psi_0 \in \ell^2(\mathbb{Z}^d)$ compactly supported,*

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Note that we cannot prove dynamical localization in the usual sense, in particular, we cannot prove that the system is an insulator.

Conjecture : there is no dynamical localization.