Topological states of matter

Sven Bachmann
The University of British Columbia

Lectures given at 'Mathematical Challenges in Quantum Mechanics'

GSSI, February 2025

Thanks!

- Alex Bols [ETH Zurich]
- Wojciech De Roeck [KU Leuven]
- Martin Fraas [UC Davis]
- Yoshiko Ogata [RIMS Kyoto]

Part I.
The sandbox: quantum Hall effect

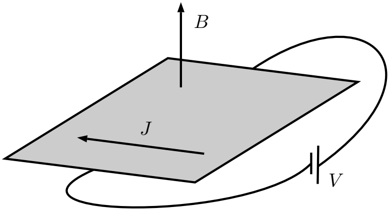
Overview

- ▷ Physics: Off-diagonal response; Integer and fractional QHE
- ▷ Metrology: Definition of the kg
- ▷ The Landau Hamiltonian
- ▷ Charge pumping
- ▷ Index of a pair of projections
- ▷ Topological phases
- ▷ Anyons

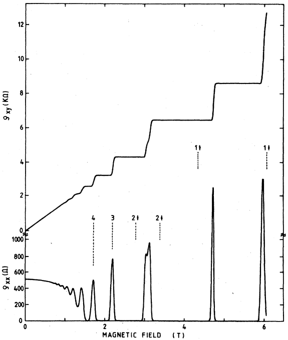
Quantum Hall effect

Example of off-diagonal response

The setting:



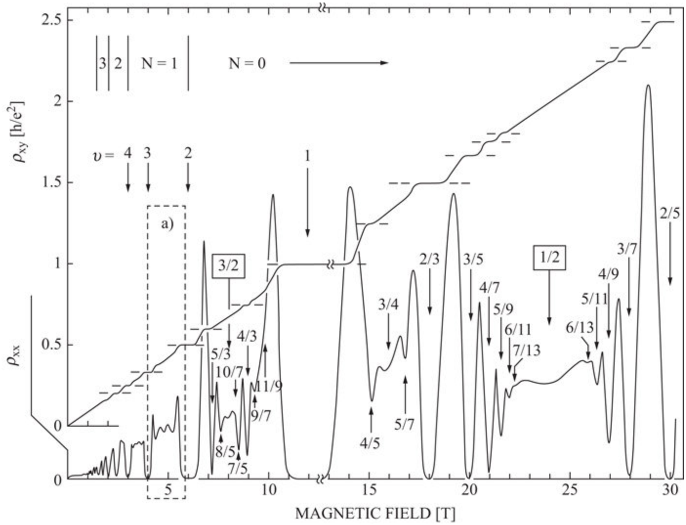
Quantized resistance:



v.Klitzing-Dorda-Pepper (1980)

$$\sigma_{\mathrm{H}} = \frac{1}{\rho_{xy}} = \frac{e^2}{h} n, \qquad n \in \mathbb{Z}$$

Fractional QHE



R.L. Willett, J.P. Eisenstein, H.L. Stormer, D.C. Tsui, A.C. Gossard, and J.H. English, Phys. Rev. Lett. 59, 1776 (1987).

The kilogram – 2018

BIPM CELEBRATING 150 1875 - 2025

SIGN IN | Français

Working together to promote and advance the global comparability of measurements

ABOUT US | 150TH ANNIVERSARY | COORDINATION | LIAISON | TECHNICAL/SCIENTIFIC | PUBLICATIONS & EVENTS

- THE SI
- DEFINING CONSTANTS
- SI BASE UNITS
- - SECOND
- - METRE
- - KILOGRAM
- - AMPERE
- - KELVIN
- - MOLE
- - CANDELA
- SI PREFIXES
- PRACTICAL REALIZATIONS
- SI BROCHURE

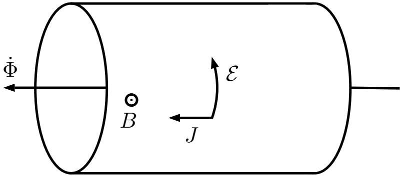
SI base unit: kilogram (kg)

The kilogram, symbol kg, is the SI unit of mass. It is defined by taking the fixed numerical value of the Planck constant h to be 6.626 070 15 x 10^{-34} when expressed in the unit J s, which is equal to kg m^2 s^{-1}, where the metre and the second are defined in terms of c and $\Delta\nu_{Cs}$.

This definition implies the exact relation h = 6.626 070 15 x 10^{-34} kg m^2 s^{-1}. Inverting this relation gives an exact expression for the kilogram in terms of the three defining constants h, $\Delta\nu_{Cs}$ and c:

$$1\ \text{kg} = \frac{h}{6.626\,070\,15 \times 10^{-34}}\ \text{m}^{-2}\ \text{s}$$

Landau Hamiltonian



Faraday's law: Time-dependent Φ yields electromotive force $\mathcal{E}$.

Vector potential in the φ direction:

$$A_\varphi = \text{Constant } B \text{ field} + \text{Flux } \Phi = -By + \frac{\Phi}{2\pi}$$

Hamiltonian:

$$H_V = \frac{1}{2}\left(-\partial_y^2 + (-\mathrm{i}\partial_\varphi - eA_\varphi)^2\right) + V(y)$$

V: boundary conditions

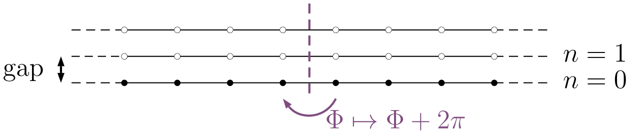
Laughlin's argument

- $\mathrm{Spec}(H_0) = (n + \frac{1}{2})eB,\ n \in \mathbb{N}$, independent of Φ.
- Eigenspaces are infinitely degenerate: 'Landau levels'
- Eigenfunctions are exponentially localized at

$$\text{const}\left(l - e\frac{\Phi}{2\pi}\right) \qquad l \in \mathbb{Z}$$

- Recall: Many-body ground state given by $P = \chi_{(-\infty,\mu]}(H_0)$

Spectral flow $\Phi \mapsto \Phi + \frac{2\pi}{e}$:

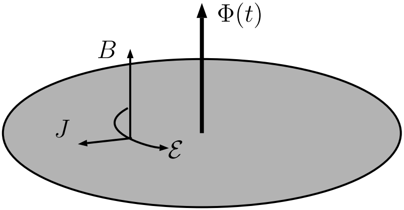


Adiabatic increase $\Phi \mapsto \Phi + \frac{2\pi}{e}$ pumps integer charge if $\mu \in$ gap

Laughlin's argument

In the punctured plane geometry:
Let

$$U = \frac{z}{|z|}$$



Then

$$2\pi\sigma_{\mathrm{H}}(P) = \mathrm{Ind}(P, UPU^*) \in \mathbb{Z}$$

Why U? If

$$H = (-\mathrm{i}\nabla - A)^2 + V,$$

then

$$UHU^* = (-\mathrm{i}\nabla - \nabla\arg(z) - A)^2 + V$$

so U inserts unit flux

Brief history

- ▷ Laughlin 1981: Flux insertion and gauge invariance
- ▷ Halperin 1982: Extended edge states
- ▷ Thouless-Kohomoto-Nightingale-den Nijs 1982: Stokes theorem
- ▷ Avron-Seiler-Simon 1983: Topology of line bundles

- ▷ Fröhlich-Kerler-Marchetti-Studer 1991+: Field theories
- ▷ Avron-Seiler-Simon 1994: Index of Fredholm operators
- ▷ Bellissard-van Elst-Schulz-Baldes 1994: Non-commutative geometry
- ▷ Aizenman-Graf 1997: Anderson localization and Hall plateaux
- ▷ ...

Index of projections

Two self-adjoint projections P, Q on $\mathcal{H}$:

$$C = P - Q, \qquad S = 1 - P - Q$$

Then

$$\text{(i) } CS + SC = 0, \qquad \text{(ii) } C^2 + S^2 = 1$$

Consequence: If $C\psi = \lambda\psi$, then by (i)

$$C(S\psi) = -\lambda(S\psi)$$

and if $S\psi = 0$ by (ii)

$$0 = \langle \psi, S^2\psi \rangle = 1 - \lambda^2$$

Conclusion: (Avron-Seiler-Simon 1994)

$$\begin{aligned} \mathrm{Tr}((P-Q)^{2n+1}) &= \sum_j \lambda_j^{2n+1} = m_1 - m_{-1} \\ &= \dim \mathrm{Ker}(P - Q - 1) - \dim \mathrm{Ker}(P - Q + 1) \in \mathbb{Z} \end{aligned}$$

Fredholm index

In the quantum Hall effect,

$$Q = UPU^*$$

Let

$$T = PUP, \qquad TT^* = PQP, \qquad T^*T = U^*QPQU$$

Hence index of projections is the Fredholm index of $T : P\mathcal{H} \to P\mathcal{H}$:

$$\begin{aligned} \dim \mathrm{Ker}(TT^*) - \dim \mathrm{Ker}(T^*T) &= \mathrm{Tr}(P - PQP) - \mathrm{Tr}(P - U^*QPQU) \\ &= \mathrm{Tr}(P - PQP) - \mathrm{Tr}(Q - QPQ) \\ &= \mathrm{Tr}((P - Q)^3) \end{aligned}$$

since

$$\begin{aligned} (P - Q)^2 P &= (P - Q)(1 - Q)P = P(1 - Q)P \\ -(P - Q)^2 Q &= (P - Q)(1 - P)Q = -Q(1 - P)Q \end{aligned}$$

Topological invariance

The index is constant under 'deformations':

- For any unitary V, let $Q = VPV^*$. Then

$$\mathrm{Ind}(P, UPU^*) = \mathrm{Ind}(Q, UQU^*)$$

- If $U - V$ is compact, then

$$\mathrm{Ind}(P, UPU^*) = \mathrm{Ind}(P, VPV^*)$$

- If U_t is a strongly continuous one-parameter group, then

$$\mathrm{Ind}(P, U_t P U_t^*) = 0$$

for all t

Summary

- Hall conductance = charge transport by flux insertion
- Charge transport = index of pair of Fermi projections

Consequences:

- Hall conductance is an integer
- Topological stability

Question:

Where are the fractions?

Answer requires interactions

An interacting Hamiltonian

Consider the Hamiltonian

$$H_\mu(\lambda) = H^0 + \lambda V - \mu N$$

where

$$H^0 = \sum_{x,y,\sigma,\sigma'} a^*_{x,\sigma} h_{\sigma,\sigma'}(x-y) a_{y,\sigma'} \qquad N = \sum_{x,\sigma} a^*_{x,\sigma} a_{x,\sigma}$$

$$V = \sum_{x,y,\sigma,\sigma'} a^*_{x,\sigma} a_{x,\sigma} v_{\sigma,\sigma'}(x-y) a^*_{y,\sigma'} a_{y,\sigma'}$$

- $a^*_{x,\sigma}$: annihilation operator, where $x \in (\mathbb{Z}/L\mathbb{Z})^2$ and $\sigma \in \{\uparrow, \downarrow\}$
- Decay of kernels:

$$\|h_{\sigma,\sigma'}(x-y)\|, \|v_{\sigma,\sigma'}(x-y)\| \leq \frac{C_N}{(1+|x-y|)^N}$$

for all $N \in \mathbb{N}$

The IQHE phase

Assumption: Spectral gap for $H^0 - \mu N$

Let $\sigma_{\mathrm{H}}(\lambda)$ be the Hall conductivity for $H_\mu(\lambda)$ in the limits

1. Infinite volume: $L \to \infty$
2. Zero temperature: $T \to 0$

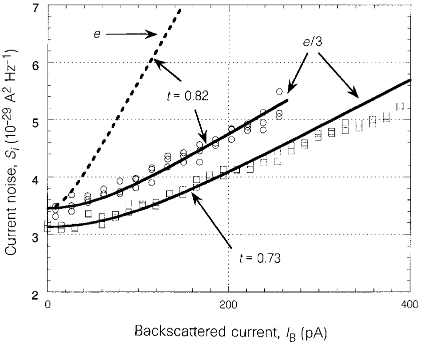
Theorem. [Giuliani-Mastropietro-Porta 2017]
For $|\lambda|$ sufficiently small,

$$\sigma_{\mathrm{H}}(\lambda) = \sigma_{\mathrm{H}}(0)$$

- General result for concrete model
- Gap assumption only for free Hamiltonian
- Methods: Cluster expansion, using Ward identities and analytic continuation
- Consequence: FQHE requires strong interactions

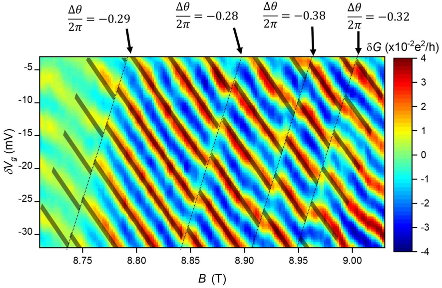
Experiments in FQHE

Fractional charge:



de-Picciotto et al 1997

Anyons:



Nakamura et al 2020

More on that later!

Part II.
Topological phases of matter

Overview

- ▷ Beyond Fermi projections
- ▷ C* algebras
- ▷ Generator of dynamics
- ▷ States
- ▷ Topological phases and topological indices
- ▷ The spectral flow

N-fermions

- Hilbert space $\mathcal{H} = \mathscr{H} \wedge \cdots \wedge \mathscr{H}$
- Hamiltonian

$$H_N = \sum_{i=1}^{N} h_i + \lambda W$$

  for example $h_i = -\frac{1}{2}\Delta_{x_i} + V(x_i)$ on $\mathscr{H} = L^2(\mathbb{R}^d)$
- Non-interacting ($\lambda = 0$) many-body ground state

$$\Phi_N = \psi_1 \wedge \cdots \wedge \psi_N$$

  where ψ_j is eigenvector for the jth lowest eigenvalue
- Equivalently: Fermi projection P_N
- P makes sense in infinite volume, $N \to \infty$ limit, finite density

If $\lambda \neq 0$, no Fermi projection $\rightsquigarrow$ no index!

General setting

- Algebra of observables $\mathcal{A}$: A C*-algebra
  - Banach algebra
  - with (conjugate linear) involution $A \mapsto A^*$
  - such that $\|AA^*\| = \|A\|^2$
- (Heisenberg) dynamics: A strongly continuous 1-parameter group of *-automorphisms τ_t
  - $t \mapsto \tau_t(A)$ is continuous for all $A \in \mathcal{A}$
  - $\tau_t(A+B) = \tau_t(A) + \tau_t(B)$ and $\tau_t(AB) = \tau_t(A)\tau_t(B)$
  - $\tau_t(A^*) = \tau_t(A)^*$
  - $\|\tau_t(A)\| = \|A\|$
- State: A bounded linear functional $\omega : \mathcal{A} \to \mathbb{C}$ such that $\omega(1) = 1$

General setting: Example

- Algebra

$$\mathcal{A} = \mathcal{B}(\mathcal{H})$$

- Dynamics

$$\tau_t(A) = \mathrm{e}^{-\mathrm{i}tH} A \mathrm{e}^{\mathrm{i}tH} \qquad H = H^*$$

- State

$$\omega(A) = \langle \Phi, A\Phi \rangle \qquad \|\Phi\| = 1$$

Algebras: Examples

- $C_0(X)$: Continuous functions vanishing at ∞ on LCHS X
  Abelian algebra $\rightsquigarrow$ classical mechanics
- Theorem. If $\mathcal{A}$ is a finite dimensional C* algebra, then

$$\mathcal{A} \simeq \bigoplus_j M_{n_j}$$

  where M_{n_j} are full matrix algebras
- The set of compact operators $K(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$
- Any norm-closed subalgebra of $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$

Fermions

- $\mathcal{H}$ the one-particle Hilbert space
- Algebra of canonical anticommutation relations $\mathcal{A}(\mathcal{H})$, generated by

$$\{1\} \cup \{a(f) : f \in \mathcal{H}\}$$

with relations

$$\{a(f), a(g)\} = 0, \qquad \{a(f)^*, a(g)^*\} = 0$$
$$\{a(f), a(g)^*\} = \langle f, g \rangle_{\mathcal{H}} \cdot 1$$

Example: $\mathcal{H} = l^2(\mathbb{Z}^d; \mathbb{C}^n)$

Notation: $a_{x,j} = a(\delta_x \otimes e_j)$ and for any $\Lambda \subset \mathbb{Z}^d$,

$\mathcal{A}_\Lambda$: Algebra generated by $\{1\} \cup \{a_{x,j} : x \in \Lambda, j \in \mathbb{N}_n\}$

Quantum spin systems

- For all $x \in \mathbb{Z}^d$: Finite dimensional Hilbert space $\mathcal{H}_x \simeq \mathbb{C}^n$
- For any finite subset $\Lambda \subset \mathbb{Z}^d$:

$$\mathcal{A}_\Lambda = \otimes_{x \in \Lambda} \mathcal{B}(\mathcal{H}_x)$$

- For $\Lambda_1 \subset \Lambda_2$: Injection $\mathcal{A}_{\Lambda_1} \hookrightarrow \mathcal{A}_{\Lambda_2}$ by

$$A \in \mathcal{A}_{\Lambda_1} \to A \otimes 1_{\Lambda_2 \setminus \Lambda_1} \in \mathcal{A}_{\Lambda_2}$$

- Algebra of local observables:

$$\mathcal{A}_{\mathrm{loc}} = \bigcup_{\Lambda} \mathcal{A}_\Lambda$$

- C* algebra

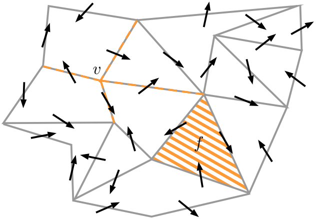
$$\mathcal{A} = \overline{\mathcal{A}_{\mathrm{loc}}}^{\|\cdot\|}$$

The toric code model

Finite volume example:

Toric code Hamiltonian (Bravyi, Kitaev 2003)
here $\mathcal{H}_x = \mathbb{C}^n$

$$H_\Lambda = -\sum_{v:\text{vertex}} A_v - \sum_{f:\text{face}} B_f$$



where

$$A_v = \prod_{x \in v} \sigma_x^1, \qquad B_f = \prod_{x \in f} \sigma_x^3$$

The graph Λ can be embedded on a compact oriented surface with genus g

Theorem. *(i) The ground state space is 4^g-dimensional.*
(ii) The spectral gap is $\gamma = 2$, uniformly in $|\Lambda|$

The infinite volume limit

Infinite graph Γ, with $(\Lambda_n)_{n\in\mathbb{N}}$ such that

$$\Lambda_n \subset \Lambda_m \qquad (n < m)$$
$$\forall x \in \Gamma, \exists n \text{ s.t. } x \in \Lambda_n$$

For any $A \in \mathcal{A}_{\mathrm{loc}}$,

$$\lim_{n\to\infty} \mathrm{i}[H_{\Lambda_n}, A] = \delta(A)$$

since $[H_{\Lambda_n}, A]$ is eventually constant

δ is an unbounded densely defined derivation of $\mathcal{A}$:

$$\delta(AB) = \delta(A)B + A\delta(B)$$

Infinite volume limit

Alternatively,

$$\lim_{n\to\infty} \mathrm{e}^{-\mathrm{i}tH_{\Lambda_n}} A \mathrm{e}^{-\mathrm{i}tH_{\Lambda_n}} = \tau_t(A)$$

exists.

In fact $\tau_t(A)$ is the unique solution of

$$\frac{d}{dt}\tau_t(A) = \tau_t(\delta(A)) \qquad \tau_0(A) = A$$

In general, no finite propagation speed:

$$\tau_t(A) \notin \mathcal{A}_{\mathrm{loc}} \text{ even if } A \in \mathcal{A}_{\mathrm{loc}}$$

Local generators

- $\mathcal{F} = \{f : [0,\infty) \to (0,\infty) : \forall n \in \mathbb{N} \ \sup_r f(r)(1+r)^n < \infty\}$
- $A \in \mathcal{A}$ is almost localized at x if there is $A_n \in \mathcal{A}_{B_n(x)}$ and $f \in \mathcal{F}$

  $$\|A - A_n\| \le \|A\| f(n)$$

  Notation: $\mathcal{A}_{\mathrm{aloc}}$ or $\mathcal{A}_x^f$
- Equivalently:

  $$A \in \mathcal{A}_x^f \quad \Longleftrightarrow \quad \|[A,B]\| \le 2\|A\|\|B\| f(|x-y|)$$

  for all $y \in \Gamma$ and $B \in \mathcal{A}_{\{y\}}$
- 0-chain: Function $h : \Gamma \to \mathcal{A}$ such that $h_x \in \mathcal{A}_x^f$, f uniform in x
- Local generator: Family

  $$H_\Lambda = \sum_{x\in\Lambda} h_x, \qquad \Lambda \subset \Gamma \text{ finite}$$

Lieb-Robinson bound

Theorem. [Lieb-Robinson 1972, Nachtergaele-Sims 2006,...]
The limit

$$\tau_t(A) = \lim_{\Lambda \to \Gamma} \mathrm{e}^{-\mathrm{i}tH_\Lambda} A \mathrm{e}^{\mathrm{i}tH_\Lambda}$$

exists, defines a dynamics on $\mathcal{A}$ and satisfies

$$\tau_t(\mathcal{A}_{\mathrm{aloc}}) \subset \mathcal{A}_{\mathrm{aloc}}$$

LGA: locally generated dynamics

Representations

Question: Can $\mathcal{A}$ be represented in Hilbert space?

Theorem. [Gelfand-Naimark 1943, Segal 1947,...]
Let ω be a state on $\mathcal{A}$. Then there is a Hilbert space $\mathcal{H}_\omega$, a representation $\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$ and a unit vector Ω_ω such that

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle$$

The representation is cyclic: $\pi_\omega(\mathcal{A})\Omega_\omega$ is dense in $\mathcal{H}_\omega$.

Representation:

$$\pi(AB) = \pi(A)\pi(B)$$
$$\pi(A + \lambda B) = \pi(A) + \lambda\pi(B)$$
$$\pi(A^*) = \pi(A)^*$$

Equivalence of representations

Two reps π_j on $\mathcal{H}_j$ are unitarily equivalent if

$$U\pi_1(A)U^* = \pi_2(A)$$

where $U : \mathcal{H}_1 \to \mathcal{H}_2$ is unitary

GNS is unique up to unitary equivalence: Given $(\mathcal{H}, \pi, \Omega), (\mathcal{H}'\pi', \Omega')$,

$$U\pi(A)\Omega = \pi'(A)\Omega'$$

Indeed:

$$\pi'(B)U\pi(A)\Omega = \pi'(BA)\Omega' = U\pi(BA)\Omega = U\pi(B)\pi(A)\Omega$$

implies

$$\pi'(B)U = U\pi(B)$$

Moreover,

$$\langle U\pi(B)\Omega, U\pi(A)\Omega\rangle = \langle\pi'(B)\Omega', \pi'(A)\Omega'\rangle = \omega(B^*A) = \langle\pi(B)\Omega, \pi(A)\Omega\rangle$$

so is an isometry.

(In)equivalence of representations

For quantum spin systems:

Theorem.
The GNS reps of ω_1, ω_2 *are equivalent iff for* $\epsilon > 0$*, there is* $r > 0$ *such that*

$$|\omega_1(A) - \omega_2(A)| < \epsilon \|A\|$$

for all $A \in \mathcal{A}_{B_r(0)^c}$

$\rightsquigarrow$ Two states are equivalent iff they are almost equal at infinity / thermodynamically equivalent / local perturbations of each other

Implementability

- A dynamics is implementable in a rep if

$$\pi(\tau_t(A)) = U_t^* \pi(A) U_t$$

- In general, not the case (orthogonality catastrophe): the states ω and $\omega \circ \tau_t$ differ at infinity
- Proposition. If $\omega = \omega \circ \tau_t$, then τ_t is implementable. Indeed, the representations

$$(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega) \text{ of } \omega \text{ and } (\mathcal{H}_\omega, \pi_\omega \circ \tau_t, \Omega_\omega) \text{ of } \omega \circ \tau_t$$

are unitarily equivalent and

$$U_t \Omega_\omega = \Omega_\omega$$

- By Stone's theorem $U_t = \mathrm{e}^{\mathrm{i}tH_\omega}$ with $H_\omega \Omega = 0$

What is a phase?

- Physicist's answer: Two states are in the same phase if one can be smoothly deformed into the other without crossing a phase transition.
- Gapped ground states: $|\psi_i\rangle$ ground state of H_i with spectral gap g_i above GS energy E_i:
  In the same phase if there is

  $$[0,1] \ni s \mapsto H(s) = H(s)^* \qquad H(0) = H_1, H(1) = H_2$$

  and $H(s)$ is uniformly gapped
  Note: The family $|\psi(s)\rangle$ of ground states is interpolating family

Gapped ground state

- ω is a ground state of τ_t if

$$-\mathrm{i}\omega(A^*\delta(A)) \geq 0$$

In GNS:

$$\langle \psi, H_\omega \psi \rangle \geq 0 \qquad \langle \Omega_\omega, H_\omega \Omega_\omega \rangle = 0$$

- ω is a gapped ground state if

$$-\mathrm{i}\omega(A^*\delta(A)) \geq g\omega(A^*A) \qquad \omega(A) = 0$$

In GNS:

$$\inf_{\psi \perp \Omega_\omega} \langle \psi, H_\omega \psi \rangle \geq g \langle \psi, \psi \rangle$$

Finite volume limits

Consider a family of vectors ψ_Λ that are ground states of H_Λ with a spectral gap g_Λ. Then

- The family of states $\langle\psi_\Lambda, (\cdot)\psi_\Lambda\rangle$ is compact
- Any limiting point ω is an algebraic ground state
- Gaps can only open :
  Theorem. [B-Dybalsky-Naaijkens 2016]
  Let E *be such that* $(E-\epsilon, E+\epsilon)\cap \mathrm{Spec}(H_\Lambda - E_\Lambda)$ *for all finite* Λ*, then* $E \notin \mathrm{Spec}(H_\omega)$
- Analogies
  math: strong resolvent convergence
  physics: edge states disappear

Gapped ground state phase

Definition. [Hastings, Chen-Gu-Wen,...]

$\omega_i, i = 0, 1$ gapped ground states of $\tau_t^{(i)}$ are in the same phase if

- $\tau_t^{(i)}$ are LGAs with generators $h^{(i)}$
- there is a smooth family of 0-chains $h(s)$ such that $\tau_t^{h(s)}$ has a gapped ground state $\omega^{(s)}$

Theorem. [Hastings-Wen 2005, B-Michalakis-Nachtergaele-Sims 2010, Moon-Ogata 2020]

There is an LGA α_s such that

$$\omega_1 = \omega_0 \circ \alpha_1$$

Vocabulary: ‘quasi-adiabatic continuation’, or ‘spectral flow’, or ‘automorphic equivalence’

Spectral flow via finite volume limit

H has a gap g above ground state energy, P is ground state projection

Consider an odd function $W \in L^1(\mathbb{R};\mathbb{R}) \cap L^\infty(\mathbb{R};\mathbb{R})$ such that

$$\begin{aligned} &\text{i.} \quad \widehat{W}(\xi) = -\mathrm{i}\xi^{-1} \qquad (|\xi| > \mathrm{gap}) \\ &\text{ii.} \quad W(t) = \mathcal{O}(t^{-\infty}) \qquad (|t| \to \infty) \end{aligned}$$

The linear map on $\mathcal{A}$

$$A \mapsto \mathcal{I}(A) := \widehat{W}(-\mathrm{ad}_H)(A) = \int_{-\infty}^{\infty} W(t) \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH} dt$$

is the inverse of $\mathrm{i}[H,\cdot]$ on off-diagonal operators

$$A = \mathcal{I}(\mathrm{i}[H,A])$$

whenever $A = PAP^\perp + P^\perp AP$

Spectral flow via finite volume

- Recall $\mathcal{I}(A) = \int W(t)\tau_t(A)dt$
- Check:

$$\int_{-\infty}^{\infty} W(t)\mathrm{e}^{\mathrm{i}tH}\mathrm{i}[H,A]\mathrm{e}^{-\mathrm{i}tH}dt = \int W(t)\mathrm{e}^{\mathrm{i}t(\lambda-\mu)}\mathrm{i}(\lambda-\mu)dP_\lambda A dP_\mu dt$$
$$= \int_{|\lambda-\mu|>g} \widehat{W}(\mu-\lambda)\mathrm{i}(\lambda-\mu)dP_\lambda A dP_\mu$$
$$= \int_{|\lambda-\mu|>g} dP_\lambda A dP_\mu = A$$

- $A \in \mathcal{A}_{\mathrm{aloc}}$ implies $\tau_t(A) \in \mathcal{A}_{\mathrm{aloc}}$ and since $|W| \in \mathcal{F}$,

$$\mathcal{I}(\mathcal{A}_{\mathrm{aloc}}) \subset \mathcal{A}_{\mathrm{aloc}}$$

- Extends to map on 0-chains:

$$\mathcal{I}(h)_x := \mathcal{I}(h_x)$$

Properties

- For any operator A, the operator

$$\overline{A} := A - \mathcal{I}(\mathrm{i}[H, A])$$

is block diagonal:

$$[\overline{A}, P] = 0$$

- Parallel transport: For family $H(s)$:

$$\mathrm{i}[\mathcal{I}(\dot{H}), P] = \mathrm{i}\mathcal{I}([\dot{H}, P]) = -\mathcal{I}(\mathrm{i}[H, \dot{P}]) = \dot{P}$$

since $P^2 = P$ implies $\dot{P}$ is off-diagonal
- Conclusion: The 0-chain $k = \mathcal{I}\left(\dot{h}\right)$ generates the spectral flow

$$s \mapsto \omega_s = \omega \circ \alpha_s$$

- Holds for limits of finite volume ground states, also for algebraic ground states

Remarks

- The generator $\mathcal{I}(\dot{H})$ replaces the standard Kato generator

$$\mathrm{i}[\dot{P}, P]$$

and it is explicitly local
- The map $\mathcal{I}$ is extremely useful:
  - For gapped systems: Proof of the adiabatic theorem [B-De Roeck-Fraas 2018, Monaco-Teufel 2019]
  - For perturbation closing the gap: Construction of non-equilibrium steady state (NEASS) [Teufel 2019,...]
  - In both settings: Validity of linear response
- Variations on the theme yield exponential clustering for gapped systems: [Hastings-Koma 2006, Nachtergaele-Sims 2006, B-Bols-De Roeck-Fraas 2021]

$$|\langle \psi, AB\psi \rangle - \langle \psi, APB\psi \rangle| \leq C(A,B) f(r) \qquad \psi = P\psi$$

where $f \in \mathcal{F}$ and r is the distance between the supports of A, B

What is a quantum phase?

- Proving that τ_t is gapped is notoriously difficult, so:
- Definition. Two pure states ω, ν are equivalent if

$$\nu = \omega \circ \alpha_1$$

  where α_s is an LGA
- (Gapped ground state phases) are equivalence classes of states
  In other words: Connected components of the state space
- Short Range Entangled states: equivalence class of the product states
  ω_0 is a product state if

$$\omega_0(A_X B_Y) = \omega_0(A_X)\omega_0(B_Y) \qquad X \cap Y = \emptyset$$

Stabilization

- Make definition independent of the 'ambient space':
  Definition. Two pure states ω_1, ω_2 on algebras $\mathcal{A}_1, \mathcal{A}_2$ are stably equivalent if there are product states ν_1, ν_2 on algebras $\mathcal{A}'_1, \mathcal{A}'_2$ such that

$$\omega_1 \otimes \nu_1 \sim \omega_2 \otimes \nu_2$$

  on $\mathcal{A}_1 \otimes \mathcal{A}'_1 \simeq \mathcal{A}_2 \otimes \mathcal{A}'_2$.
  Notation: $\omega_1 \overset{\mathrm{s}}{\sim} \omega_2$
- A state is stably SRE if it is stably equivalent to a product state.

Remark: The set of stable equivalence classes is a monoid, with the class of the product state ω_0 being the identity

Natural question: Are there 'invertible elements'?

- Definition. A state ω is invertible if there is a state $\bar{\omega}$ such that

$$\omega \otimes \bar{\omega} \sim \omega_0$$

Free fermions

Back to the CAR algebra $\mathcal{A}(\mathcal{H})$

A pure gauge invariant quasi-free state on $\mathcal{A}(\mathcal{H})$ is defined by

$$\omega_P(a^*(f_n)\dots a^*(f_1)a(g_1)\dots a(g_m)) = \delta_{n,m}\det\left(\langle g_i, Pf_j\rangle_{i,j=1}^n\right),$$

where $P = P^* = P^2$ (physically: the Fermi projection)

Since $\mathcal{A}(\mathcal{H}_1 \oplus \mathcal{H}_2) = \mathcal{A}(\mathcal{H}_1) \wedge \mathcal{A}(\mathcal{H}_2)$, we have the stacking property

$$\omega_{P_1} \otimes \omega_{P_2} = \omega_{P_1 \oplus P_2}$$

Proposition. [B-Bols-Rahnama 2024]

If P is translation invariant, then $\sigma_{\mathrm{H}}(P) = 0$ implies ω_P is stably SRE.

Furthermore: Any quasi-free state is invertible and $\sigma_{\mathrm{H}}(P) \in \mathbb{Z}$, see [Kapustin-Sopenko 2020]

Some questions

- ▷ Stably SRE states are trivial. Role of local symmetries?
  $\rightsquigarrow$ Symmetry Protected Topological (SPT) phases
- ▷ Difference between stably SRE and invertible?
- ▷ Classification of non-invertible phases?
- ▷ Construction of indices of gapped phases?
- ▷ Higher 'homotopy groups', classification of cycles?

Also:

- ▷ Relationship to TQFTs?
- ▷ Proof of gaps?

Part III.
Some results on quantum Hall

Overview of existing rigorous results

For lattice systems:

- One dimension: Classification of SPT phases
- One dimension: Classification of SPT pumps
- One dimension: Parametrized phases without symmetry
- Two dimensions: Quantization of Hall conductance
- Two dimensions: Solvable models of intrinsic topological orders
- Superselection sectors, anyons and the quantum Hall effect
- Generalizations of 'Hall conductance' for general groups and dimensions
- ...

Charge conserving Hamiltonian

Finite setting: A discrete torus $\Lambda = (\mathbb{Z}/L\mathbb{Z})^2$

Finite-range Hamiltonian

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X) \qquad \text{in} \quad \mathcal{A}(l^2(\Lambda))$$

with

$$\Phi(X) = \Phi(X)^\dagger \qquad \Phi(X) = 0 \text{ if } \mathrm{diam}(X) > R$$

Assumption: Charge conservation ($\mathrm{U}(1)$-symmetry)

$$[\Phi(X), Q_\Lambda] = 0 \qquad Q_\Lambda = \sum_{x \in \Lambda} a_x^* a_x$$

Example

$$H = \sum_{d(x,y)=1} \big(t(x,y)(a_x^* a_y + a_y^* a_x) + U(x,y) q_x q_y \big) + \sum_x V(x) q_x$$

Gapped ground state space

Spectral assumptions: uniformly in volume $|\Lambda|$

- Spectral gap: $E_1 - E_0 \geq g > 0$, where E_0 is the ground state energy
- $\mathrm{Rank}(P) = p$, where P is the ground state projection

Invariant subspace: $[Q_\Lambda, H] = 0$ implies

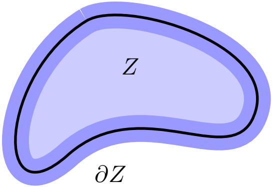
$$Q_\Lambda P = P Q_\Lambda P$$

But not for any $Z \subsetneq \Lambda$: charge fluctuations across ∂Z.

Charge fluctuations

Charge conservation implies

$$\mathrm{supp}([H, Q_Z]) \text{ supported along } \partial Z$$



Then,

- $K_{\partial Z} = \mathcal{I}([H, Q_Z]) \in \mathcal{A}_{\mathrm{aloc}}(\partial Z)$
- $K_{\partial Z}$ describes charge fluctuations: if $\overline{Q}_Z = Q - K_{\partial Z}$, then

$$[\overline{Q}_Z, P] = 0 \quad \text{namely} \quad \overline{Q}_Z P = P \overline{Q}_Z P$$

Loop operators:

$$\begin{aligned} \mathrm{e}^{-2\pi \mathrm{i} \overline{Q}_Z} &= \mathrm{e}^{-2\pi \mathrm{i}(Q_{\partial Z} - K_{\partial Z})} \mathrm{e}^{-2\pi \mathrm{i}(Q_Z - Q_{\partial Z})} \\ &= \mathrm{e}^{-2\pi \mathrm{i}(Q_{\partial Z} - K_{\partial Z})} \in \mathcal{A}_{\mathrm{aloc}}(\partial Z) \end{aligned}$$

since $\mathrm{Spec}(Q_Z - Q_{\partial Z}) \subset \mathbb{Z}$.

Loops and boundaries

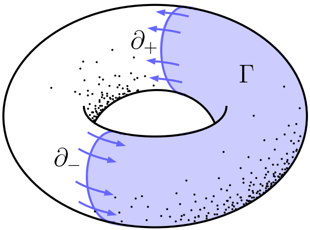
$$\mathrm{e}^{-2\pi \mathrm{i}\overline{Q}_Z} P = P \mathrm{e}^{-2\pi \mathrm{i}\overline{Q}_Z} P$$

Now Λ is the torus, and $Q = Q_\Lambda$

$$V = \mathrm{e}^{-2\pi \mathrm{i}\overline{Q}} = \mathrm{e}^{-2\pi \mathrm{i}\overline{Q}_-} \mathrm{e}^{-2\pi \mathrm{i}\overline{Q}_+} = V_- V_+$$

Gap assumption implies exponential clustering:

$$P V_- V_+ P = P V_- P V_+ P$$



$V_\pm$ are 'Wilson loops':

$$[P, V_\pm] = 0$$

but $[P, \overline{Q}_-] \neq 0$.
Importantly $\partial_- \cup \partial_+ = \partial\Gamma$, but ∂_- is not a boundary

Loops and boundaries

Assume topological order

$$P\overline{Q}_Z P \propto P$$

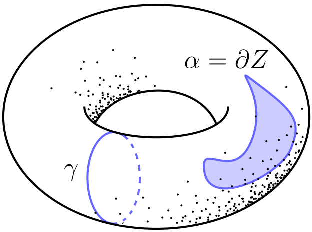
then

- If α is a boundary

$$V_\alpha P = P \mathrm{e}^{2\pi \mathrm{i} \overline{Q}_Z} P = \mathrm{e}^{2\pi \mathrm{i} P\overline{Q}_Z P} \propto P$$

  V_α acts trivially on $\mathrm{Ran} P$

- If γ is not a boundary,
  V_γ is a nontrivial unitary on $\mathrm{Ran} P$

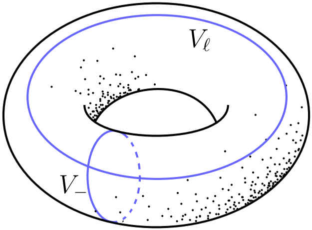


Algebra of loops

Theorem. [B-Bols-De Roeck-Fraas 2020]
For any gapped system with a $\mathrm{U}(1)$ *charge and a topologically ordered ground state space,*

$$V_\ell^* V_- V_\ell V_-^* P = \mathrm{e}^{2\pi\mathrm{i}\frac{q}{p}} P$$

where $q \in \mathbb{Z}$ *and* $p = \mathrm{Rank}(P)$.



Rational rotation algebra

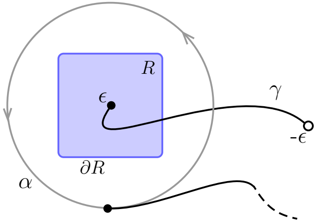
Proof by analyzing

$$\lambda \mapsto \det\left(\left(\mathrm{e}^{\mathrm{i}\lambda(V_\ell^*\overline{Q}V_\ell)_-}\mathrm{e}^{-\mathrm{i}\lambda\overline{Q}_-}\right)P\right)$$

Fractionalization & braiding

For an open path γ

$$\varphi = V_\gamma \Omega \qquad (\Omega = P\Omega)$$



is a state of a pair of excitations
Independent on γ since

$$V_\gamma V_{-\gamma'} \Omega = V_{\gamma - \gamma'} \Omega = \Omega$$

whenever γ, γ' have the same endpoints

- Fractional charge (Laughlin, Saminadayar, Reznikov,...)

$$\langle \varphi, Q_R \varphi \rangle - \langle \Omega, Q_R \Omega \rangle = \langle \Omega, (V_\gamma^* Q_R V_\gamma - Q_R)\Omega \rangle = \frac{q}{p}$$

- Braiding, anyons (..., Wen, Fröhlich-Kerler,...)

$$V_\alpha \varphi = V_\gamma (V_\gamma^* V_\alpha V_\gamma V_\alpha^*)\Omega = \mathrm{e}^{2\pi \mathrm{i} \frac{q}{p}} \varphi$$

Quantized transport

More can be said:

$$\frac{d}{d\lambda} \mathrm{e}^{\mathrm{i}\lambda (V_\ell^* \overline{Q} V_\ell)_-} \mathrm{e}^{-\mathrm{i}\lambda \overline{Q}_-} P \Big|_{\lambda=0} = \mathrm{Tr}((V_\ell^* \overline{Q} V_\ell - \overline{Q})_- P)$$
$$= \mathrm{Tr}((V_\ell^* Q V_\ell - Q)_- P)$$

and $(V_\ell^* Q V_\ell - Q)_-$ is the operator of charge transported across the fiducial line ∂_-.

The theorem shows quantized charge transport:

$$p^{-1} \mathrm{Tr}((V_\ell^* Q V_\ell - Q)_- P) = \frac{q}{p} \in \mathbb{Q}$$

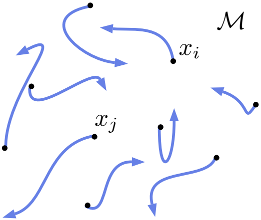
Note V_ℓ is finite volume, many-body analog of $U = \frac{z}{|z|}$

Abelian anyons

- Configuration space of N point particles on manifold $\mathcal{M}$

$$\Gamma_N = \mathcal{M}^N \setminus \{x_i = x_j, 1 \le i \ne j \le N\}$$

Non-trivial topology in low dimensions



- Quantum mechanical pure state: one-dimensional projector

$$P_\psi = P_\psi^* = P_\psi^2 \qquad \psi \in L^2(\Gamma_N)$$

Of course:

$$P_{e^{i\theta}\psi} = P_\psi \quad \text{for all} \quad \theta \in [0, 2\pi]$$

- As particles move around: Path $s \mapsto \gamma_s \in \Gamma_N$ and corresponding

$$s \longmapsto P_s = P_{\psi_s}$$

Loops in configuration space

If γ is a loop

$$\gamma_0 = \gamma_1$$

then

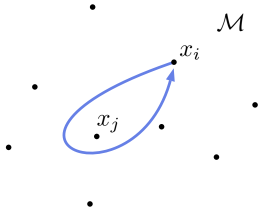
$$P_{\psi_0} = P_{\psi_1}$$

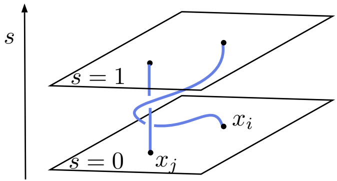
Holonomy

$$\psi_1 = \mathrm{e}^{\mathrm{i}\eta_\gamma}\psi_0 \qquad \eta_\gamma \in [0, 2\pi)$$

and η_γ depends only on the homotopy class of γ.

Typical non-trivial loop in 2d





Anyons and the braid group

We have a one-dimensional representation

$$\eta : \pi_1(\Gamma_N) \to \mathrm{U}(1) \qquad \gamma \mapsto \eta_\gamma$$

In this context: $\pi_1(\Gamma_N)$ is called the coloured braid group

Anyons carry a non-trivial representation

- Theoretical possibility:
  Leinaas-Myrheim 1977, Goldin-Menikoff-Sharp 1981, Wilczek 1982
- Concrete wavefunctions (quantum Hall effect):
  Laughlin 1988
- Specific lattice models:
  Kitaev 2003, Lewin-Wen 2005

Today: General and explicit construction of anyonic quasi-particles

Finite volume: Summary

Assumptions:

- Uniform spectral gap
- Finite ground state degeneracy p
- Local topological order

Consequences:

- 'Flux insertion' cycles through ground states
- σ_{H} is well-defined and $\sigma_{\mathrm{H}} = \frac{p}{q}$
- Quasi-particle excitations are anyons

Infinite volume

- ▷ Setting for QHE, anyons, in infinite volume?
- ▷ What replaces spectral gap, ground state degeneracy and topological order?

$\rightsquigarrow$ Superselection sectors

Superselection sectors: idea

Goal: to classify 'particle states'

- To be defined with respect to a reference vacuum state
- Up to unitary equivalence
- The particle is localized

Examples:

- Gauss' law in electrodynamics: The presence of a charge can be detected at infinity
- The anyons of QHE: An infinite string γ in $V_\gamma \Omega$ can be detected at infinity

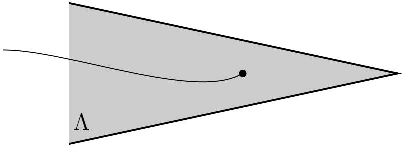
See

- QFT fundamentals: Doplicher-Haag-Robert 1971 and Buchholz-Fredenhagen 1982
- Non-relativistic lattice systems: Naaijkens 2011, Ogata 2022,...

DHR criterion

- Pick a representation π_0 is $\mathcal{A}$
  Concretely: the GNS of a ground state ω_0
- Classify representations π that satisfy the superselection criterion:

$$\forall \text{ cone } \Lambda : \quad \pi \upharpoonright \mathcal{A}_{\Lambda^c} \simeq \pi_0 \upharpoonright \mathcal{A}_{\Lambda^c}$$



- The particle is localized at the apex, the string is not observable
- The particle can be moved by the action of a local observable (in $\pi(\mathcal{A})$), not removed

Example of toric code

Recall

$$H = \sum_{v:\text{vertex}} (1 - A_v) + \sum_{f:\text{face}} (1 - B_f)$$

where

$$A_v = \prod_{x \in v} \sigma_x^1, \qquad B_f = \prod_{x \in f} \sigma_x^3$$

Since

$$[A_v, B_f] = 0$$

so finite volume ground states characterized by

$$A_v \Omega = 0 \qquad B_f \Omega = 0 \qquad \text{for all } v, f$$

Elementary excitations:

$$V_\gamma \Omega, \qquad V_\gamma = \prod_{x \in \gamma} \sigma_x^1$$

Example of toric code

Theorem. [Alicki-Fannes-Horodecki 2007, Fiedler-Naaijkens 2015]
On $\mathbb{Z}^2$, there is a unique gapped ground state ω_0 which is frustration-free, namely

$$\omega_0(A_v) = \omega_0(B_f) = 1$$

For γ extending to infinity and γ_n its truncation to the first n sites, define

$$\rho_\gamma(A) = \lim_{n\to\infty} V_\gamma^* A V_\gamma$$

Proposition. *The representations*

$$\pi_0 \circ \rho_\gamma$$

satisfy the superselection criterion.

In fact, they are the GNS reps of the algebraic ground states $\omega_0 \circ \rho_\gamma$

Sketch of proof

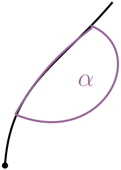
- For $\alpha = \partial X$,

$$V_\alpha = \prod_{v \in X} A_v$$

  hence

$$\pi_0(V_\alpha)\Omega = \Omega$$

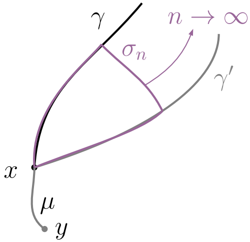
  $\rightsquigarrow \omega_0 \circ \rho_\gamma$ depends only on γ only through initial point



- Transporters:
  - Move initial point $x \to y$ by $\pi_0(\prod_{x\in\mu} \sigma^1_x)$
  - Move infinite string by

$$V_{\gamma\to\gamma'} = \mathrm{w-}\lim_{n\to\infty} \pi_0\left(\prod_{x\in\sigma_n} V_{\sigma_n}\right)$$

  The limit exists and is unitary



Sketch of proof

Let ρ_γ be given.
Pick a cone Λ' and a path $\gamma' \in \Lambda'$:

$$V_{\gamma\to\gamma'}\big(\pi_0 \circ \rho_\gamma\big)V^*_{\gamma\to\gamma'} = \pi_0 \circ \rho_{\gamma'}$$

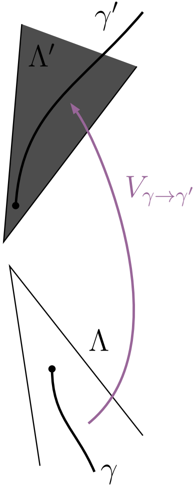
and because $\gamma' \in \Lambda'$

$$\rho_{\gamma'} \restriction \mathcal{A}_{\Lambda'^c} = \mathrm{id} \restriction \mathcal{A}_{\Lambda'^c}$$

hence

$$\pi_0 \circ \rho_\gamma \restriction \mathcal{A}_{\Lambda^c} \simeq \pi_0 \restriction \mathcal{A}_{\Lambda^c}$$

Meaning of superselection criterion:
The particle and its string can be hidden in any cone



Construction generalized to Kitaev's 'quantum double models' [Bols-Vadnerkar 2024]

Back to QHE

$\Gamma \subset \mathbb{Z}^2$: Upper half plane

$$\beta^{\Gamma}_{\phi}(A) = \lim_{\Lambda \to \Gamma} \mathrm{e}^{\mathrm{i}\phi \bar{Q}_\Lambda} A \mathrm{e}^{-\mathrm{i}\phi \bar{Q}_\Lambda} \qquad \bar{Q}_\Lambda = Q_\Lambda - K_\Lambda$$

Since $K_\Lambda \in \mathcal{A}_{\mathrm{aloc}}(\partial\Lambda)$ and $\mathrm{Spec}(Q_\Lambda) \subset \mathbb{N}$,

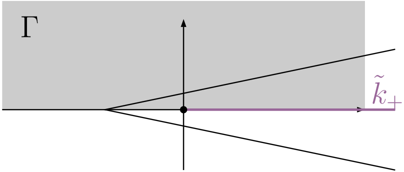
$$\beta^{\Gamma}_{2\pi}(A) = \tau^{\tilde{k}}_{2\pi}(A)$$

where $\tilde{k}$ is a 0 chain that is supported along the line $\partial\Lambda$

Define

$$\rho = \tau^{\tilde{k}_+}_{2\pi}$$

where $(\tilde{k}_+)_x = \tilde{k}_+ \chi_{x_1 > 0}$



Exercise: Construct $\tilde{k}$

Since $\tau_{2\pi}^{q^\Gamma} = \mathrm{id}$,

$$\beta_{2\pi}^{\Gamma} = \beta_{2\pi}^{\Gamma} \circ (\tau_{2\pi}^{q^\Gamma})^{-1} = \mathrm{id} + \int_0^{2\pi} \partial_\phi \left(\beta_\phi^{\Gamma} \circ (\tau_\phi^{q^\Gamma})^{-1} \right) d\phi.$$

Now

$$\partial_\phi \left(\beta_\phi^{\Gamma} \circ (\tau_\phi^{q^\Gamma})^{-1} \right) = \beta_\phi^{\Gamma} \circ \left(\delta_{\bar{q}^\Gamma} - \delta_{q^\Gamma} \right) \circ (\tau_\phi^{q^\Gamma})^{-1}.$$

With $\bar{q}^\Gamma - q^\Gamma = -k^\Gamma$:

$$\partial_\phi \left(\beta_\phi^{\Gamma} \circ (\tau_\phi^{q^\Gamma})^{-1} \right) = - \left(\beta_\phi^{\Gamma} \circ (\tau_\phi^{q^\Gamma})^{-1} \right) \circ \tau_\phi^{q^\Gamma} \circ \delta_{k^\Gamma} \circ (\tau_\phi^{q^\Gamma})^{-1},$$

so we pick

$$\tilde{k}(\phi) = -\tau_\phi^{q^\Gamma}(k^\Gamma)$$

QHE superselection sectors

- Start with QH Hamiltonian δ and assume the existence of gapped ground state ω

$$\frac{\omega(A^*\delta(A))}{\omega(A^*A)} \geq g > 0 \qquad \omega(A) = 0$$

- Lemma. $\pi_\omega \circ \rho$ satisfies the superselection criterion
- Also: One can define $\theta(\rho) \in U(1)$ corresponding physically to the braiding holonomy

QHE

Theorem. [B-Corbelli-Fraas-Ogata 2024]

- $\theta(\rho) = \mathrm{e}^{-\mathrm{i}(2\pi)^2\sigma_{\mathrm{H}}}$
- If there are p equivalence classes of representations satisfying the superselection criterion, then

$$2\pi\sigma_{\mathrm{H}} = \frac{q}{p'}$$

  for some $p' \leq p$.

See also Kapustin-Sopenko 2020

In a TQFT setting: Fröhlich-Kerler 1991, Fröhlich-Studer 1993,...

Concluding summary

- ▷ IQHE well understood, non-interacting fermions
- ▷ FQHE mathematically poorly understood, but progress
- ▷ More general question: ‘topological’ phases and their classification
- ▷ σ_{H} is a ‘topological invariant’;
  other ‘topological indices’ & their physical meaning
- ▷ Emergence of anyons, fractional charges