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Elements of Mathematical foundations of Quantum Mechanics

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ref: Mathematical Methods in QM by G. Teschl

> Any norm must satisfy
> i) $\|f\|_X \geq 0$, $\forall f \in X$ and $\|f\|_X = 0 \iff f = 0$;
> ii) $\|\alpha f\|_X = |\alpha| \, \|f\|_X$, $\forall f \in X$, $\alpha \in \mathbb{C}$;
> iii) $\|f+g\|_X \leq \|f\|_X + \|g\|_X$, $\forall f, g \in X$

X, Y normed space $\|\cdot\|_X$, $\|\cdot\|_Y$

- $\mathcal{L}(X,Y)$ denotes the set of **linear** maps between X and Y

  $\mathcal{L}(X)$ is a shortcut for $\mathcal{L}(X,X)$

Let $A \in \mathcal{L}(X,Y)$. $\mathcal{D}(A) \subseteq X$ is the domain for which A is well-defined and $\mathrm{Ran}(A)$ is its image.

$$A\,\mathcal{D}(A) = \mathrm{Ran}(A) \subseteq Y$$

Moreover, $\mathrm{Ker}(A) := \{ f \in \mathcal{D}(A) \mid Af = 0 \}$ denotes the Kernel of A.

PRO

- A is injective iff $\mathrm{Ker}(A) = \{0\}$.
- A is invertible if it is injective and $\overline{\mathrm{Ran}A} = Y$
- An invertible operator is boundedly invertible if it is surjective.

- $\mathcal{L}(X, \mathbb{C})$ is the space of linear **functionals** defined in X, according to our definitions. This space is denoted also by X^*, the dual of X.

DEF
If every Cauchy sequence converges, then the normed space is complete and it is called a **Banach** space.

N.B.
> Every convergent sequence is also a Cauchy sequence, the converse is not trivial!

DEF
$\mathcal{L}(X,Y)$ can be equipped with the so-called **operator norm** $\|A\|_{\mathcal{L}(X,Y)} := \sup_{\substack{f \in X: \\ \|f\|_X = 1}} \|Af\|_Y$

THEO (Hahn-Banach)

Given $A \in \mathcal{L}(X,Y)$ with $\mathcal{D}(A) \subset X$, there always exists $\tilde{A} \in \mathcal{L}(X,Y)$ satisfying

- $\mathcal{D}(\tilde{A}) \supset \mathcal{D}(A)$;
- $\tilde{A}f = Af$, $\forall f \in \mathcal{D}(A)$.

Moreover $\tilde{A}$ can be chosen s.t. $\|\tilde{A}\|_{\mathcal{L}(X,Y)} = \|A\|_{\mathcal{L}(X,Y)}$.

N.B.
> These two properties define an **extension** of A. This theorem ensures that there is always a (norm-preserving) extension.

DEF

- $A \in \mathcal{L}(X,Y)$ is **continuous** if there exists $\delta_\epsilon > 0$ s.t. $\|f-g\|_X < \delta_\epsilon$ and $\|Af - Ag\|_Y < \epsilon$ regardless the value of $\epsilon > 0$.

- $\mathcal{B}(X,Y)$ is the set of linear and bounded maps between X and Y.

  $(\mathcal{B}(X) := \mathcal{B}(X,X))$ The Kernel of any bounded operator is closed.

PRO

For any $A \in \mathcal{L}(X,Y)$ one has $A \in \mathcal{B}(X,Y) \iff A$ is continuous.

$\mathcal{B}(X,Y)$ equipped with the operator norm $\|\cdot\|_{\mathcal{L}(X,Y)}$ is a Banach space if Y is Banach.

THEO (BLT - Bounded, linear transformation)

Given $A \in \mathcal{B}(X,Y)$ with Y a Banach space and A densely defined ($\mathcal{D}(A)$ dense in X) there exists a **unique** extension of A, denoted by $\tilde{A}$ s.t. $\mathcal{D}(\tilde{A}) = X$ and $\|\tilde{A}\|_{\mathcal{L}(X,Y)} = \|A\|_{\mathcal{L}(X,Y)}$

According to this theorem there is no ambiguity in providing a densely defined linear, bounded map in a normed space, since there exists only one possible extension that is everywhere well defined.

THEO (Banach-Steinhaus)

Let X be a Banach space and Y a normed space.
Given a family $\{A_\alpha\} \subseteq \mathcal{B}(X,Y)$ satisfying $\|A_\alpha f\| \leq c_f$ $\forall \alpha$ and for any $f \in X$ fixed, then

$$\sup_\alpha \|A_\alpha\|_{\mathcal{L}(X,Y)} < +\infty.$$

Let $\mathfrak{h}$ be a vector space over $\mathbb{C}$. A map $\langle\cdot,\cdot\rangle_\mathfrak{h} : \mathfrak{h}\times\mathfrak{h} \to \mathbb{C}$ is said sesquilinear if

i) $\langle \alpha_1 f_1 + \alpha_2 f_2, g\rangle_\mathfrak{h} = \alpha_1^* \langle f_1, g\rangle_\mathfrak{h} + \alpha_2^* \langle f_2, g\rangle_\mathfrak{h}$ $\quad \forall \alpha_1,\alpha_2 \in \mathbb{C},\ f_1, f_2, g \in \mathfrak{h}$;

ii) $\langle f, \alpha_1 g_1 + \alpha_2 g_2\rangle_\mathfrak{h} = \alpha_1 \langle f, g_1\rangle_\mathfrak{h} + \alpha_2 \langle f, g_2\rangle_\mathfrak{h}$ $\quad \forall \alpha_1,\alpha_2 \in \mathbb{C},\ f, g_1, g_2 \in \mathfrak{h}$;

DEF

If a sesquilinear form is positive ($\langle f,f\rangle_\mathfrak{h} > 0$ $\forall f \neq 0$) and symmetric ($\langle f,g\rangle_\mathfrak{h} = \langle g,f\rangle_\mathfrak{h}^*$) then it is called an **inner product**. A vector space endowed with a scalar product is said **pre-Hilbert** space.

PRO (Couchy-Schwarz-Bunjakowski inequality)

Let $f, g \in \mathfrak{h}$ pre-Hilbert space. Then there holds

$$|\langle f,g\rangle_\mathfrak{h}| \leq \|f\|_\mathfrak{h} \cdot \|g\|_\mathfrak{h}$$

N.B. A pre-Hilbert space is a normed space since one always has an induced norm defined by $\|\cdot\|_\mathfrak{h} := \sqrt{\langle\cdot,\cdot\rangle_\mathfrak{h}}$

This proposition implies that, given a convergent sequence $\{f_n\}$, $\|f_n - f\|_\mathfrak{h} \xrightarrow[n\to+\infty]{} 0$ we also have $\langle f_n, g\rangle_\mathfrak{h} \xrightarrow[n\to+\infty]{} \langle f, g\rangle$, namely the map $\langle\cdot, g\rangle$ is continuous for any $g \in \mathfrak{h}$. Clearly the same holds for $\langle f, \cdot\rangle$.

THEO (Jordan-Von Neumann)

A given norm can be associated to a scalar product iff there holds the **parallelogram identity**, i.e.

$$\|f+g\|_X^2 + \|f-g\|_X^2 = 2\|f\|_X^2 + 2\|g\|_X^2, \qquad \forall f, g \in X.$$

If it is the case, then also the **polarization identity** holds

$$\langle f,g\rangle_X := \frac{1}{4}\left(\|f+g\|_X^2 - \|f-g\|_X^2 + i\|f-ig\|_X^2 - i\|f+ig\|_X^2 \right).$$

DEF

If a pre-Hilbert space is complete according to the norm $\sqrt{\langle\cdot,\cdot\rangle_\mathfrak{h}}$, then it is called a **Hilbert space**.

THEO (Riesz)

Given a Hilbert space $\mathfrak{h}$, for any continuous linear functional $l \in \mathcal{B}(\mathfrak{h}, \mathbb{C})$ there exists a **unique** vector $\varphi_l \in \mathfrak{h}$ s.t.

- $l(\psi) = \langle \varphi_l, \psi\rangle_\mathfrak{h}$
- $\|\varphi_l\|_\mathfrak{h} = \|l\|_{\mathfrak{h}^*}$

The weak topology of a Hilbert space is therefore represented in terms of scalar products

$\boxed{\psi_n \xrightarrow[n\to\infty]{} \psi}$ iff $l(\psi_n) \xrightarrow[n\to\infty]{} l(\psi)$ $\forall l \in \mathfrak{h}^*$ or, equivalently $\langle\varphi, \psi_n\rangle_\mathfrak{h} \xrightarrow[n\to\infty]{} \langle\varphi,\psi\rangle_\mathfrak{h}$ $\forall \varphi \in \mathfrak{h}$

weak convergence

N.B. We already saw that ~~norm~~ convergence $\Rightarrow$ weak conv

③ **DEF**
A family of vectors $\{\varphi_j\}_{j\in\mathbb{N}} \subset \mathfrak{h}$, with $\mathfrak{h}$ complex Hilbert space is a base (complete, orthonormal system) if

- $\langle \varphi_i, \varphi_j \rangle_{\mathfrak{h}} = \delta_{ij}$;
- $\overline{\text{span}\{\varphi_j\}_{j\in\mathbb{N}}} = \mathfrak{h}$ ⟵ N.B. This means that $\forall \psi \in \mathfrak{h}$ $\exists \{c_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ s.t.

$$\psi = \sum_{j\in\mathbb{N}} c_j \varphi_j$$

namely $\| \psi - \sum_{j=0}^{n} c_j \varphi_j \|_{\mathfrak{h}} \xrightarrow[n\to+\infty]{} 0$.

$\{c_j\}$ are called Fourier coefficients and there holds

$$c_j = \langle \varphi_j, \psi \rangle_{\mathfrak{h}}.$$

A Hilbert space with a countable base is said **separable**.

PRO (Parseval's identity)

$$\|\psi\|^2_{\mathfrak{h}} = \sum_{j\in\mathbb{N}} |c_j|^2.$$

DEF
A bijective operator $U \in \mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2)$ is said to be **unitary** if $\forall \psi \in \mathfrak{h}_1$

$\|U\psi\|_{\mathfrak{h}_2} = \|\psi\|_{\mathfrak{h}_1}$ or, equivalently ⟵ The two conditions are equivalent because of the polarization identity.

if $\langle U\varphi, U\psi \rangle_{\mathfrak{h}_2} = \langle \varphi, \psi \rangle_{\mathfrak{h}_1}$

for any $\varphi, \psi \in \mathfrak{h}_1$.

REMARK

- Every separable (infinite-dimensional), complex Hilbert space is unitarily equivalent to $\ell_2(\mathbb{N})$. Indeed, consider $U: \mathfrak{h} \to \ell_2(\mathbb{N})$, $\psi \mapsto \{c_j\}_{j\in\mathbb{N}}$ that is injective ($c_j = 0\ \forall j \Rightarrow \psi = 0$) and surjective since φ_j is a base for $\mathfrak{h}$.

  Moreover, the Parseval's identity reads $\|U\psi\|^2_{\ell_2(\mathbb{N})} = \|\psi\|^2_{\mathfrak{h}}$ (where $\|U\psi\|^2_{\ell_2(\mathbb{N})} = \sum_{j=0}^{+\infty} |c_j|^2$)

  ⇒ Hence U is unitary.

- Every bounded operator $A \in \mathcal{B}(\mathfrak{h})$ is completely characterized by its **matrix elements**

$$\langle \varphi_j, A\varphi_k \rangle_{\mathfrak{h}} =: A_{jk}$$

  with $\{\varphi_j\}_{j\in\mathbb{N}}$ a base for $\mathfrak{h}$. However this does not mean that $\mathcal{B}(\mathfrak{h})$ is separable if $\dim(\mathfrak{h}) = \infty$.

- $\{\varphi_j\}_{j\in\mathbb{N}}$ is an example of weakly converging sequence that has no limit in norm topology, since $\langle \varphi_j, \psi \rangle_{\mathfrak{h}} \xrightarrow[j\to\infty]{} 0$ for any $\psi \in \mathfrak{h}$, since $\{\langle \varphi_j, \psi \rangle_{\mathfrak{h}}\}_{j\in\mathbb{N}} \in \ell_2(\mathbb{N})$

  $\|\varphi_j\| = 1$, $\|\varphi_n - \varphi_m\|^2 = 2$

DEF
One can define on $\mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2)$ the **adjoint operator** of $A \in \mathcal{B}(\mathfrak{h}_2, \mathfrak{h}_1)$ as follows

$$\langle \varphi, A^*\psi \rangle_{\mathfrak{h}_2} := \langle A\varphi, \psi \rangle_{\mathfrak{h}_1}, \quad \forall \varphi \in \mathfrak{h}_2,\ \psi \in \mathfrak{h}_1$$

PRO
The adjoint map of a given operator satisfies

i) $(\alpha A + \beta B)^* = \alpha^* A^* + \beta^* B^*$, $\quad \alpha, \beta \in \mathbb{C}$, $A, B \in \mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2)$

ii) $A^{**} = A$, $\quad A \in \mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2)$

iii) $(AB)^* = B^* A^*$ $\quad B \in \mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2)$, $A \in \mathcal{B}(\mathfrak{h}_2, \mathfrak{h}_3)$ ⟵ N.B. The product of operators must be intended as a composition

iv) $\|A\|_{\mathcal{L}(h_1,h_2)} = \|A^*\|_{\mathcal{L}(h_2,h_1)}$ and $\|A\|^2_{\mathcal{L}(h_1,h_2)} = \|A^*A\|_{\mathcal{L}(h_1)} = \|AA^*\|_{\mathcal{L}(h_2)}$

Notice that the anti-linear map $^*: A \longmapsto A^*$ is continuous, since $\|A\|_{\mathcal{L}(h_1,h_2)} = \|A^*\|_{\mathcal{L}(h_2,h_1)}$.

DEF

Given $A \in \mathcal{B}(h)$ it is said

- Normal, in case $AA^* = A^*A$
- Self-adjoint, if $A^* = A$
- Unitary, whenever $A^*A = AA^* = \mathbb{1}$ (identity map $\mathbb{1}: f \longmapsto f$)

} these operators are also normal

- Orthogonal projection, if $A^2 = A = A^*$
- Non-negative, if $\exists\, B \in \mathcal{B}(h_1, h)$ s.t. $A = BB^*$

} these operators are also self-adjo

PRO

A is normal iff $\|A\psi\|_h = \|A^*\psi\|_h$, $\forall \psi \in h$

DEF

A sequence of operators $\{A_m\}_{m\in\mathbb{N}} \subseteq \mathcal{L}(h_1,h_2)$

- Converges strongly to $A \in \mathcal{L}(h_1,h_2)$, $A_m \xrightarrow[m\to+\infty]{s} A$ if $\|A_m\psi - A\psi\|_{h_2} \xrightarrow[m\to\infty]{} 0$, $\forall \psi \in \mathcal{D}(A) \subseteq \mathcal{D}(A_m)\ \forall m$
- Converges weakly to $A \in \mathcal{L}(h_1,h_2)$, $A_m \xrightarrow[m\to+\infty]{w} A$ if $A_m\psi \xrightarrow[m\to\infty]{} A\psi$, $\forall \psi \in \mathcal{D}(A) \subseteq \mathcal{D}(A_m)\ \forall m$

N.B. While A must be bounded in case $\|A_m - A\|_{\mathcal{L}(h_1,h_2)} \longrightarrow 0$, this is not the case for the strong-topology.

DEF if $K \in \mathcal{B}(h_1,h_2)$ is s.t. $\forall \{\psi_n\}_{n\in\mathbb{N}} \subseteq h_1$ satisfying $\psi_n \rightharpoonup \psi$ one has $K\psi_n \longrightarrow K\psi$, then we say that K is compact.

- Notice that $\|A_m - A\|_{\mathcal{L}(h_1,h_2)} \longrightarrow 0 \iff \sup_{\|\psi\|_{h_1}=1} \|(A_m - A)\psi\|_{h_2} \longrightarrow 0$

$\implies \|(A_m - A)\psi\|_{h_2} \longrightarrow 0\ \ \forall \psi \in h_1 \implies |\langle\varphi, (A_m - A)\psi\rangle_{h_2}| \leq \|\varphi\|_{h_2} \|(A_m-A)\psi\|_{h_2}$

$\implies$ For any fixed $\varphi \in h_1$ $\langle\varphi, (A_m - A)\psi\rangle_{h_2} \longrightarrow 0$

In other words, the convergence in operator norm implies the strong convergence which implies the weak convergence as well.

- The adjoint map is continuous w.r.t. the weak convergence

$\circledcirc$ $A_m \xrightarrow{w} A \Rightarrow \langle A_m^*\varphi, \psi\rangle_{h_1} \longrightarrow \langle\varphi, A\psi\rangle_{h_2} = \langle A^*\varphi, \psi\rangle_{h_1} \Rightarrow A_m^* \xrightarrow{w} A^*$

The same holds for the strong convergence only if $\{A_m\}$ and A are normal

$\circledcirc$ $\|(A_m - A)\psi\|_h \longrightarrow 0 \Rightarrow \|(A_m - A)^*\psi\|_h \longrightarrow 0$ (here $A \in \mathcal{B}(h)$, but can be extended in $\mathcal{L}(h)$)

DEF

$\{U(t)\}_{t\in\mathbb{R}}$ is said a strongly-continuous one-parameter unitary group if $U(t)$ is unitary and

- $U(0) = \mathbb{1}$
- $U(t+s) = U(t)U(s) = U(s)U(t)$ ($\Rightarrow U(-t) = U(t)^*$, pick $s = -t$)
- $U(t) \xrightarrow[t\to t_0]{s} U(t_0)$

To each strongly-continuous, one-parameter, unitary group can be associated a generator

$$G\psi = i \lim_{t\to 0} \frac{(U(t) - \mathbb{1})\psi}{t}$$

(the convergence occurs in the sense of the norm induced by the scalar product of the Hilbert space).

In other words, $\frac{i(U(t) - \mathbb{1})}{t} \xrightarrow[t\to 0]{s} G$.

5) Assume that $G \in \mathcal{B}(h)$. Then, one has that G is self-adjoint since

$$G^* \psi = -i \lim_{t\to 0} \frac{U(-t) - \mathbb{1}}{t} \psi = i \lim_{t'\to 0} \frac{U(t') - \mathbb{1}}{t'} \psi = G\psi, \quad \forall \psi \in h.$$

But what if $G \in \mathcal{L}(h) \setminus \mathcal{B}(h)$? Can we extend the definition of self-adjointness? Yes...

N.B. For unbounded operator we don't have BLT theorem, hence, an operator A is not ambiguous only in a given domain of definition $\mathcal{D}(A)$. We denote this operator by $A \restriction \mathcal{D}(A)$.

DEF

An operator $A \restriction \mathcal{D}(A) \in \mathcal{L}(h)$ densely defined, is said **symmetric** if

$$\langle \varphi, A\psi \rangle_h = \langle A\varphi, \psi \rangle_h, \quad \forall \varphi, \psi \in \mathcal{D}(A)$$

PRO

$A \restriction \mathcal{D}(A) \in \mathcal{L}(h)$ is symmetric iff $\langle \psi, A\psi \rangle \in \mathbb{R}$ for each $\psi \in \mathcal{D}(A)$.

DEF

Given $A \restriction \mathcal{D}(A) \in \mathcal{L}(h_1, h_2)$, $\tilde{A} \restriction \mathcal{D}(\tilde{A}) \in \mathcal{L}(h_1, h_2)$ is an **extension** of $A \restriction \mathcal{D}(A)$ (and we denote $A \subseteq \tilde{A}$) if $\tilde{A} \restriction \mathcal{D}(A) = A \restriction \mathcal{D}(A)$ and $\mathcal{D}(\tilde{A}) \supseteq \mathcal{D}(A)$.

Clearly we have $A \restriction \mathcal{D}(A) = \tilde{A} \restriction \mathcal{D}(\tilde{A})$ only when $\tilde{A} \subseteq A$ and $A \subseteq \tilde{A}$.

DEF

Given $T \restriction \mathcal{D}(T) \in \mathcal{L}(h_1, h_2)$ we say that $T \restriction \mathcal{D}(T)$ is **closed** if $\forall \{\psi_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ with $\|\psi_n - \psi\|_{h_1} \to 0$, the fact $\|T\psi_n - \varphi\|_{h_2} \to 0$ implies $\psi \in \mathcal{D}(T)$ and $\varphi = T\psi$.

In other words, $T \restriction \mathcal{D}(T)$ is closed if for any convergent sequence in h_1 that makes convergent the sequence $\{T\psi_n\}_{n\in\mathbb{N}} \subset h_2$ in h_2, one finds the limit of ψ_n (that a priori is only in h_1) is in $\mathcal{D}(T)$.

DEF

An operator $A \restriction \mathcal{D}(A) \in \mathcal{L}(h_1, h_2)$ is **closable** if there exists an extension $\tilde{A} \restriction \mathcal{D}(\tilde{A}) \in \mathcal{L}(h_1, h_2)$ that is closed. Moreover, $\bar{A} \restriction \mathcal{D}(\bar{A}) \in \mathcal{L}(h_1, h_2)$ is the **closure** of $A \restriction \mathcal{D}(A)$, i.e. its "smallest" closed extension, namely, $A \subseteq \bar{A}$ and $\overline{\mathcal{G}(A)} = \mathcal{G}(\bar{A})$, where $\mathcal{G}(\cdot)$ is the graph

$$\mathcal{G}(A) := \{(\psi, A\psi) \mid \psi \in \mathcal{D}(A)\}$$

PRO

Any closed operator has closed graph and closed kernel.

N.B. Here the topology that selects the closed sets is the one induced by the so-called **graph-norm**, i.e.

$$\|\psi\|^2_{\mathcal{G}(A)} := \|\psi\|^2_{h_1} + \|A\psi\|^2_{h_2}, \quad \psi \in \mathcal{D}(A).$$

DEF

Given $A \restriction \mathcal{D}(A) \in \mathcal{L}(h_1, h_2)$, its **adjoint** is defined by

$$\begin{cases} \mathcal{D}(A^*) = \{\psi \in h_2 \mid \exists\, \Psi \in h_2 : \langle \psi, A\varphi \rangle = \langle \Psi, \varphi \rangle, \ \forall \varphi \in \mathcal{D}(A)\} \\ A^*\psi = \Psi \end{cases}$$

▶ This definition is well-posed only if $\mathcal{D}(A)$ is dense in h_1. Otherwise there would be a subspace of h_1 orthogonal to $\mathcal{D}(A)$ and one could replace $\Psi = A^*\psi$ with any $\Psi + \varphi_\perp$, where $\varphi_\perp \in \mathcal{D}(A)^\perp$.

PRO

For any densely defined $A \restriction \mathcal{D}(A) \in \mathcal{L}(h_1, h_2)$, the operator $A^* \restriction \mathcal{D}(A^*) \in \mathcal{L}(h_2, h_1)$ is closed.

PRO

Given $A\restriction\mathcal{D}(A)\in\mathcal{L}(h_1,h_2)$ densely defined,

i) $(\alpha A)^*=\alpha^* A^*$, $\forall\alpha\in\mathbb{C}$;

ii) $(A+B)^*\supseteq A^*+B^*$, $(AB)^*\supseteq B^*A^*$

iii) $Ker(A^*)=Ran(A)^\perp$;

iv) $A^{**}=\bar{A}$, in case A is closable (iff A^* is densely defined). Moreover, $(\bar{A})^*=A^*$;

v) If $Ker(A)=\{0\}$ and $Ran(A)$ is dense in h_2, then $(A^*)^{-1}=(A^{-1})^*$;

vi) If A is closable and $Ker(\bar{A})=\{0\}$, then $(\bar{A})^{-1}=\overline{(A^{-1})}$.

N.B. $\mathcal{D}(A+B)=\mathcal{D}(A)\cap\mathcal{D}(B)$

- Observe that, given $B\supseteq A$, densely defined, the condition in the domain of the adjoint must hold in a larger set for B with respect to A, hence $A^*\supseteq B^*$ the domain is smaller.
- Let $A\restriction\mathcal{D}(A)\in\mathcal{L}(h)$ be a symmetric operator. There holds

$$A\subseteq\bar{A}\subseteq A^*.$$

The operator $A\restriction\mathcal{D}(A)$ is self-adjoint when $A=A^*$ and this occurs when there are no proper extension. Indeed suppose $A\restriction\mathcal{D}(A)$, $B\restriction\mathcal{D}(B)$ symmetric operators s.t. $A=A^*$ and $B\supseteq A$. Then, $A\subseteq B\subseteq B^*\subseteq A^*=A \Rightarrow A=B$.

DEF

If a symmetric operator $A\restriction\mathcal{D}(A)\in\mathcal{L}(h)$ is s.t. $\bar{A}$ is self-adjoint, then A is said to be **essentially self-adjoint**.

THEO (Criterion for (essentially) self-adjointness)

Let $A\restriction\mathcal{D}(A)\in\mathcal{L}(h)$ be a symmetric operator. Then

i) if $\exists z\in\mathbb{C}$: $Ker(A^*+z^*)=Ker(A^*+z)=\{0\}$, $A\restriction\mathcal{D}(A)$ is essentially self-adjoint;

ii) if $\exists z\in\mathbb{C}$: $Ran(A+z^*)=Ran(A+z)=h$, $A\restriction\mathcal{D}(A)$ is self-adjoint.

N.B. The theorem is equivalently phrased for essentially self-adjoint operators by exploiting $\overline{Ran(A)}=Ker(A^*)^\perp$.

DEF

Given $A\restriction\mathcal{D}(A)\in\mathcal{L}(h)$ and $B\restriction\mathcal{D}(B)\in\mathcal{L}(h)$, with $\mathcal{D}(B)\supseteq\mathcal{D}(A)$, we say that B is A-bounded if there exist $a\geq0$ and $b>0$ s.t.

$$\|B\psi\|_h\leq a\|A\psi\|_h+b\|\psi\|_h,\quad\forall\psi\in\mathcal{D}(A)$$

N.B. a is said A-bound. Clearly, if $a=0$ then B is simply bounded

THEO (Kato-Rellich)

Given $A\restriction\mathcal{D}(A)\in\mathcal{L}(h)$ self-adjoint and $B\restriction\mathcal{D}(B)\in\mathcal{L}(h)$ symmetric, A-bounded with A-bound $a<1$, then the operator $A+B\restriction\mathcal{D}(A)$ is self-adjoint.

DEF

- $A\restriction\mathcal{D}(A)\in\mathcal{L}(h)$ self-adjoint is **non-negative** if $\langle\psi,A\psi\rangle_h\geq0$ $\forall\psi\in h$.
- Moreover, given $B\restriction\mathcal{D}(B)$ A-bounded, with A-bound <1, $A\geq B$ stands for $A-B$ non-negative
- For the criterion of (essentially) self-adjointness, z can be chosen in $(-\infty,a)$ if $A\restriction\mathcal{D}(A)$ satisfies $A\geq a$ (that's to say $A-a\mathbb{1}\geq0$)

7) Now we can give sense to the generator H (self-adjoint operator) of the dynamics of a system, represented by any strongly-continuous one-parameter unitary group $\{U(t)\}_{t\in\mathbb{R}}$:

N.B. $H = \left(\frac{d}{dt}U(t)\right)\Big|_{t=0}$

$$\frac{\mathbb{1} - U(t)}{it/\hbar} \xrightarrow[t\to 0]{s} H.$$

In this situation H is called the Hamiltonian of the system.

Let $\psi_0 \in \mathcal{D}(H)$ be the initial state. Then $i\hbar\frac{d}{dt}\psi(t) = H\psi(t)$, the **Shrödinger equation** is solved by $\psi(t) = U(t)\psi_0$.

- In case instead one considers the Hamiltonian as a fundamental object (rather than the dynamics) one solves the same Shrödinger equation by imposing $\psi(t) = e^{-iHt/\hbar}\psi_0$.
$\Rightarrow U(t) = e^{-iHt/\hbar}$

N.B. Two different dynamics with the same generator can only differ by a phase, e.g.

$$U(t) = e^{-iHt/\hbar}, \quad U'(t) = e^{-iH(t-t')/\hbar}$$

DEF
An **observable** is a maximally-defined, symmetric operator, hence, a self-adjoint operator.

THEO (Closed Graph)
$A \upharpoonright \mathfrak{h}_1 \in \mathcal{L}(\mathfrak{h}_1, \mathfrak{h}_2)$. Then,
$A \in \mathcal{B}(\mathfrak{h}_1, \mathfrak{h}_2) \iff A$ is closed.

THEO (Hellinger-Toeplitz)
$A \upharpoonright \mathfrak{h} \in \mathcal{L}(\mathfrak{h})$ is symmetric. Then, $A \in \mathcal{B}(\mathfrak{h})$.

DEF
Let $A \upharpoonright \mathcal{D}(A) \in \mathcal{L}(\mathfrak{h})$ fulfils $A \geq \gamma \in \mathbb{R}$. Next, define $\|\cdot\|_A$ as the norm induced by the scalar product
$\langle \varphi, \psi \rangle_A := \langle \varphi, (A + 1 - \gamma)\psi \rangle_{\mathfrak{h}}$, $\forall \varphi, \psi \in \mathcal{D}(A)$.
Then, the completion of $\mathcal{D}(A)$ according to $\|\cdot\|_A$ is said **form domain** of A, $\mathcal{Q}(A)$.

Example
Consider $-\Delta \upharpoonright H^2(\mathbb{R}^n) \in \mathcal{L}(L^2(\mathbb{R}^n))$. One has $-\Delta \geq 0$, hence $\gamma = 0$ and since $\langle \varphi, -\Delta\psi \rangle_{L^2(\mathbb{R}^n)} = \langle \nabla\varphi, \nabla\psi \rangle$
$\Rightarrow \mathcal{Q}(-\Delta) = H^1(\mathbb{R}^n) \supsetneq H^2(\mathbb{R}^n)$

THEO (Friedrichs's extensions)
Let $A \upharpoonright \mathcal{D}(A) \in \mathcal{L}(\mathfrak{h})$ be symmetric and such that $\langle \psi, A\psi \rangle_{\mathfrak{h}} \geq \gamma \|\psi\|^2_{\mathfrak{h}}$, $\forall \psi \in \mathcal{D}(A)$.
Then $\exists!$ self-adjoint extension $\tilde{A} \supseteq A$ s.t. $\tilde{A} \geq \gamma$ and $\mathcal{D}(\tilde{A}) \subseteq \mathcal{Q}(A)$.

It is interesting to understand when a given quadratic form is associated to a self-adjoint operator (i.e. an observable). Hence, given a sesquilinear form $s: \mathcal{Q} \times \mathcal{Q} \to \mathbb{C}$ we can always associate a quadratic form $q: \mathcal{Q} \to \mathbb{C}$, by setting $q(\psi) = s(\psi, \psi)$ and the converse can be done through the polarization identity. If $q: \mathcal{Q} \to \mathbb{R}$, then s is symmetric ($s(\varphi,\psi) = \overline{s(\psi,\varphi)}$) and q is said **hermitian**.

N.B. Here, $\mathcal{Q}$ is a dense subspace of $\mathfrak{h}$

DEF
- A hermitian quadratic form is **lower-bounded** if $\exists \gamma \in \mathbb{R}: \; q(\psi) \geq \gamma \|\psi\|^2_{\mathfrak{h}}$, $\forall \psi \in \mathcal{Q}$.
It is non-negative if $\gamma = 0$.
- It is closed if the closure of $\mathcal{Q}$ with respect to $\|\cdot\|_q := q(\cdot) + (1-\gamma)\|\cdot\|^2_{\mathfrak{h}}$ is equal to itself.

In other words, q is closed in $\mathfrak{h}$ if $\forall \{\psi_m\}_{m\in\mathbb{N}} \subseteq \Omega$ Cauchy sequence wrt $\|\cdot\|_q$ one has

$$\|\psi_m\|_{\mathfrak{h}} \to 0 \implies \|\psi_m\|_q \to 0 .$$

DEF
Given $A \upharpoonright \mathcal{D}(A) \in \mathcal{L}(\mathfrak{h})$ with $A \geq \gamma \in \mathbb{R}$ we set $q_A : \mathcal{D}(A) \to \mathbb{R}$ as $q_A(\psi) = \langle \psi, A\psi \rangle_{\mathfrak{h}}$.
Another quadratic form $q : \Omega \to \mathbb{R}$ with $\Omega \supseteq \mathcal{D}(A)$ is called **relatively form bounded** wrt q_A if $\exists\, a \geq 0$ and $b > 0$ s.t.

$$|q(\psi)| \leq a\, q_{A-\gamma}(\psi) + b\|\psi\|^2_{\mathfrak{h}}, \quad \forall \psi \in \mathcal{D}(A).$$

PRO
Let $q_1 : \Omega_1 \to \mathbb{R}$ be a closed, hermitian quadratic form that is bounded from below and let $q_2 : \Omega_2 \to \mathbb{R}$ be relatively form bounded wrt q_1 with form bound less than 1.
Then, $q_1 + q_2 : \Omega_1 \to \mathbb{R}$ is closed.

THEO (KLMN - Kato, Lions, Lax, Milgram, Nelson)
Let $q : \Omega \to \mathbb{R}$ be a lower-bounded and close, hermitian quadratic form.
Then $\exists!$ self-adjoint operator A such that $q(\psi) = \langle \psi, A\psi \rangle_{\mathfrak{h}}$ $\forall \psi \in \mathcal{D}(A)$.
Denote by s the sesquilinear form associated to q by the polarization identity, so that

$$\mathcal{D}(A) = \{\psi \in \Omega \mid \exists\, \Psi \in \mathfrak{h} : s(\varphi, \psi) = \langle \varphi, \Psi \rangle_{\mathfrak{h}} \ \forall \varphi \in \Omega\}$$

$$A\psi = \Psi .$$

COR
If $q : \Omega \to \mathbb{R}$ is also bounded (i.e. $|q(\psi)| \leq c\|\psi\|^2_{\mathfrak{h}}$ $\forall \psi \in \Omega$ with $\|q\| := \sup_{\|\psi\|_{\mathfrak{h}}=1} |q(\psi)|$)
then, the associated self-adjoint operator is also bounded, with $\|A\|_{\mathcal{L}(\mathfrak{h})} = \|q\|$

N.B.
This implies that, for symmetric operators, one has

$$\|A\|_{\mathcal{L}(\mathfrak{h})} = \sup_{\|\psi\|=1} |\langle \psi, A\psi \rangle|$$

DEF
Let $A \upharpoonright \mathcal{D}(A) \in \mathcal{L}(\mathfrak{h}_1, \mathfrak{h}_2)$ be a closed, densely defined operator.

- The **resolvent set** is
$$\rho(A) := \{z \in \mathbb{C} \mid (A-z)^{-1} \in \mathcal{B}(\mathfrak{h}_2, \mathfrak{h}_1)\}$$
- The **spectrum** is
$$\sigma(A) = \rho(A)^c = \mathbb{C} \setminus \rho(A).$$

N.B.
By the closed graph theorem we know that the closed operator $(A-z)^{-1}$ is bounded if it is defined on all $\mathfrak{h}_2$. Hence one needs to check bijectivity.

DEF

- Given $\psi \in \mathfrak{h}_1$ with $\psi \neq 0$ satisfying
$\mathrm{Ker}(A-z) \ni \psi$, we say that ψ is a eigenfunction of A with eigenvalue z.
- The map $R_A : \rho(A) \longrightarrow \mathcal{B}(\mathfrak{h}_2, \mathfrak{h}_1)$, $z \longmapsto (A-z)^{-1}$ is the resolvent operator of A.

it fulfils $R_A(z)^* = R_{A^*}(z^*)$ and the so-called first-resolvent identity

$$R_A(z_1) - R_A(z_2) = (z_1 - z_2) R_A(z_1) R_A(z_2) = (z_1 - z_2) R_A(z_2) R_A(z_1).$$

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PRO

N.B. The spectrum is a closed set in $\mathbb{C}$.

- The resolvent set is open in $\mathbb{C}$;
- $\|R_A(z)\|_{\mathcal{L}(h_2,h_1)} \geq \dfrac{1}{\text{dist}(z, \sigma(A))}$, $\forall z \in \rho(A)$;
- $\sum_{j=0}^{m} (z-z_0)^j R_A(z_0)^{j+1} \xrightarrow[m\to+\infty]{s} R_A(z)$, $\forall z, z_0 \in \rho(A)$: $\|R_A(z_0)\|_{\mathcal{L}(h_2,h_1)} > \dfrac{1}{|z-z_0|}$;
- if A is bounded, $\{z \in \mathbb{C} \mid \|A\|_{\mathcal{L}(h_1,h_2)} < |z|\} \subset \rho(A)$ and $R_A(z) = -\sum_{j\in\mathbb{N}_0} \dfrac{A^j}{z^{j+1}}$.

DEF

Given $A \upharpoonright D(A) \in \mathcal{L}(h)$ we say that $\{\psi_n\}_{n\in\mathbb{N}} \subset D(A)$ with $\|\psi_n\|_h = 1$ $\forall n$ is a **Weyl sequence** for A :

$$\|(A-z)\psi_n\|_h \xrightarrow[n\to\infty]{} 0, \text{ for some } z \in \mathbb{C}.$$

PRO

If there exists a Weyl sequence for A with parameter $z \in \mathbb{C}$, then one has $z \in \sigma(A)$. Moreover, if $\{\psi_n\}_{n\in\mathbb{N}}$ does not contain any convergent subsequence, the $\{\psi_n\}_{n\in\mathbb{N}}$ is called a **singular Weyl sequence** and $z \in \sigma_{ess}(A)$.

Notice that $\sigma(A) = \sigma_{disc}(A) \cup \sigma_{ess}(A)$, where $\sigma_{disc}(A)$ is made of eigenvalues with finite multiplicity.

THEO (Weyl)

Let $A \upharpoonright D(A) \in \mathcal{L}(h)$ be self-adjoint and K a compact, self-adjoint operator. Then

$\sigma_{ess}(A+K) = \sigma_{ess}(A)$.

PRO

- Let $A \upharpoonright D(A) \in \mathcal{L}(h)$ be injective with dense range.
  Then, $\sigma(A^{-1}) \setminus \{0\} = (\sigma(A) \setminus \{0\})^{-1}$ and $A\psi = z\psi \iff A^{-1}\psi = z^{-1}\psi$.
- Let $A \in \mathcal{B}(h)$. Then, $\sigma(AA^*) \setminus \{0\} = \sigma(A^*A) \setminus \{0\}$.

THEO

Let $A \upharpoonright D(A) \in \mathcal{L}(h)$ be a symmetric operator. Then

i) A is self-adjoint $\iff \sigma(A) \subseteq \mathbb{R}$. Moreover, $\|R_A(z)\|_{\mathcal{L}(h)} \leq \dfrac{1}{|\operatorname{Im} z|}$;

ii) $A \geq \gamma \in \mathbb{R} \iff \sigma(A) \subseteq [\gamma, +\infty)$. Moreover, $\|R_A(z)\|_{\mathcal{L}(h)} \leq \dfrac{1}{\gamma - \lambda}$, $\lambda < \gamma$;

iii) All eigenvalues are real and corresponding eigenfunctions are orthogonal.

iv) $\inf \sigma(A) = \inf_{\substack{\psi \in D(A) \\ \|\psi\|_h = 1}} \langle \psi, A\psi \rangle$, $\sup \sigma(A) = \sup_{\substack{\psi \in D(A) \\ \|\psi\|_h = 1}} \langle \psi, A\psi \rangle$ provided A self-adjoint.