Analytical Transport of Ultra High Energy Protons and their Secondary Products

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Transport equation

Cosmological transport equation:

$$\frac{\partial n(E,t)}{\partial t} + 3H(t)n(E,t) - \frac{\partial}{\partial E}\left[n(E,t)b(E,t)\right] = \frac{\widetilde{Q}(E,t)}{a^3(t)}$$

- $b(E,t) = -\frac{dE}{dt}$: energy losses equation (adiabatic + interactions).
- $\widetilde{Q} \propto \widetilde{\mathcal{L}} \times q(E)$: injection term per unit comoving volume from astrophysical CR accelerators.
- $\widetilde{Q} \propto \widetilde{n}_p \times R_{p\gamma_t}$: injection term for secondary products of proton interactions with background target photons

From protons to secondary products

$$\widetilde{Q}^{\text{astro}}_{\text{proton}}(E_p, z) \rightarrow \widetilde{n}_p(E_p, z) \rightarrow \begin{cases} \widetilde{Q}_{\gamma\text{-rays}}(E_\gamma^{\text{VHE}}, z) \rightarrow \phi_\gamma^{\text{diff}}(E_\gamma) & E_\gamma \approx 0.1\text{-}100 \text{ GeV} \\ \widetilde{Q}_\nu(E_\nu(1+z), z) \rightarrow \phi_\nu(E_\nu) & E_\nu \gtrsim 100\text{TeV} \end{cases}$$

Solution

- The equation can be rewritten as:

$$\frac{dn(E,z)}{dz} + \left|\frac{dt}{dz}\right| \left[3H(z) - \frac{\partial b(E',z)}{\partial E'}\Big|_{E'=E_g(E,z)}\right] n(E,z) = \left|\frac{dt}{dz}\right| (1+z)^3\, \widetilde{Q}(E_g(E,z),z)$$

- With solution:

$$\frac{n(E,z)}{(1+z)^3} = \int_z^{z_m} dz_g \left|\frac{dt}{dz_g}\right| \widetilde{Q}(E_g(E,z,z_g),z_g) \mathrm{EXP}\left(\int_z^{z_g} ds \left|\frac{dt}{ds}\right| \frac{\partial b(E',s)}{\partial E'}\Big|_{E'=E_g(E,z,s)}\right) = \widetilde{n}(E,z)$$

- Energy losses: adiabatic + interactions

$$b(E,z) = \frac{dE}{dz} = \left[\frac{1}{1+z} + \left|\frac{dt}{dz}\right| b_\gamma(E,z)\right]$$

- $E_g(E,z,z_g)$: solution of the energy losses equation with the observed energy as initial data.
- Sterile particles suffer only adiabatic losses: $E_g(E,z,z_g) = \frac{1+z_g}{1+z}E$.
- Injection term: $\widetilde{Q}(E,z) = \frac{\widetilde{\mathcal{L}}}{E_0^2}(\gamma-2)(1+z)^m \left(\frac{E}{E_0}\right)^{-\gamma} \times \begin{cases} 1 & E \le E_{\mathrm{max}} \\ \mathrm{EXP}\left(1 - \frac{E}{E_{\mathrm{max}}}\right) & E > E_{\mathrm{max}} \end{cases}$

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Proton interactions

$$\beta(E,z) = \frac{1}{E}\frac{dE}{dt} = \int d\epsilon'\, \epsilon' \eta(\epsilon')\sigma(\epsilon') \int d\epsilon \frac{n_\gamma(\epsilon,z)}{\epsilon^2}$$

Pair production:

$$E_{\rm th} = \frac{M_e}{2E_\gamma}(M_p + M_e) \approx 2 \cdot 10^{18}\ \text{eV}$$

$$p_{\rm CR} + \gamma_t \rightarrow p'_{\rm CR} + e^- + e^+$$

Photo-pion production:

$$E_{\rm th} = \frac{M_\Delta^2 - M_p^2}{2E_\gamma(1-\cos\theta)} \approx 2 \cdot 10^{20}\ \text{eV} \quad \text{on CMB}$$

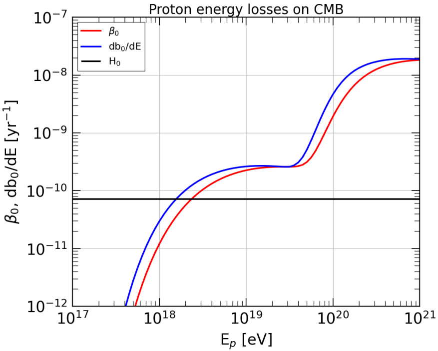
$$p_{\rm CR} + \gamma_{\rm t} \rightarrow \Delta^+ \rightarrow \pi_0 + p'_{\rm CR}$$

$$p_{\rm CR} + \gamma_{\rm t} \rightarrow \Delta^+ \rightarrow \pi_+ + n_{\rm CR}$$

$$p_{\rm CR} + \gamma_{\rm t} \rightarrow \Delta^+_* \rightarrow p'_{\rm CR} + \pi_+ + \pi_-$$

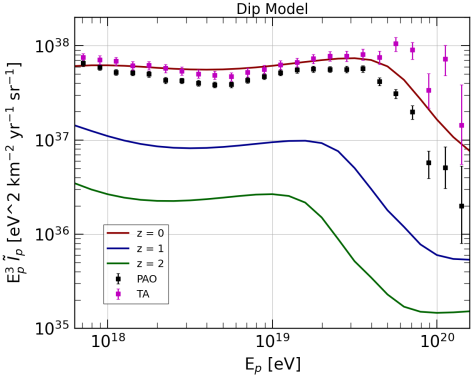
ONLY CMB APPROXIMATION (V. Berezinsky, PRD, 2006):

- $\beta(E,z) = (1+z)^3\, \beta(E(1+z), z=0)$
- $b(E,z) = (1+z)^2\, b(E(1+z), z=0)$
- $\frac{db(E,z)}{dE} = (1+z)^3 \frac{db(E(1+z),z=0)}{dE}$

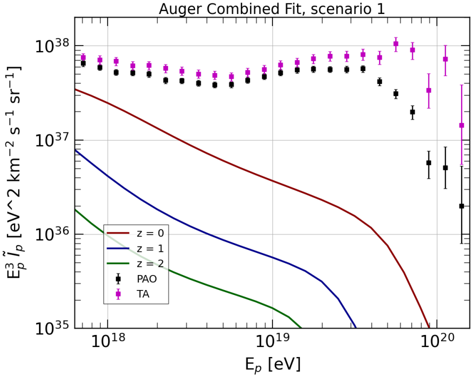


Diffuse proton spectrum

- $m = 0,\ z_{\rm max} = 6$
- $E_0 = 10^{17}$ eV, $E_{\rm max} = 10^{22}$ eV
- $\mathcal{L} \approx 2 \cdot 10^{45}$ erg Mpc^{-3} yr^{-1}, $\gamma = 2.6$



- $m = 0,\ z_{\rm max} = 6$
- $E_0 = 10^{17.8}$ eV, $E_{\rm max} = 10^{22}$ eV
- $\mathcal{L} \approx 3 \cdot 10^{44}$ erg Mpc^{-3} yr^{-1}, $\gamma = 3.34$



Data from: Maurin et al., A&A 569, A32 (2014); Maurin et al., Univ. 6, 102 (2020).

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Production of secondary particles

$$p_{\rm CR} + \gamma_{\rm t} \rightarrow \Delta^+ \rightarrow \pi_0 + p'_{\rm CR} \Rightarrow \pi_0 \rightarrow \gamma + \gamma$$

$$p_{\rm CR} + \gamma_{\rm t} \rightarrow \Delta^+ \rightarrow \pi_+ + n_{\rm CR} \Rightarrow \begin{cases} n_{\rm CR} \rightarrow p'_{\rm CR} + e^- + \bar{\nu}_e \\ \pi_+ \rightarrow \mu^+ + \nu_\mu \Rightarrow \mu^+ \rightarrow e^+ + \bar{\nu}_\mu + \nu_e \end{cases}$$

$$p_{\rm CR} + \gamma_{\rm t} \rightarrow \Delta^+_* \rightarrow p'_{\rm CR} + \pi_+ + \pi_- \Rightarrow \begin{cases} \pi_+ \rightarrow \mu^+ + \nu_\mu \Rightarrow \mu^+ \rightarrow e^+ + \bar{\nu}_\mu + \mu_e \\ \pi_- \rightarrow \mu^- + \bar{\nu}_\mu \Rightarrow \mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu \end{cases}$$

$$p_{\rm CR} + \gamma_t \rightarrow p'_{\rm CR} + e^- + e^+$$

- VHE neutrinos and γ-rays are produced through pions decay.
- Electron / Positron pair are produced mainly through the pair production, but also from the charged pions decay.
- Leptons and γ-rays produce EM cascades through multiple pair production & Inverse Compton interactions on CMB and EBL target photons $\rightarrow$ Calorimetric estimate of the total energy injected in EM cascade with universal shape fixed by the target fields.
- Neutrinos suffer only adiabatic energy losses (at least for z $\lesssim$ 10)

Photopion Cross Section modeling

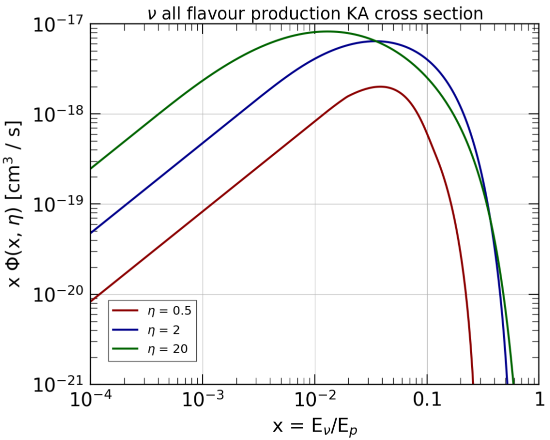
Distribution of final products:

- We rely on parametrized function Φ_i fitted from Monte Carlo simulation of proton - photon interaction performed by SOPHIA (Mücke, CPC, 02/2000).
- Describe the energy distribution of the secondary products (S. R. Kelner, PRD 78, 034013 (2008))

$$\frac{dN}{dE_i} = \int d\epsilon \frac{dE_p}{E_p} n_p(E_p) n_\gamma(\epsilon) \Phi_i(\eta, x)$$

$$\Phi_i(\eta, x) = c \frac{d\sigma(\eta, x)}{dx}, \; x = \frac{E_i}{E_p}, \; \eta = \frac{4\epsilon E_p}{m_p^2 c^4}$$

- Parameters are given as a function of the ratio $\rho = \frac{\eta(\epsilon)}{\eta(\epsilon_{\rm th})}$



Diffuse neutrino spectrum

Solution of the transport equation:

$$\phi_\nu(E_\nu) = \frac{c}{4\pi}\int_0^{z_{max}} dz_g \left|\frac{dt}{dz_g}\right| (1+z_g)\widetilde{Q}(E_\nu(1+z_g), z_g)$$

$$E_{g,\nu}(E_\nu, z, z_g) = E_\nu \frac{(1+z_g)}{(1+z)}$$

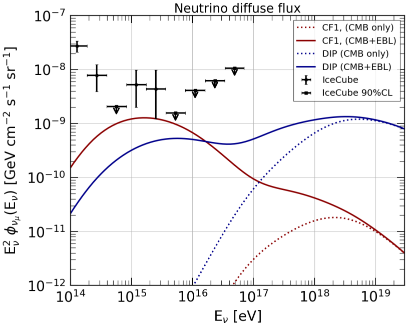
With injection term:

$$\widetilde{Q}(E_\nu, z) = \int_{E_\nu}^{+\infty} \frac{dE_p}{E_p}\widetilde{n}_p(E_p, z) \times R_\nu(E_\nu, E_p, z)$$

$$R_\nu(E_\nu, E_p, z) = \int_{\epsilon_{th}(E_p)} d\epsilon n_\gamma(\epsilon, z)\Sigma_{\nu_i}\Phi_i(E_p, E_\nu, \epsilon)$$

Neutrino Mixing:

$$\phi_{\nu_\mu} \approx \frac{1}{3}\phi_{\nu\text{- all flav.}}$$



Data from: C. Kopper, PoS ICRC2017, 981 (2017).

Gamma-ray cascade

Propagation of the cascades

$$\text{PP: } \gamma + \gamma_{\text{EBL}} \to e^+ + e^-, \; E_{\gamma,th}^{\text{EBL}} < E_{\gamma,th}^{\text{CMB}}$$

$$\text{IC: } e + \gamma_{\text{CMB}} \to \gamma_{\text{HE}} + e, \; n_{\text{CMB}} >> n_{\text{EBL}}$$

Dichromatich model

$$\epsilon_{\text{CMB}}(z) = (1+z) \cdot 6.3 \cdot 10^{-4} \text{ eV}, \; \epsilon_{\text{EBL}}(z) \approx 1 \text{ eV}.$$

Cascade development

- $\widetilde{\omega}_{\gamma\text{-rays}}(z)$: total EM emissivity from protons at a given epoch.
- $\epsilon_{\text{EBL}}^{\text{PP}}(z) = \frac{m_e^2}{\epsilon_{\text{EBL}}(z)}$: PP threshold on EBL.
- $\epsilon_{\text{X}}(z) = \frac{\epsilon_{\text{EBL}}^{\text{PP}}(z)}{3} \frac{\epsilon_{\text{CMB}}(z)}{\epsilon_{\text{EBL}}(z)}$: energy of CMB photons upscattered through IC by an e^-/e^+ produced at the minimum energy by PP.

Universal shape of the cascade, $\tau_{\text{casc}}(E_{\text{s}}, z_{\text{s}}) \sim \left(\sigma(E_{\text{s}}) \cdot c \cdot n_\gamma^0(z_{\text{s}})\right)^{-1} << H^{-1}(z_{\text{s}})$

$$\widetilde{Q}_{\gamma\text{-rays}}(E,z) = \frac{\widetilde{\omega}_{\gamma\text{-rays}}(z)}{\epsilon_{\text{X}}^2(z)\left[2 + \ln \frac{\epsilon_{\text{EBL}}^{\text{PP}}(z)}{\epsilon_{\text{X}}(z)}\right]} \times \begin{cases} \left(\frac{E_\gamma}{\epsilon_{\text{X}}(z)}\right)^{-3/2} & \text{if: } E_\gamma \leq \epsilon_{\text{X}}(z) \\ \left(\frac{E_\gamma}{\epsilon_{\text{X}}(z)}\right)^{-2} & \text{if: } \epsilon_{\text{X}}(z) < E_\gamma \leq \epsilon_{\text{EBL}}^{\text{PP}}(z) \\ 0 & \text{if: } E_\gamma > \epsilon_{\text{EBL}}^{\text{PP}}(z) \end{cases} \tag{1}$$

Gamma-ray cascade

Cascade development

- Stage I: leading particles with $E_\gamma > E^{PP}_{\text{t, CMB}}$ lose small fraction of energy in each collision $f \approx [1/\log 2E\epsilon/m_e]$.
- Stage II: multiplication regime with VHE γ-rays producing pairs on EBL, that upscatter CMB photons.
- Stage III: low energy regime with no more γ-rays with enough energy to produce pairs on EBL; the cascade only proceed through the IC of remnant electrons and positrons on the CMB.

Universal shape:

- Kinematic of the Inverse Compton Scattering: $E^{'}_\gamma = \frac{4}{3}\Gamma^2_e \epsilon_{\text{CMB}} \propto E^2_e$
- Number of photons in the cascade: $dn_\gamma(E_\gamma) = q_e(E_e)\frac{dE_e}{E_\gamma}$
- Conservation of the total energy in the multiplication regime: $E_i = Eq(E) = E(q_e(E) + q_\gamma(E)) \Rightarrow q_e(E_e) \approx \frac{2}{3}\frac{E_i}{E_e}$
- Pairs are not produced in the low-energy regime: $q_e(E_e) = \text{const.}$
- $\epsilon^{PP}_{t,EBL} = \frac{m^2_e}{\epsilon_{EBL}} \approx 4 \cdot 10^{11}\ \text{eV} \Rightarrow E_e = \frac{\epsilon^{PP}_{t,EBL}}{2} \approx 2 \cdot 10^{11}\ \text{eV} \Rightarrow E_\gamma = \frac{4}{3}\frac{E^2_e}{m^2_e}\epsilon_{CMB} = \frac{m^2_e}{\epsilon^2_{EBL}}\frac{\epsilon_{CMB}}{3} \approx 100\ \text{MeV}$

Diffuse Gamma-rays background

Solution of the transport equation:

$$\phi_\gamma(E_\gamma) = \frac{c}{4\pi}\int_0^{z_{max}} dz_g \left|\frac{dt}{dz_g}\right| (1+z_g)\widetilde{Q}_{\gamma\text{-rays}}(E_\gamma(1+z_g), z_g)$$

$$E_{g,\gamma}(E_\gamma, z, z_g) = E_\gamma \frac{(1+z_g)}{(1+z)}$$

Injection from photo-pion

$$\widetilde{Q}(E_\gamma, z) = [\widetilde{\omega}_\pi(z) + \widetilde{\omega}_{ee}(z)] \times f(E_\gamma, z)$$

$$\widetilde{\omega}_\pi(z) = \int dEE\left[\widetilde{Q}_{e^-}(E,z) + \widetilde{Q}_{e^+}(E,z) + \widetilde{Q}_\gamma(E,z)\right]$$

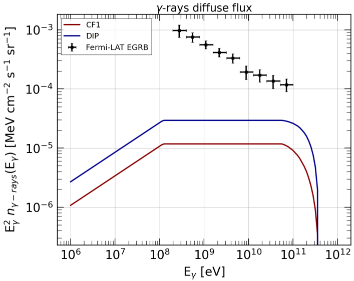
$$\widetilde{Q}(E,z) = \int_E^{+\infty} \frac{dE_p}{E_p}\widetilde{n}_p(E_p,z) \times R_i(E, E_p, z)$$

$$R_i(E, E_p, z) = \int_{\epsilon_{th}(E_p)} d\epsilon n_\gamma(\epsilon, z)\Phi_i(E_p, E, \epsilon)$$

Injection from pair production

All the energy goes into pairs:

$$\widetilde{\omega}_{ee}(z) = \int dE_p \widetilde{n}_p(E_p, z) E_p \beta_{ee}(E_p, z)$$



Data from: A.A. Abdo, Physical Review Letters 104.10 (2010).

Gamma-ray cascade

Reference values

$$\epsilon^{PP}_{\mathrm{CMB}} = \frac{m_e^2}{\epsilon_{\mathrm{CMB}}} \approx 400\ \mathrm{TeV}, \qquad \epsilon^{PP}_{\mathrm{EBL}} = \frac{m_e^2}{\epsilon_{\mathrm{EBL}}} \approx 200\ \mathrm{GeV}, \qquad \epsilon_{\mathrm{X}}(z) = \frac{\epsilon^{\mathrm{PP}}_{\mathrm{EBL}}(z)}{3}\frac{\epsilon_{\mathrm{CMB}}(z)}{\epsilon_{\mathrm{EBL}}(z)} \approx 100\ \mathrm{MeV}$$

Timescale

- Adiabatic: $H_0 \approx 7 \cdot 10^{-11}\,\mathrm{yr}^{-1}$
- IC Scattering on CMB ($\epsilon_{CMB} \approx 6 \cdot 10^{-4}$ eV, $n_{CMB} \approx 400\,\mathrm{cm}^{-3}$):

$$\tau_{IC}^{-1} = c\sigma_T n_{CMB} \approx 3 \cdot 10^{-4}\,\mathrm{yr}^{-1} \ \text{Thompson regime:}\ E_e \lesssim \frac{m_e^2}{\epsilon_{CMB}} \approx 400\,\mathrm{TeV},\ E'_\gamma \approx 3\,\mathrm{GeV}\,E^2_{e,TeV}$$

$$\tau_{IC}^{-1} \approx \frac{3}{8}\frac{\sigma_T}{\Gamma_e}\frac{m_e}{\epsilon_{CMB}} c n_{CMB} \approx 1 \cdot 10^{-4}\,\mathrm{yr}^{-1}\left(\frac{E_e}{E_{KN}}\right)^{-1} \ \text{KN regime:}\ E_e \gtrsim \frac{m_e^2}{\epsilon_{CMB}} \approx 400\,\mathrm{TeV},\ E'_\gamma \approx E_e$$

- Pair Production on EBL ($\epsilon_{EBL} \approx 10^{-2}$ - 1 eV, $n_{EBL} \approx 1 - 10^{-2}\,\mathrm{cm}^{-3}$):

$$\tau_{PP}^{-1} \approx c\sigma_{PP} n_{EBL} \approx 2 \cdot 10^{-9} - 10^{-7}\,\mathrm{yr}^{-1} \ \text{at the } \sigma\text{-peak:}\ E_\gamma \approx 100 \text{ - } 1\ \mathrm{TeV}),\ \sigma_{PP} = \frac{\sigma_T}{4}\ \ (E_\gamma \epsilon_t = 4m_e^2)$$

- Path-lenght: $\langle l_{IC} \rangle \approx$ kpc, $\langle l_{PP,EBL} \rangle \approx 1 - 100\,\mathrm{Mpc}$, $\langle l_{PP,CMB} \rangle \approx 10\,\mathrm{kpc}\,E_{PeV}$

End of the calculations

presentation

Cosmology

UHECR travels cosmological distances $\rightarrow$ Cosmological model of the expansion of the Universe

Metric in an expanding Universe

$$g_{\mu\nu} dx^{\mu} dx^{\nu} = ds^2 = c^2 dt^2 - a^2(t) d\vec{x}^2$$

Λ-CDM model

- FLRW metric.
- Cosmological constant Λ associated with dark energy.
- Cold - Non Baryonic Dark matter
- Cosmological Equation of State: $\frac{P}{\rho} = w$.

FLRW metric

- Solving General Relativity equation:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- Isotropic stress-energy tensor $T_{\mu\nu}$ and metric.

$$a^{-1}(t) = 1 + z; \quad -\frac{dt}{dz} = \frac{1}{(1+z)H(z)}; \quad H(z) = H_0 \sqrt{(1+z)^3 \Omega_M + \Omega_\Lambda}; \quad \begin{cases} H_0 \approx 70 \, \text{km Mpc}^{-3} \, \text{s}^{-1} \\ \Omega_M \approx 0.3 \, \text{(Dark+Baryonic)} \\ \Omega_\Lambda \approx 0.7 \end{cases}$$

Continuity equation

Noether Theorem: continuity equation $\rightarrow$ energy losses to be added "by hand"

$$\frac{1}{\sqrt{g}} g^{\mu\nu} \partial_\mu \left(\sqrt{g} J_\nu \right) = Q$$

Time component, J_0 = n

$$\frac{1}{a^3(t)} \frac{\partial}{\partial t} \left(a^3(t) n(E,t) \right) = \frac{\partial n(E,t)}{\partial t} + 3H(t) n(E,t)$$

Spatial component

- For a source at a given location, exists a preferential direction $\hat{e}$:

$$J_i = -a(t)\widehat{e} c n(E,t) \Rightarrow \frac{-a^2(t)}{a^3(t)} \left(-a^4(t) c \widehat{e} \right) \cdot \left(\frac{\partial n(E,x)}{\partial x_1} + \frac{\partial n(E,x)}{\partial x_2} + \frac{\partial n(E,x)}{\partial x_3} \right) = a^{-1}(t) c \widehat{e} \cdot \nabla_x n(E,x,t)$$

- For identical, uniformly distributed sources the net current $J_i = 0$ everywhere.

Boltzmann equation

Boltzmann equation in General Relativity:

Relativistic gas in gravitational field: $p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^i_{\mu\nu} p^\mu p^\nu \frac{\partial f}{\partial p^i} = (C(f,p))_{\text{coll}} + Q_s$

FLRW metric: $\Rightarrow \quad \frac{\partial f}{\partial t} - 2\frac{\dot{a}(t)}{a(t)} p^i \frac{\partial f}{\partial p^i} = (C(f,p))_{\text{coll}} + Q_s$

First-order Momentum average: $\Rightarrow \quad \frac{\partial n}{\partial t} + 3H(t)n - \frac{\partial}{\partial E}[EH(t)n] = \langle (C(f,p))_{\text{coll}} \rangle_p + \langle Q_s \rangle_p$

Collisional term:

- Collision with a distribution of target: $C = \int dp_* (f_*' f' - f_* f) c(1 - \beta cos\theta)\sigma d\Omega$
- The target distribution is not modified: $f_*' = f_* \Rightarrow \int dp_* f_* = n_\gamma(\epsilon) d\epsilon$
- Energy loss: $\int (f' - f) c(1 - \beta cos\theta)\sigma d\Omega n_\gamma(\epsilon) d\epsilon = \frac{\partial}{\partial E}(f \int c(1 - \beta cos\theta)\sigma d\Omega n_\gamma(\epsilon) d\epsilon dE)$
- Taking the momentum average: $\Rightarrow \frac{\partial}{\partial E}(n(E)b(E))$;
- with $b(E) = \int c(1 - \beta cos\theta)\sigma(E,\epsilon) n_\gamma(\epsilon) d\epsilon dE d\Omega$

Formal solution

- I'll show the methodology to obtain the formal solution of the transport equation for the physical density in the comoving coordinates; the equation reads:

$$\frac{\partial n(E,t)}{\partial t} + 3H(t)n(E,t) - \frac{\partial}{\partial E}\left[n(E,t)b(E,t)\right] = \frac{\widetilde{Q}(E,t)}{a^3(t)}$$

- Exploiting the derivative wrt. to the energy:

$$-\frac{\partial}{\partial E}\left[n(E,t)b(E,t)\right] = -\frac{\partial n(E,t)}{\partial E} b(E,t) - \frac{\partial b(E,t)}{\partial E} n(E,t) = \frac{\partial n(E,t)}{\partial E}\frac{dE}{dt} - \frac{\partial b(E,t)}{\partial E} n(E,t)$$

- The total derivative wrt. time can be collected along the "energy-path" $E_g(E,t,t')$ described by the solution of the energy losses equation $dE/dt = b(E,t)$:

$$\frac{dn(E,t)}{dt} = \frac{\partial n(E,t)}{\partial t} + \frac{\partial n(E,t)}{\partial E}\frac{dE}{dt}$$

- Finally, we move the equation into redshift multiplying by the Jacobian $|dt/dz|$

$$\frac{dn(E,z)}{dz} + \left|\frac{dt}{dz}\right| \left[3H(z) - \left.\frac{\partial b(E',z)}{\partial E'}\right|_{E'=E_g(E,z)}\right] n(E,z) = \left|\frac{dt}{dz}\right| (1+z)^3\, \widetilde{Q}(E_g(E,z),z)$$

Formal solution

- The Green function of the associated homogeneous equation:

$$G(E,z,z') = \mathrm{EXP}\left(-\int_z^{z'} ds \left|\frac{dt}{ds}\right| \left[3H(s) - \frac{\partial b(E',s)}{\partial E'}\Big|_{E'=E_g(E,z,s)}\right]\right)$$

$$= \frac{(1+z)^3}{(1+z')^3}\mathrm{EXP}\left(\int_z^{z'} ds \left|\frac{dt}{ds}\right| \frac{\partial b(E',s)}{\partial E'}\Big|_{E'=E_g(E,z,s)}\right)$$

- From the homogeneous equation one can also obtain the derivative of the Green function:

$$\frac{dG}{dz} = -\left|\frac{dt}{dz}\right| \left[3H(z) - \frac{\partial b(E',z)}{\partial E'}\Big|_{E'=E_g(E,z)}\right] G$$

- The solution of the homogeneous equation:

$$n^{\mathrm{homo}}(E,z,z_s) = A(z_s)G(E,z,z_s) = A(z_s)\frac{(1+z)^3}{(1+z_s)^3}\mathrm{EXP}\left(\int_z^{z_s} ds \left|\frac{dt}{ds}\right| \frac{\partial b(E',s)}{\partial E'}\Big|_{E'=E_g(E,z,s)}\right)$$

- This solution represents a density present at redshift z_s that propagates to redshift $z < z_s$

Formal solution

- Our equation is a first degree, ordinary, complete differential equation with variable coefficients
- A common solving method for this class of ODEs, given their Green function, is to make the ansatz:

$$n(E, z, z_s) = A(z, z_s)G(E, z, z_s) \qquad \Rightarrow \frac{dn}{dz} = \frac{dA}{dz}G + A\frac{dG}{dz}$$

- Substituting into the original equation a particular solution is obtained:

$$\frac{dA(z, z_s)}{dz} = \left|\frac{dt}{dz}\right| (1+z)^3\, \widetilde{Q}(E_g(E, z), z)G^{-1}(E, z, z_s) \Rightarrow$$

$$A(z, z_s) = \int_z^{z_s} dz_g \left|\frac{dt}{dz_g}\right| (1+z_g)^3\, \widetilde{Q}(E_g(E, z, z_g), z_g)G^{-1}(E, z_g, z_s) \Rightarrow$$

$$n(E, z, z_s) = \int_z^{z_s} dz_g \left|\frac{dt}{dz_g}\right| (1+z_g)^3\, \widetilde{Q}(E_g(E, z, z_g), z_g)\frac{G(E, z, z_s)}{G(E, z_g, z_s)}$$

- It is easy to see that $G(E, z, z_s)G^{-1}(E, z_g, z_s) = G(E, z, z_g)$

Formal solution

- The full solution is given by the sum of the homogeneous and particular ones:

$$n(E,z) = A(z_{max})G(E,z,z_{max}) + \int_z^{z_{max}} dz_g \left|\frac{dt}{dz_g}\right| (1+z_g)^3\, \widetilde{Q}(E_g(E,z,z_g),z_g)G(E,z,z_g)$$

- Imposing as initial data that the density is zero at $z = z_{max}$: $G(E, z_{max}, z_{max}) = 1 \Rightarrow A(z_{max}) = 0 \Rightarrow$

$$n(E,z) = \int_z^{z_{max}} dz_g \left|\frac{dt}{dz_g}\right| \cancel{(1+z_g)^3}\widetilde{Q}(E_g(E,z,z_g),z_g)\frac{(1+z)^3}{\cancel{(1+z_g)^3}}\mathrm{EXP}\left(\int_z^{z_g} ds \left|\frac{dt}{ds}\right| \frac{\partial b(E',s)}{\partial E'}\Big|_{E'=E_g(E,z,s)}\right)$$

- The term $(1+z_g)^3$ comes from the physical injection term.
- The term $(1+z)^3/(1+z_g)^3$ comes from $3H(t)$ (expanding volume element).
- Solving the equation for the comoving density with the same methodology:

$$\widetilde{n}(E,z) = \int_z^{z_{max}} dz_g \left|\frac{dt}{dz_g}\right| \widetilde{Q}(E_g(E,z,z_g),z_g)\mathrm{EXP}\left(\int_z^{z_g} ds \left|\frac{dt}{ds}\right| \frac{\partial b(E',s)}{\partial E'}\Big|_{E'=E_g(E,z,s)}\right) = \frac{n(E,z)}{(1+z)^3}$$

- Injection term: $\widetilde{Q}(E,z) = \frac{\widetilde{\mathcal{L}}}{E_0^2}(\gamma-2)(1+z)^m\left(\frac{E}{E_0}\right)^{-\gamma} \times \begin{cases} 1 & E \leq E_{\max} \\ \mathrm{EXP}\left(1 - \frac{E}{E_{\max}}\right) & E > E_{\max} \end{cases}$

Point source:

Transport equation:

$$\frac{\partial n(E,\bar{x},t)}{\partial t} + \left[\frac{c\hat{e}}{a(t)} \cdot \nabla_{\bar{x}} + 3H(t)\right] n(E,\bar{x},t) - \frac{\partial}{\partial E}\left(n(E,\bar{x},t)b(E,t)\right) = \frac{Q(E,t)}{a^3(t)}\delta^3(\bar{x}-\bar{x}_s)$$

$$\frac{\partial n(E,\bar{r},t)}{\partial t} + \left[\left(c\hat{e} + H(t)\vec{r}\right) \cdot \nabla_{\bar{r}} + 3H(t)\right] n(E,\bar{r},t) - \frac{\partial}{\partial E}\left(n(E,\bar{r},t)b(E,t)\right) = Q(E,t)\delta^3(\bar{r}-\bar{r}_s)$$

- The solution is found with the same methodology, but in Fourier space to deal with the δ-function.
- The solution depends on the source location $\bar{x}_s \to z_s$.

$$n^{\rm PS}(E) = \frac{(1+z_s)Q(E_g(E,z_s))}{4\pi c d_L^2}\frac{dE_g}{dE}(E,z_s)$$

$$d_L = (1+z)^2 d_A$$

$$\frac{dV}{dz} = \left|\frac{dt}{dz}\right| 4\pi c d_A^2 (1+z)^3 = \left|\frac{dt}{dz}\right| 4\pi c d_L^2 (1+z)^{-1}$$

Integrating over a uniform sources distribution:

$$\Rightarrow n(E) = \int dV\, n_s(z) n^{\rm PS}(E) = \int_0^{z_{max}} dz \left|\frac{dt}{dz}\right| \tilde{n}_s(z) Q(E_g(E,z)) \frac{dE_g}{dE}(E,z)$$

Diffusive regime and magnetic horizon

Transport equation:

$$J_\mu = D(E)\nabla_{\bar{x}} n(E) \Rightarrow \frac{\partial n(E)}{\partial t} + \left[\frac{D(E)}{a^2(t)} \cdot \nabla^2_{\bar{x}} + 3H(t)\right] n(E) - \frac{\partial}{\partial E}\left(n(E)b(E)\right) = \frac{Q(E)}{a^3(t)}\delta^3(\bar{x} - \bar{x}_s)$$

$$n^{PS}_{\rm Diff}(E) = \int_0 dz \left|\frac{dt}{dz}\right| Q(E_g(E,z))\frac{dE_g}{dE}(E,z)\frac{EXP\left(-\frac{d_L^2}{(1+z)^2 4\lambda(E_g,z)}\right)}{\sqrt{4\pi\lambda(E_g,z)}}, \; \lambda(E_g,z) = \int_0^z ds\left|\frac{dt}{ds}\right|(1+s)^2 D(E_g(E,s))$$

Integrating over the source's distribution:

$$n(E) = \int dx_g 4\pi x_g^2 \tilde{n}_s(z) n^{PS}_{\rm Diff}(E,z) \propto \int dx_g \frac{4\pi x_g^2}{(4\pi\lambda)^{3/2}} \mathrm{EXP}\left(-\frac{x_g^2}{4\lambda}\right) = 1$$

Magnetic Horizon:

- The spectrum for a uniform source's distribution is universal.
- In the diffusive case, this is valid as long as the diffusive length is larger than the mean separation between sources: $l_{\rm Diff} \approx \frac{D(E)}{c} \gtrsim \langle d_s \rangle$

Solution for nuclei

Does NOT CONSERVE the nucleus type

$$(A, Z) + \gamma \to (A-1, Z-1) + p$$

$$\text{threshold: } \epsilon' \approx 5\,\text{MeV} \Rightarrow E_{\text{th}} \approx \frac{Am_p}{2}\frac{5\,\text{MeV}}{\epsilon_t} \Rightarrow E_{\text{th}} \approx 3 \cdot A \cdot 10^{18}\,\text{eV on CMB}$$

- Photodisintegration conserves the energy/nucleon $\to$ write the transport in terms of Lorentz factor.
- It can be treated as a *decay* in the transport equation, with characteristic timescale:

$$\frac{1}{E}\frac{dE}{dt} = \frac{1}{A}\frac{dA}{dt} = \frac{1}{\tau_A(\Gamma, z)} = \frac{c}{2\Gamma^2}\int d\epsilon' \epsilon' \sigma_A^{\text{phdis}}(\epsilon') \int d\epsilon \frac{n_\gamma(\epsilon, z)}{\epsilon^2}$$

- The transport equation becomes:

$$\frac{dn_A(\Gamma, z)}{dz} + \left[3H(z) + \frac{1}{\tau_A(\Gamma, z)} - \frac{\partial b_{A\gamma}(\Gamma, z)}{\partial \Gamma}\right]\left|\frac{dt}{dz}\right| n_A(\Gamma, z) = (1+z)^3 \left|\frac{dt}{dz}\right| \widetilde{Q}(\Gamma, z)$$

- With solution:

$$n_A(\Gamma) = \int_0^{z_{max}} dz_g \left|\frac{dt}{dz_g}\right| (1+z)^3\, \widetilde{Q}(\Gamma_g(\Gamma, z_g), z_g) \frac{d\Gamma_g}{d\Gamma}(\Gamma, s) e^{-\eta(\Gamma, A, z_g)}$$

Photodisintegration

Photo-hadronic model

- PSB model: empirically determined cross sections for single nucleon (GDR) and multi nucleon emissions.
- TALYS: Computational tool which simulates nuclear reaction, including α-particle emission

$$\eta(\Gamma, A, z, z_g) = \int_z^{z_g} ds \left| \frac{dt}{dz} \right| \frac{1}{\tau_A(\Gamma_g(\Gamma, z, s), s)}$$

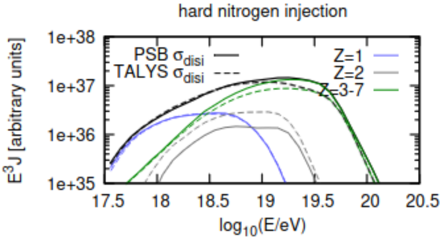
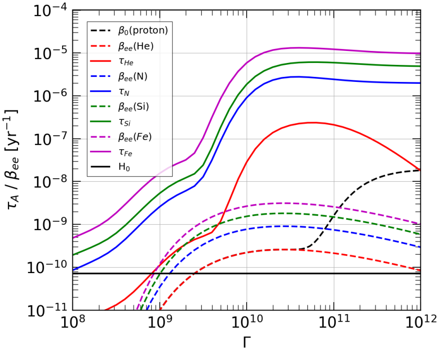


Figure: R. Alves Batista, JCAP, 10/2015



Chain of differential equations

- Transport equation for secondary nuclei:

$$\frac{\partial n_A(\Gamma,z)}{\partial z} + \left[3H(z) + \frac{1}{\tau_A(\Gamma,z)} - \frac{\partial b_{A\gamma}(\Gamma,z)}{\partial \Gamma}\right]\left|\frac{dt}{dz}\right| n_A(\Gamma,z) = (1+z)^3 \left|\frac{dt}{dz}\right| \frac{\widetilde{n}_{A+1}(\Gamma,z)}{\tau_{A+1}(\Gamma,z)}$$

- Transport equation for secondary protons:

$$\frac{\partial n(\Gamma,z)}{\partial z} + \left[3H(z) - \frac{\partial b_{\gamma}(\Gamma,z)}{\partial \Gamma}\right]\left|\frac{dt}{dz}\right| n(\Gamma,z) = (1+z)^3 \left|\frac{dt}{dz}\right| \Sigma_A \frac{\widetilde{n}_{A}(\Gamma,z)}{\tau_{A}(\Gamma,z)}$$

- For secondary nuclei, the injection term is: $\widetilde{Q}_A(\Gamma_g) = -\frac{d}{dt}\widetilde{n}_{A+1}(\Gamma_g)$.
- For secondary protons, the injections term depends on all the other species.
- To take care of all the nuclear species, a chain of ODE should be solved:
  - Protons and primary nuclei ODE's should contain the injection from both the sources and the other species.
  - Secondary nuclei has only the injection term from photodisintegration.