Self-Force in Scalar-Tensor Theories of Gravity: Perturbative Approach **Beyond Linear Order**

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Collaborators: Thomas Sotiriou and Andrea Maselli

University of Nottingham

22nd September 2023

Overview

1. **Why** we desire high accuracy scalar-tensor self-force waveforms

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2. Derive field equations and equations of motion up to second-order

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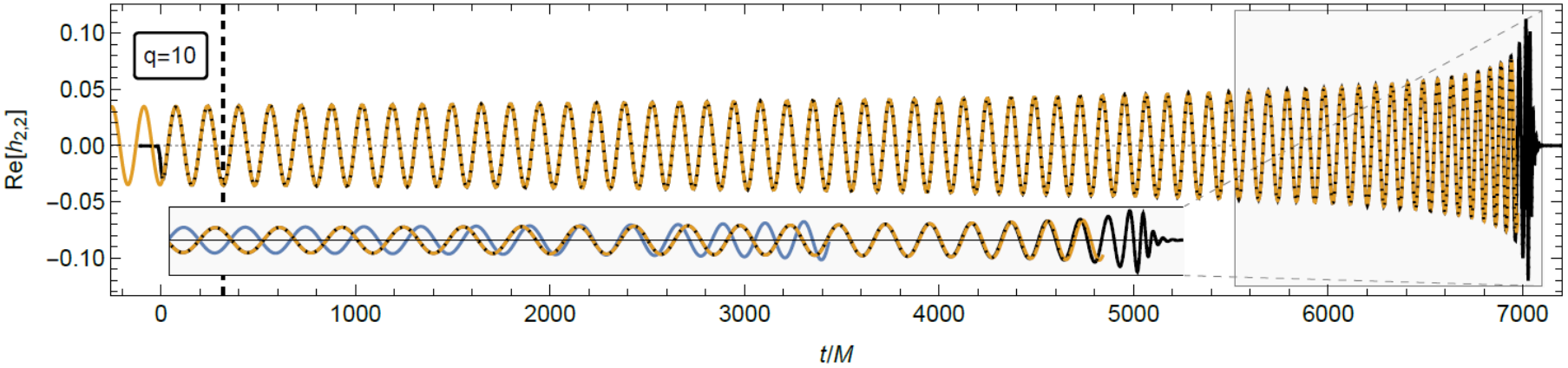
2. Derive field equations and equations of motion up to second-order
3. How easy will implementation be?

1st post-adiabatic (PA) waveforms in GR



[Wardell et al. PRL. 130.24 (2023): 241402]

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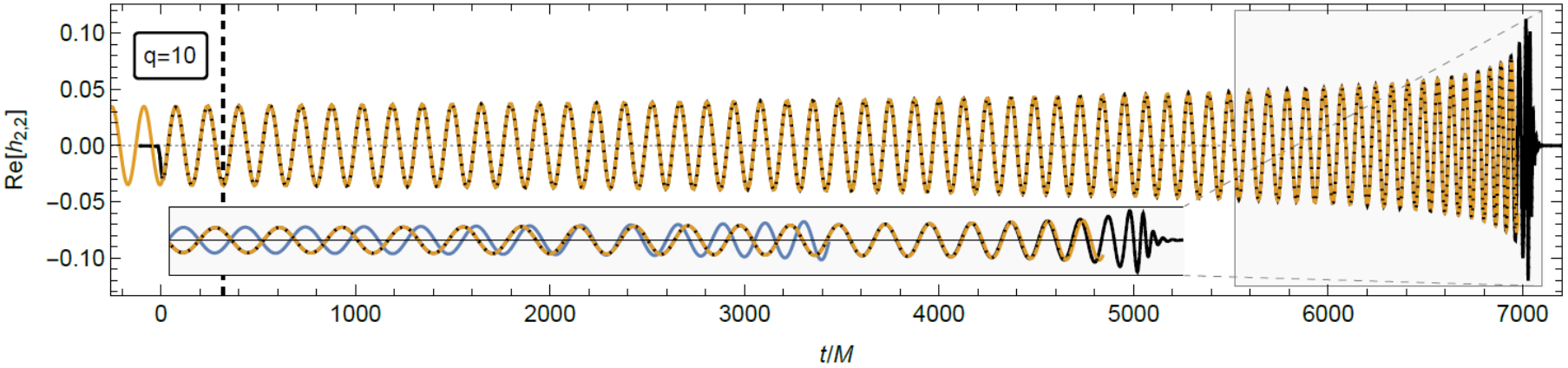

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- **But**, to test GR we need waveforms to compare to...

Same methods

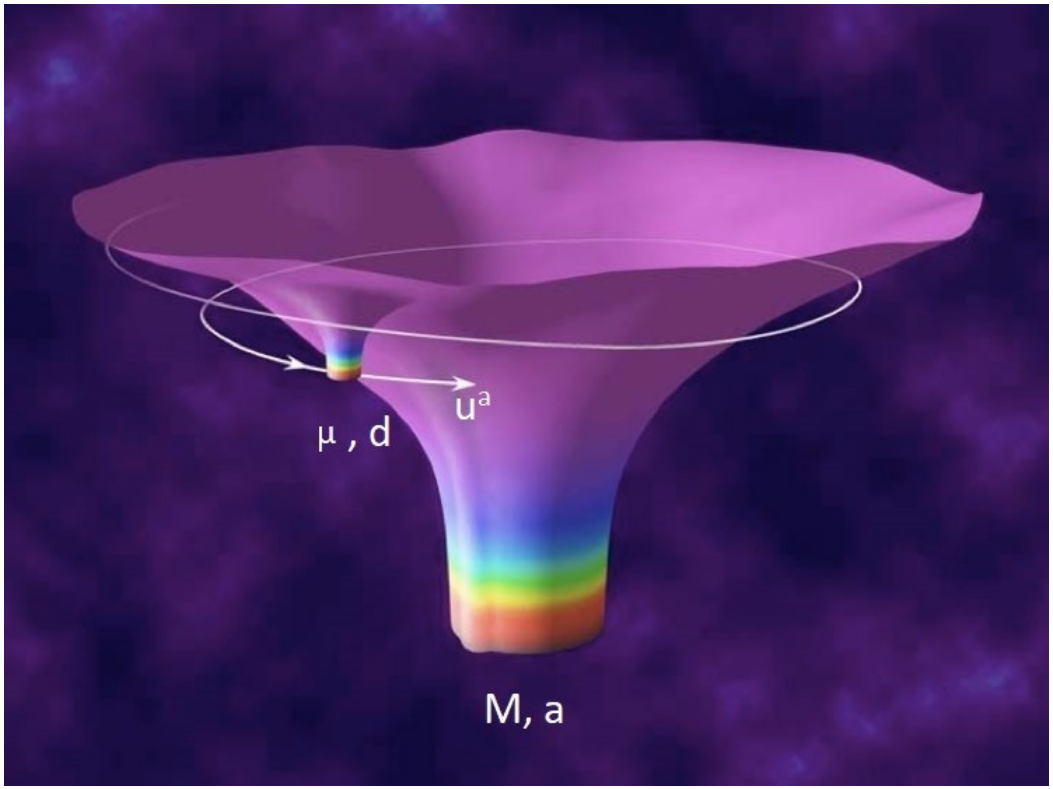
Same accuracy

New theories

New waveforms

Perturbation theory and self-force

[Source: NASA website]



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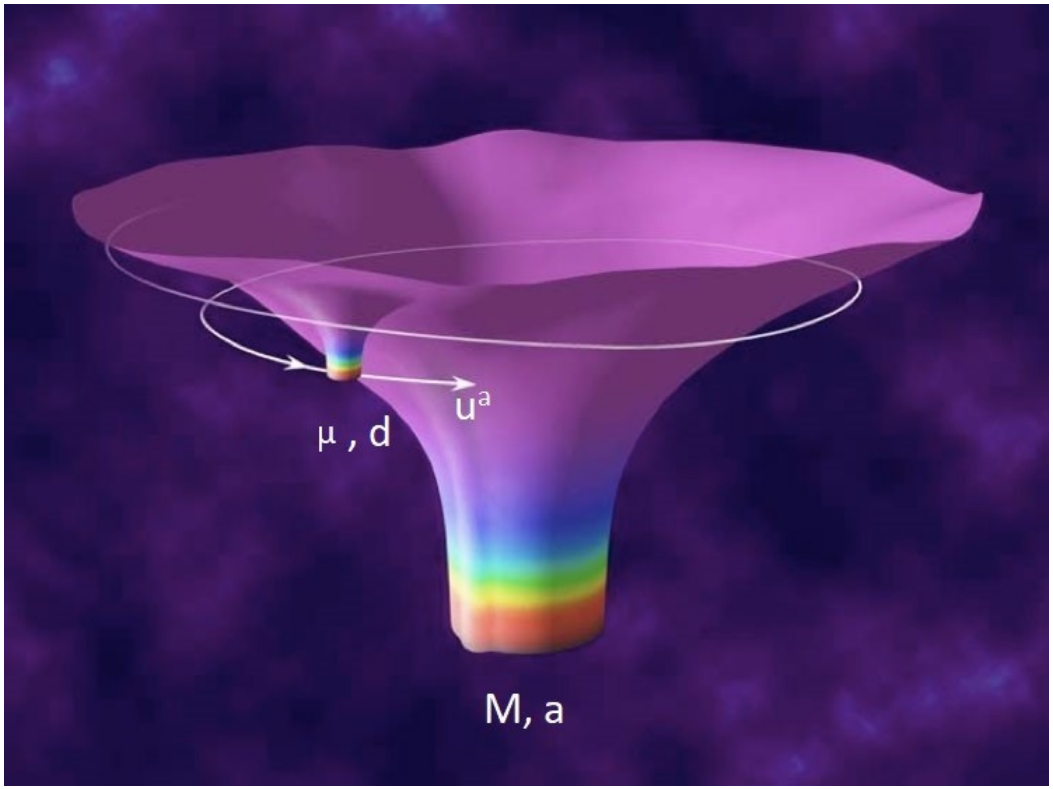

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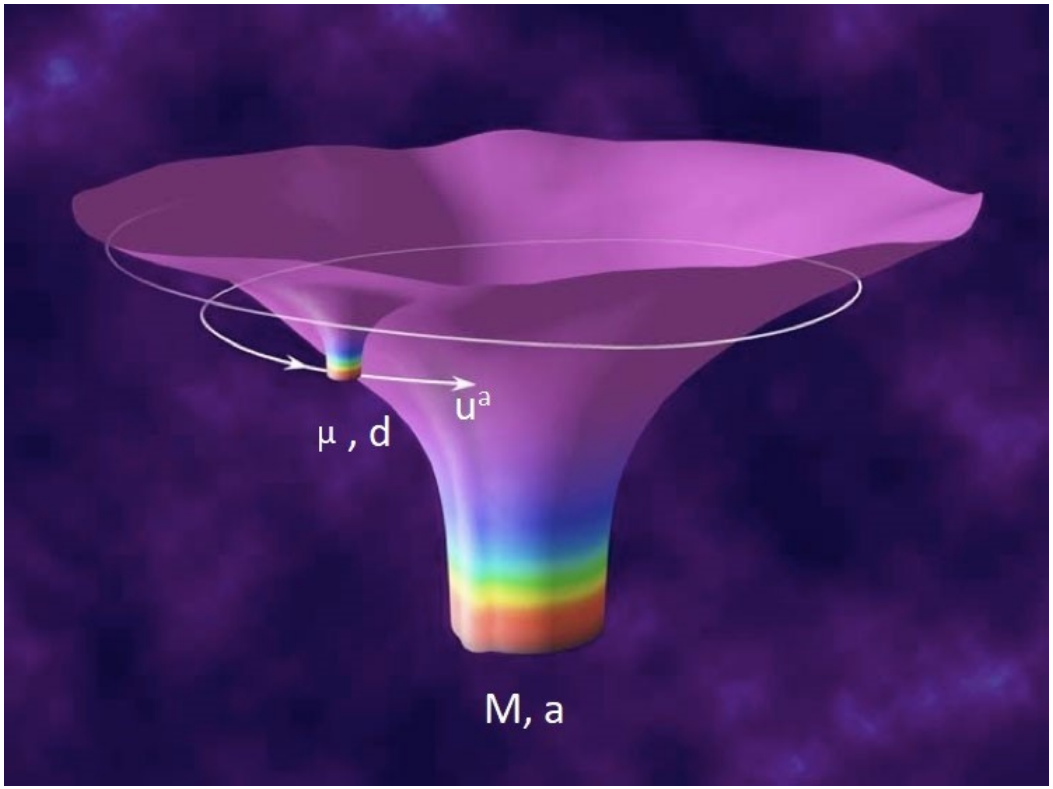

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$$\varepsilon \sim \frac{\mu}{M}$$

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$$
\begin{aligned}
g_{ab} =& g^{(0)}_{ab} + \varepsilon^1 h^{(1)}_{ab} + \varepsilon^2 h^{(2)}_{ab} + \mathcal{O}(\varepsilon^3), \\
\varphi =& \qquad \varepsilon^1 \varphi^{(1)} + \varepsilon^2 \varphi^{(2)} + \mathcal{O}(\varepsilon^3),
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Self-force from the action in GR

$$S[\mathbf{g}_{ab}, \Psi] = S_0[\mathbf{g}_{ab}] + S_{\mathrm{m}}[\mathbf{g}_{ab}, \Psi],$$

[Detweiler & Whiting, PRD 67, 024025 (2003)],
[Detweiler, PRD 85, 044048 (2012)],
[Upton & Pound, PRD 103, 124016 (2021)]

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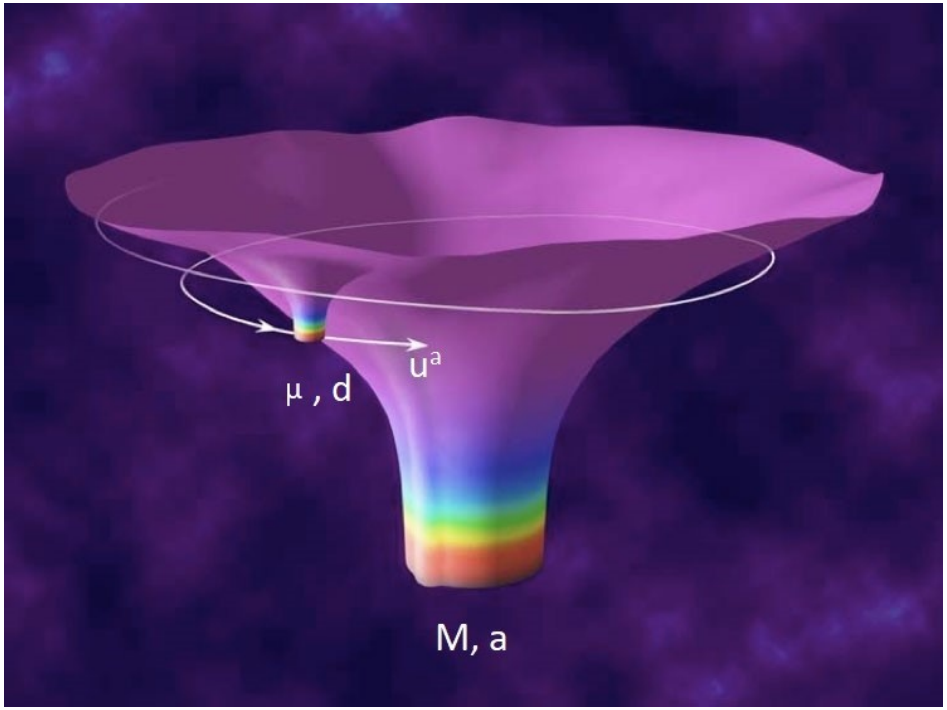

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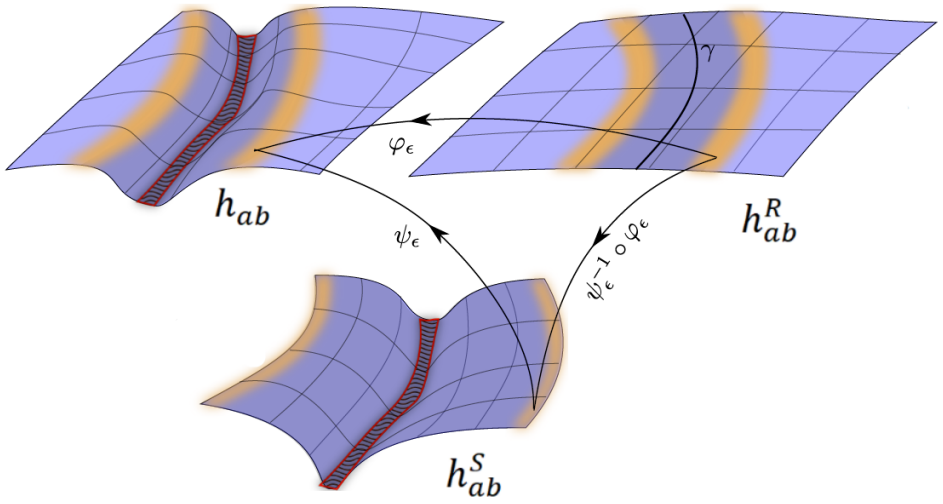

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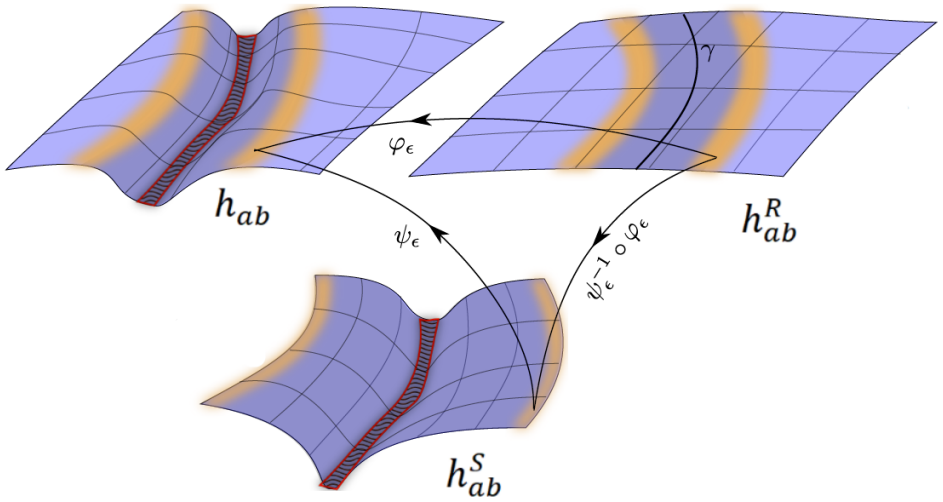

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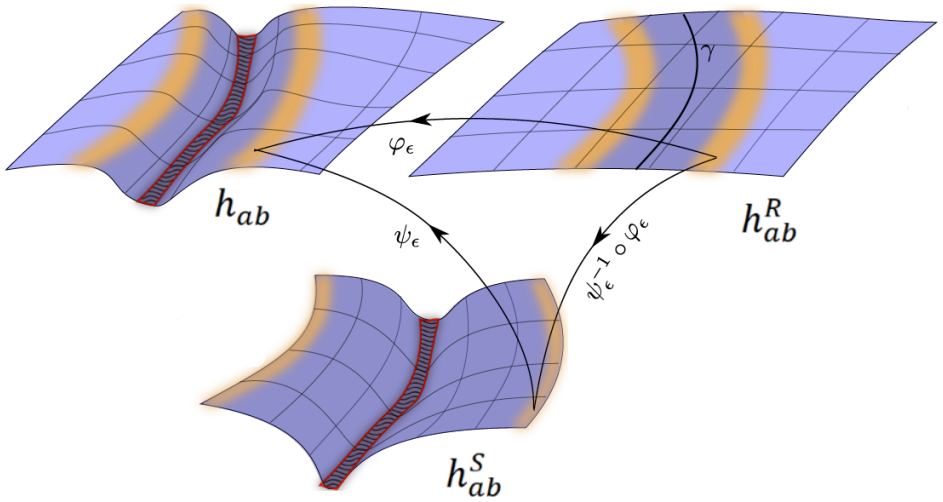

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Perturbative expansion:

$$\delta G_{ab}[h^{(1)}_{cd}] = T^{(1)}_{ab}$$
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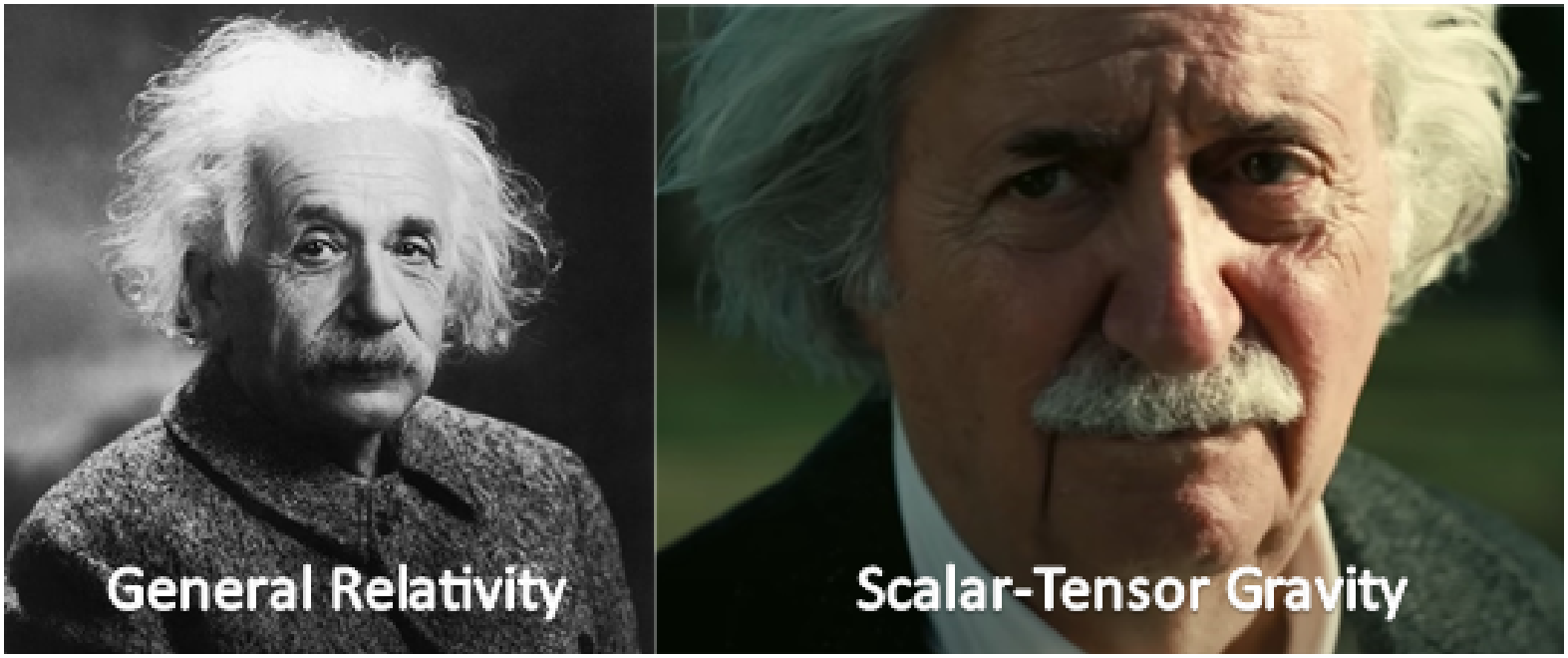
[Mino & Tanaka, PRD 55, 3457 (1997)], [Quinn & Wald, PRD 56, 3381 (1997)],

[Pound, PRL 109, 051101 (2012)], [Gralla, PRD 85, 124011 (2012)]

Scalar-tensor theories of gravity



Scalar-tensor action

$$S[\mathbf{g}_{ab}, \varphi, \Psi] = S_0[\mathbf{g}_{ab}, \varphi] + \alpha S_{\mathrm{c}}[\mathbf{g}_{ab}, \varphi] + S_{\mathrm{m}}[\mathbf{g}_{ab}, \varphi, \Psi],$$

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Non trivial φ and $\mathbf{g}_{ab}$ coupling: $\alpha S_{\mathrm{c}}[\mathbf{g}_{ab}, \varphi]$

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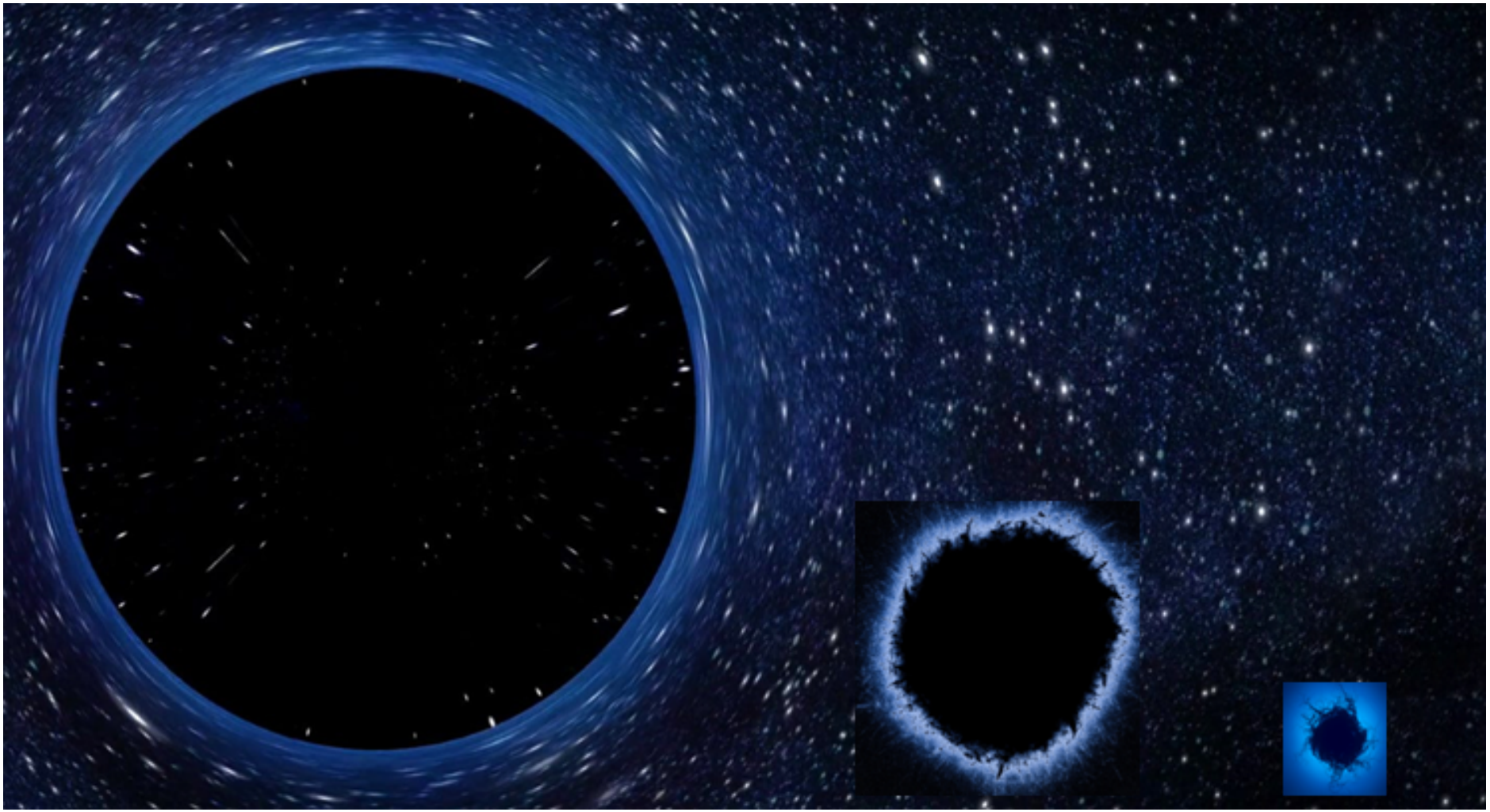
$$\Rightarrow \alpha S_{\mathrm{c}}[\mathbf{g}_{ab}, \varphi] \approx \mathcal{O}\left(\varepsilon^3\right)$$

The Bigger the Bolder



[Keck, Caltech. Getty Images. Andriy_A / Shutterstock.]

Point scalar charge action

$$S_{\mathrm{m}} = -\int_\gamma m[\varphi]\sqrt{\mathbf{g}_{ab}\mathbf{u}^a\mathbf{u}^b}d\tau$$

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- But, no part more challenging than GR


Every Barbie Career
Beauty Inside A Box


Every Theory Model
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3. Implementation builds on GR calculation (and *no more* difficult)