Elements of Mathematical Foundations of Quantum Mechanics

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1 Exercises

1.1 Banach Spaces

Exercise 1.1 (Inverse Triangular Inequality)**.** *Given a normed space $(X, \|\cdot\|_X)$, show that for any $f, g \in X$ there holds*

$$\|f - g\|_X \geq |\, \|f\|_X - \|g\|_X|.$$

Exercise 1.2. *Let $(X, \|\cdot\|_X)$ be a Banach space. Show that the norm is continuous, namely, given $f \in X$*

$$\forall \{f_n\}_{n\in\mathbb{N}} \subset X \;\; s.t. \;\; f_n \xrightarrow[n\to\infty]{} f, \quad one\ has \quad \|f_n\|_X \xrightarrow[n\to\infty]{} \|f\|_X\,.$$

Find an example for which the converse is false.

Exercise 1.3. *Let $A \in \mathscr{B}(Y, Z)$ and $B \in \mathscr{B}(X, Y)$ with X, Y, Z normed spaces. Show that*

$$\|AB\|_{\mathscr{L}(X,Z)} \leq \|A\|_{\mathscr{L}(Y,Z)}\, \|B\|_{\mathscr{L}(X,Y)}\,.$$

Exercise 1.4 (Absolute Convergence)**.** *Let $(X, \|\cdot\|_X)$ be a Banach space and suppose $\{f_n\}_{n\in\mathbb{N}} \subset X$ be a sequence satisfying $\sum_{n\in\mathbb{N}} \|f_n\|_X < +\infty$. Then prove that $\sum_{n\in\mathbb{N}} f_n := \lim\limits_{N\to\infty} \sum_{n=0}^{N} f_n$ exists.*

Exercise 1.5. *Let the function f be represented by a power series with convergence radius $R > 0$*

$$f(z) = \sum_{j\in\mathbb{N}} f_j z^j, \qquad |z| < R.$$

Moreover, let $A \in \mathscr{B}(X)$ be s.t. $\|A\|_{\mathscr{L}(X)} < R$. Show that

$$f(A) = \sum_{j\in\mathbb{N}} f_j A^j$$

exists and defines a bounded linear operator (Hint: exploit Exercises 1.3 and 1.4).

1.2 Hilbert Spaces

Exercise 1.6. *Given $\mathfrak{H}$ Hilbert space, show that*

$$\|f + g\|_{\mathfrak{H}} = \|f\|_{\mathfrak{H}} + \|g\|_{\mathfrak{H}} \qquad iff \qquad f = \alpha g,\ \alpha \geq 0 \quad \vee \quad g = 0.$$

Exercise 1.7. *The operator*

$$S\colon \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N}), \qquad (a_1, a_2, a_3, \dots) \longmapsto (0, a_1, a_2, \dots)$$

satisfies $\|Sa\|_{\ell^2(\mathbb{N})} = \|a\|_{\ell^2(\mathbb{N})}$ *. Is it unitary? Compute* S^*.

Exercise 1.8 (Jacobi Operator)**.** *Let* a *and* b *be some real-valued sequences in* $\ell^\infty(\mathbb{Z})$*. Consider the operator*

$$J : f_n \longmapsto a_n f_{n+1} + a_{n-1} f_{n-1} + b_n f_n, \qquad \{f_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}).$$

Prove that J *is a bounded, self-adjoint operator.*

Exercise 1.9. *Show that* $P\colon L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \qquad f(x) \longmapsto \frac{1}{2}\left(f(x) + f(-x)\right)$ *is a projection. Compute its range and kernel.*

Exercise 1.10. *Let* $A \in \mathscr{B}(\mathfrak{H})$*. Show that* A *is normal iff*

$$\|A\psi\|_{\mathfrak{H}} = \|A^*\psi\|_{\mathfrak{H}}, \qquad \forall\, \psi \in \mathfrak{H}.$$

Exercise 1.11. *Let* $U \in \mathscr{B}(\mathfrak{H})$*. Prove that* U *is unitary iff* $U^{-1} = U^*$.

Exercise 1.12. *Given* $A \in \mathscr{L}(\mathfrak{H}_1, \mathfrak{H}_2)$ *closable and densely defined operator,* $\psi \in \mathscr{D}(\overline{A})$ *and* $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(A)$ *s.t.* $\psi_n \xrightarrow[n\to\infty]{} \psi$*. Then, prove that* $A\psi_n \xrightarrow[n\to\infty]{} \overline{A}\psi$.

Exercise 1.13. *Suppose* $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathfrak{H}$ *and* $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathfrak{H}$ *s.t.* $\psi_n \xrightarrow[n\to\infty]{} \psi$ *and* $\varphi_n \xrightarrow[n\to\infty]{} \varphi$*. Then show that*

$$\lim_{n\to\infty} \langle \psi_n,\, \varphi_n \rangle_{\mathfrak{H}} = \langle \psi,\, \varphi \rangle_{\mathfrak{H}}$$

(Hint: every weakly convergent sequence is bounded).

Exercise 1.14. *Let* $\{\varphi_j\}_{j\in\mathbb{N}}$ *be some orthonormal base for a complex Hilbert space* $\mathfrak{H}$*. Show that* $A \in \mathscr{B}(\mathfrak{H})$ *is completely characterized by its matrix elements*

$$A_{jk} := \langle \varphi_j,\, A\varphi_k \rangle_{\mathfrak{H}}.$$

Then, write also the action of A *assuming that* $\{\varphi_j\}_{j\in\mathbb{N}}$ *is a base of eigenvectors for* A.

Exercise 1.15. *Let* M *be a multiplication operator in* $L^2(X, d\mu)$ *by a* μ*-measurable function* $f : X \longrightarrow \mathbb{C}$*. Find proper conditions on* f *in such a way that*

- M *is bounded;*
- M *is self-adjoint.*

Write the domain of self-adjointness for M.

Exercise 1.16. *Let* $A = -\frac{d^2}{dx^2}, \mathscr{D}(A) = \{f \in H^2(0, \pi) \,|\, f(0) = f(\pi) = 0\}$ *and let* $\psi(x) = \frac{x(\pi - x)}{2\sqrt{\pi}}$*. Find the error in the following statement: since* A *is symmetric, one has* $1 = \langle A\psi,\, A\psi \rangle_{L^2(0,\pi)} = \langle \psi,\, A^2\psi \rangle_{L^2(0,\pi)} = 0$.

2 Problems

Problem 2.1. *Suppose to have a quantum particle confined in the segment* $[0, L]$ *whose kinetic energy is given by the operator* $K = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}, \mathscr{D}(K) = C_c^\infty(0, L)$*. Characterize every self-adjoint extension for* K.
Then take into account the self-adjoint extensions implementing periodic-boundary conditions, Dirichlet and Neumann boundary conditions and compute the corresponding eigenvectors.
Let $\psi(x) = \sqrt{\frac{2}{L}}$*, if* $x \in \left(\frac{L}{4}, \frac{3L}{4}\right)$ *and* 0 *otherwise be a prepared state for three distinct systems described by each of the previously considered self-adjoint extensions.*

- *What is the expectation of the position of the particle in this state?*
- *What is the probability to measure the lowest possible energy? And what about the first excited state?*
- *What is the probability of finding the system in a given superposition of the first two energy levels (i.e. with probability q in the ground state and probability $1-q$ in the first excited state) after some time t has passed?*

Problem 2.2. *Consider a quantum particle in the Hilbert space $L^2(\mathbb{R}_+)$ whose energy is described by the Hamiltonian*

$$H = -\frac{\hbar}{2m}\frac{d^2}{dx^2} - \mathrm{h}\,\mathbb{1}_{[0,L]}, \qquad \mathrm{h} > 0.$$

For which domains H is self-adjoint?
Take account of the Dirichlet boundary condition at the boundary and assume negative energies in order to find the conditions the eigenfunctions of H must satisfy. Compute the probability for which the particle can be found outside the well in terms of the energy.

Problem 2.3. *A quantum particle in a crystal (consider it as a fixed point) can be found in two independent states $|\uparrow\rangle, |\downarrow\rangle$ whose energy levels are $a\hbar$ and $-a\hbar$, respectively for some frequency $a > 0$.*
Given the initial state $\frac{1}{2}|\uparrow\rangle - \frac{\sqrt{3}}{2}|\downarrow\rangle$, compute the probability for the state to remain unchanged at a given time t and find when this quantity is maximized.
Now consider an observable S whose action consists in exchanging the two independent states, i.e.

$$S|\uparrow\rangle = |\downarrow\rangle, \qquad S|\downarrow\rangle = |\uparrow\rangle$$

and find its expectation as t varies.
In a following moment the system is found to be in the state $\frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$, since a measurement of the observable S obtained as outcome the value -1. What is the probability for this state to be in the lowest energy level? The next measure for the energy applied to the same state yields the value $-a\hbar$, but what is the probability for S to be still -1? Instead, what about the value $+1$?
Explain what is going on. (Hint: prove the following statement)

> *Given $A, B \in \mathscr{B}(\mathfrak{H})$ two self-adjoint operators there holds*
>
> $$\nexists \text{ common o.n.b. of eigenvectors} \iff AB \neq BA.$$