# Exploring Orbital Trajectories and Stability through Variational Method (218)

Margaux Introna ♠ Susanna Terracini ♣ Massimiliano Vasile ♢
University of Trento ♠ University of Turin ♣ Strathclyde University ♢
♠margaux.introna@unitn.it ♣susanna.terracini@unito.it
♢massimiliano.vasile@strath.ac.uk

### Introduction

In recent years, significant advancement in celestial mechanics has emerged through the variational method applied to the $n$-body problem, leading to the discovery of novel orbital trajectories. This approach involves: firstly, discretising the problem; then, examining critical points of the Lagrangian action associated with the $n$-body problem as they represent periodic solutions of the dynamical system and they can be numerically found by merging evolutionary and deterministic algorithms.

### The Problem

Once selected the planar three body problem as main model, the first goal is to search for symmetrical periodic orbits using the method of nonlinear programming. Following [1], and [3] only collision free symmetries that yield to coercive action functionals are considered. In order to define a symmetry, one selects finite groups G, which act on:

- the time circle $\mathcal{T} \subset \mathbb{R}^2$;
- the Euclidean space $\mathbf{E}$;
- the set of indices $\mathbf{n} = \{1, 2, 3\}$.

In this research, the Line group is chosen; it consists of a reflection on the time circle $\mathcal{T}$, a reflection on the plane $\mathbf{E}$, and a trivial mapping on the set of indices. In detail, set $r_i(t) = (x_i(t), y_i(t))$ and its derivatives $\dot{r}_i(t) = (\dot{x}_i(t), \dot{y}_i(t))$ with $i = 1, \dots, n = 3$, one can write the action functional as:

$$\mathcal{A}_\omega = \int_{\mathcal{T}} L_\omega(r(t), \dot{r}(t))\, \mathrm{d}t, \quad \omega \in \mathbb{R},$$

where, set $J$ the complex unit, $m_i$ mass of $i$-th body and $\alpha = 1$,

$$L_\omega(r(t), \dot{r}(t)) = L_\omega = K_\omega + U$$
$$K_\omega(r(t), \dot{r}(t)) = \sum_{i=1}^{3} \frac{1}{2} m_i |\dot{r}_i - J\omega r_i|^2$$
$$U(r(t)) = \frac{m_1 m_2}{|r_1 - r_2|^\alpha} + \frac{m_1 m_3}{|r_1 - r_3|^\alpha} + \frac{m_2 m_3}{|r_2 - r_3|^\alpha}.$$

Setting the period $T = 2\pi$ and as all masses can have different values, the sought solutions can be written as Fourier series:

$$x_i(t) = a_{0_i} + \sum_{k=1}^{\infty} \left( a_i^k \cos(kt) + b_i^k \sin(kt) \right)$$
$$y_i(t) = c_{0_i} + \sum_{k=1}^{\infty} \left( c_i^k \cos(kt) + d_i^k \sin(kt) \right),$$

with some symmetry set conditions on the coefficients. In particular, setting at $t = 0$ and $t = \pi$ that the masses are collinear on a fixed line $l \subset E$, if the line $l$ coincides with the axis of abscissae, then one has $b_i^k = c_i^k = c_{0_i} = 0$. Using the frame with origin at the centre of mass, it follows that:

$$m_1 \mathbf{r_1} + m_2 \mathbf{r_2} + m_3 \mathbf{r_3} = 0$$
$$a_3^k = -(m_1 a_1^k + m_2 a_2^k) / m_3$$
$$d_3^k = -(m_1 d_1^k + m_2 d_2^k) / m_3.$$

Therefore one needs to specify $2\,(2k + 1)$ coefficients only.

### The Method

**First Step.** In order to find them, the main idea is to combine two algorithms: a multi-population adaptive version of inflationary differential evolution one (called MP-AIDEA, see [2]) and a domain decomposition one. MP-AIDEA consists of an inflationary differential evolution algorithm which combines basic differential evolution with some of the restart and local search mechanisms of monotonic basin hopping. On the other hand, using a domain decomposition algorithm guarantees a full exploration of the solution space. The results obtained represent feasible and physically meaningful solutions of the dynamical system as they are periodic orbits that satisfy the associated differential equations.

**Second Step.** Subsequently, attention shifts towards assessing the functional stability of the problem. Orbit trajectories identified in the previous phase are mapped as critical points, allowing for an examination of the stability or instability within their respective neighbourhoods. This involves treating the problem as a dynamical system, analysing the gradient of the action functional $\mathcal{A}$, denoted as $\eta' = -\nabla\mathcal{A}(\eta)$. This approach not only characterises the minima discovered earlier but also delineates the basin of attraction for each minimum, thanks to the algorithm's analysis of the initial input data or starting point.

**Third Step.** Lastly, one focuses on analysing the boundaries of these basins of attraction. It has been established that when two boundaries converge and then separate, the point of separation asymptotically converges to a critical point that is not a minimum. Using conventional algorithms such as the Newton method enables the numerical approximation of these new critical points, which differ from the previously identified minima.

### Conclusion || Improvement

Despite its theoretical nature, this methodology holds practical applications in Astrodynamics, particularly in mission design and the deployment of satellite constellations into orbit.

### References

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