Oligochromatic extreme mass-ratio inspirals (E-EMRIs)

Pau Amaro Seoane

GSSI, 21 Sep 2023

Universitat Politècnica de València

https://astro-gr.org
amaro@upv.es

Extreme-Mass Ratio Inspirals

- Stellar-mass object spiraling into $10^4 - 10^6\ M_\odot$.
- This range of masses corresponds to relaxed nuclei.
- With LISA z $\sim$ 1, 4.

  [Amaro-Seoane 2018, Babak et al +Amaro-Seoane 2017, Amaro-Seoane et al 2007]
- Rates are very low: 10^{-5}, 10^{-6} per year.
  (stellar-mass BHs and MW)
- Take into account the impact of asymmetry between pro- and retrograde orbits in the location of the LSO helps, if MBH is Kerr.

  [Amaro-Seoane, Sopuerta & Freitag 2013]
- In any case, we don't expect EMRIs at the Galactic Centre, right?

Not really.

Early-stage EMRIs, E-EMRIs

- The evolution of an EMRI **can be divided in three stages.**

Early-stage EMRIs, E-EMRIs

- The evolution of an EMRI **can be divided in three stages.**

(1) Monochromatic EMRIs

Early-stage EMRIs, E-EMRIs

- The evolution of an EMRI **can be divided in three stages.**

(1) Monochromatic EMRIs

$\sim 10^5$**yrs away from merger**. We can consider them mono.

Early-stage EMRIs, E-EMRIs

- The evolution of an EMRI **can be divided in three stages.**

(1) Monochromatic EMRIs

$\sim 10^5$**yrs away from merger**. We can consider them mono.

(2) Oligochromatic EMRIs

Early-stage EMRIs, E-EMRIs

- The evolution of an EMRI **can be divided in three stages.**

(1) Monochromatic EMRIs

$\sim 10^5$**yrs away from merger**. We can consider them mono.

(2) Oligochromatic EMRIs

$\sim 10^{3-4}$**yrs away from merger**. Will cover a very short frequency range.

Early-stage EMRIs, E-EMRIs

- The evolution of an EMRI **can be divided in three stages.**

(1) Monochromatic EMRIs

$\sim 10^5$**yrs away from merger**. We can consider them mono.

(2) Oligochromatic EMRIs

$\sim 10^{3-4}$**yrs away from merger**. Will cover a very short frequency range.

(3) Polychromatic EMRIs

Early-stage EMRIs, E-EMRIs

- The evolution of an EMRI **can be divided in three stages.**

(1) Monochromatic EMRIs

$\sim 10^5$**yrs away from merger**. We can consider them mono.

(2) Oligochromatic EMRIs

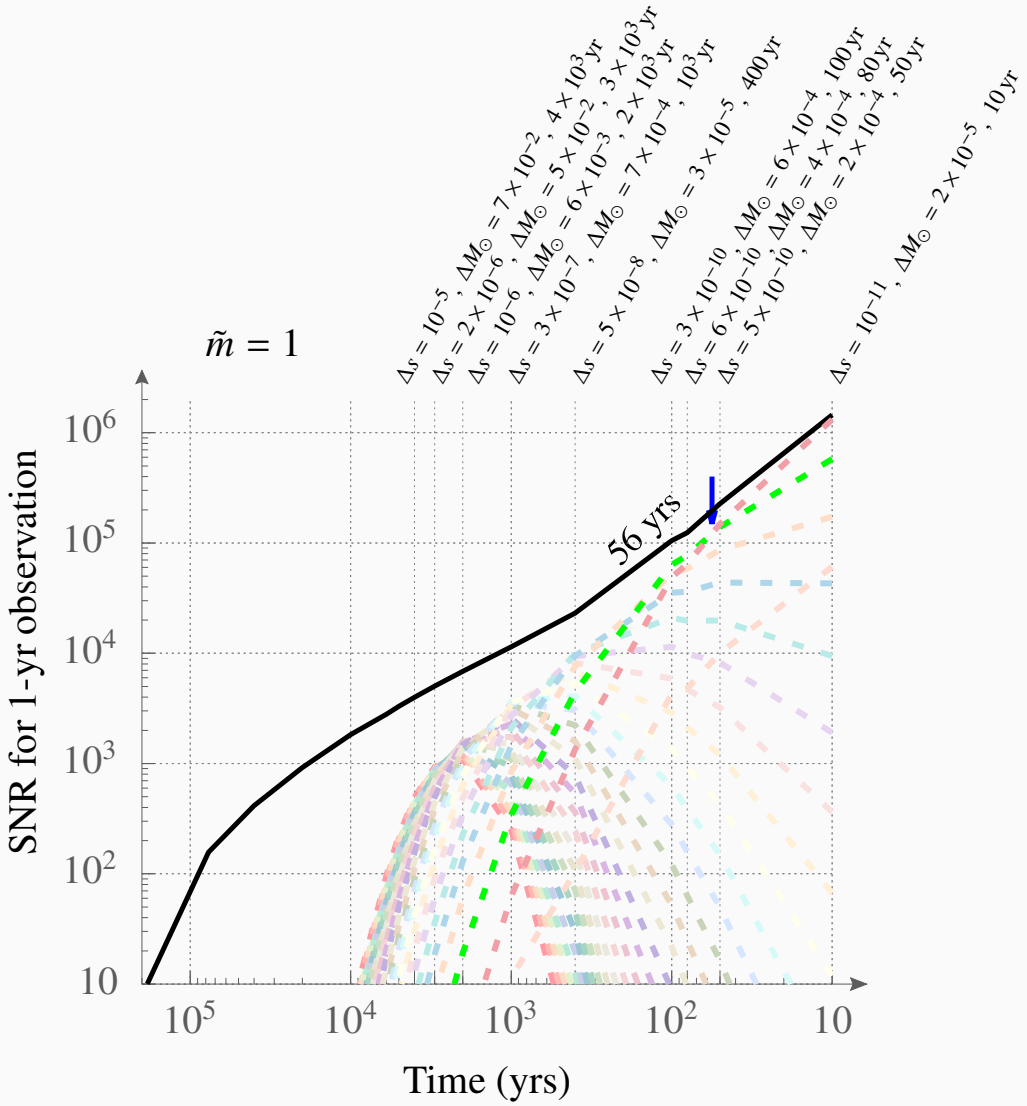
$\sim 10^{3-4}$**yrs away from merger**. Will cover a very short frequency range.

(3) Polychromatic EMRIs

"the ones we have been talking about all along", (using Bernard's words).

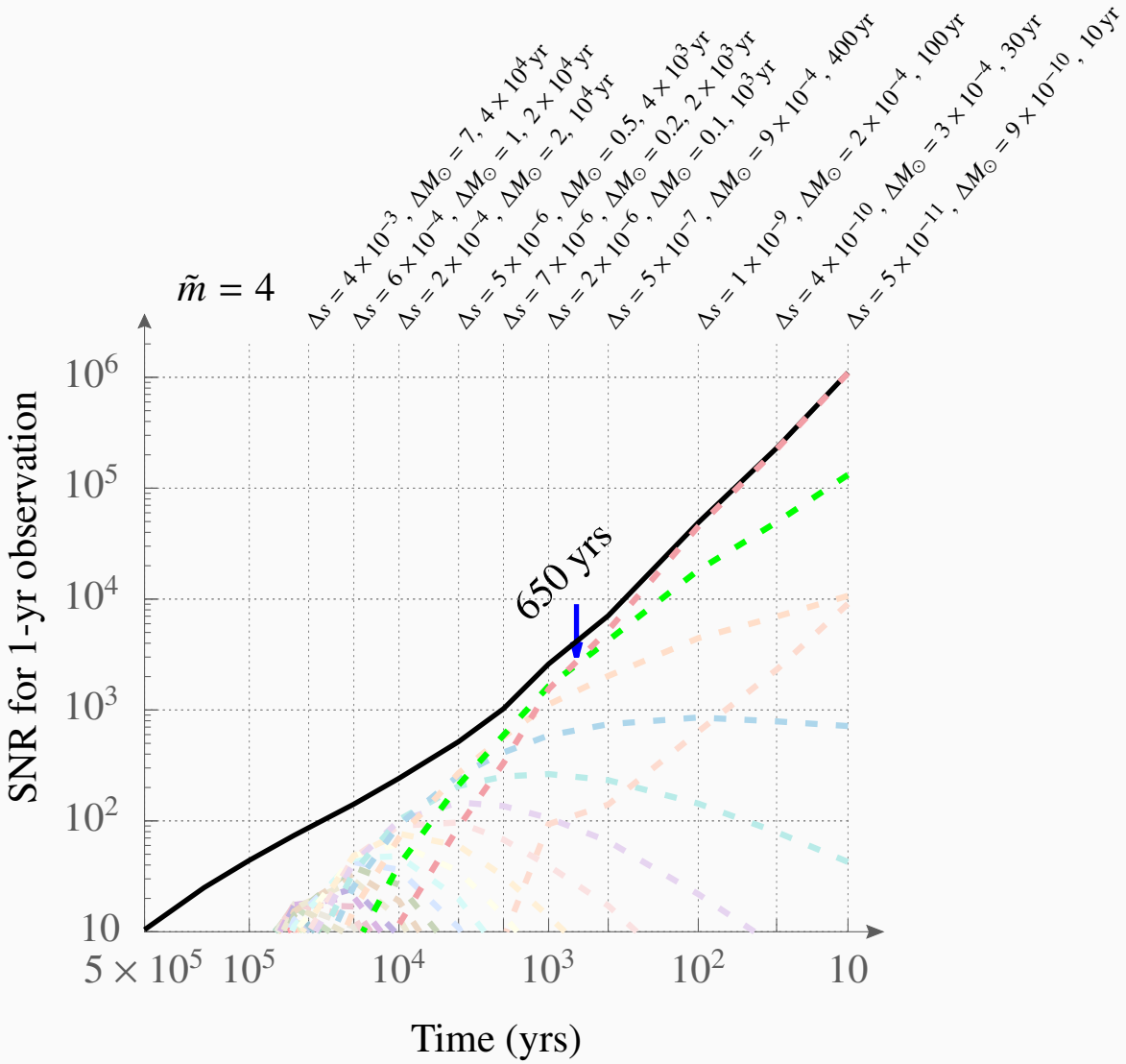
Humour me...

Evolution of an EMRI in the early stages



185000 yr before plunge, an E-EMRI would be already on band with SNR$>$ 10. Waveforms à la Barack and Cutler.

Evolution of an EMRI in the early stages



500000 yr before plunge, an E-EMRI would be already on band with SNR$>$ 10 *[PAS, Lin & Tzanavaris (TBS)]*

Fine… and the event rate?

Event rate calculation: Almost copy and paste from AS 2019

- The event rate in phase-space can be calculated as follows

$$\dot{\Gamma}_{\rm EMRI} \simeq \int_{a_{\rm min}}^{a_{\rm crit}} \frac{dn_{\rm bh}(a)}{T_{\rm rlx}(a)\ \ln\left(\theta_{\rm lc}^{-2}\right)}$$

- We need to determine **four quantities**

1. The loss-cone angle
2. The number of bh
3. The relaxation time as a function of the radius
4. The critical radius $a_{\rm crit}$

(1) The loss-cone angle

- It can be approximated as

$$\theta_{\rm lc} \simeq \frac{1}{\sqrt{J_{\rm max}/J_{\rm lc}}}$$

$$J_{\rm lc} \simeq \frac{4\,G}{c}\,M_{\rm BH},\; J_{\rm max}^2 = GM_{\rm BH}a$$

[Alexander & Livio 2001]

- So that

$$\theta_{\rm lc}^2 \simeq \sqrt{\frac{8\,R_{\rm S}}{a}}$$

(2) The number of BH as a function of the radius

- Assuming the power-law distribution,

$$n_{\mathrm{bh}}(a) \sim f_{\mathrm{sub}} \cdot N_0 \left(\frac{a}{R_0}\right)^{3-\gamma}$$

- Differentiating,

$$dn_{\mathrm{bh}}(a) = f_{\mathrm{sub}}\,(3-\gamma)\frac{N_0}{R_0}\left(\frac{a}{R_0}\right)^{2-\gamma} da$$

(3) The relaxation time as a function of the radius

- Relaxation due to the most massive stellar species, **stellar-mass black holes**

$$T_{\mathrm{rlx}}(a) = T_0 \left(\frac{a}{R_0} \right)^{\gamma - 3/2}$$

- With

$$T_0 \simeq \frac{4.26}{(3-\gamma)(1+\gamma)^{3/2}} \frac{\sqrt{R_0^3 (GM_{\mathrm{BH}})^{-1}}}{\ln(\Lambda)\, N_0} \left(\frac{M_{\mathrm{BH}}}{m_{\mathrm{bh}}} \right)^2$$

(4) The critical semi-major axis

- From its definition, **it is the threshold between stellar dynamics and the GW-dominated regime**

$$T_{\mathrm{rlx,\,peri}} = C\,T_{\mathrm{GW}}(a,\,e)$$

- And

$$\frac{8\,GM_{\mathrm{BH}}}{c^2} = (1-e)a\mathcal{W}(\iota,\,\mathrm{s})$$

$\mathcal{W}(\iota,\,\mathrm{s})$ takes into account the asymmetry between pro- and retrograde orbits for the location of the LSO for a Kerr MBH *[Amaro-Seoane et al 2013]*.

The function depends on the spin of the MBH a and the inclination of the orbit ι.

(4) The critical semi-major axis

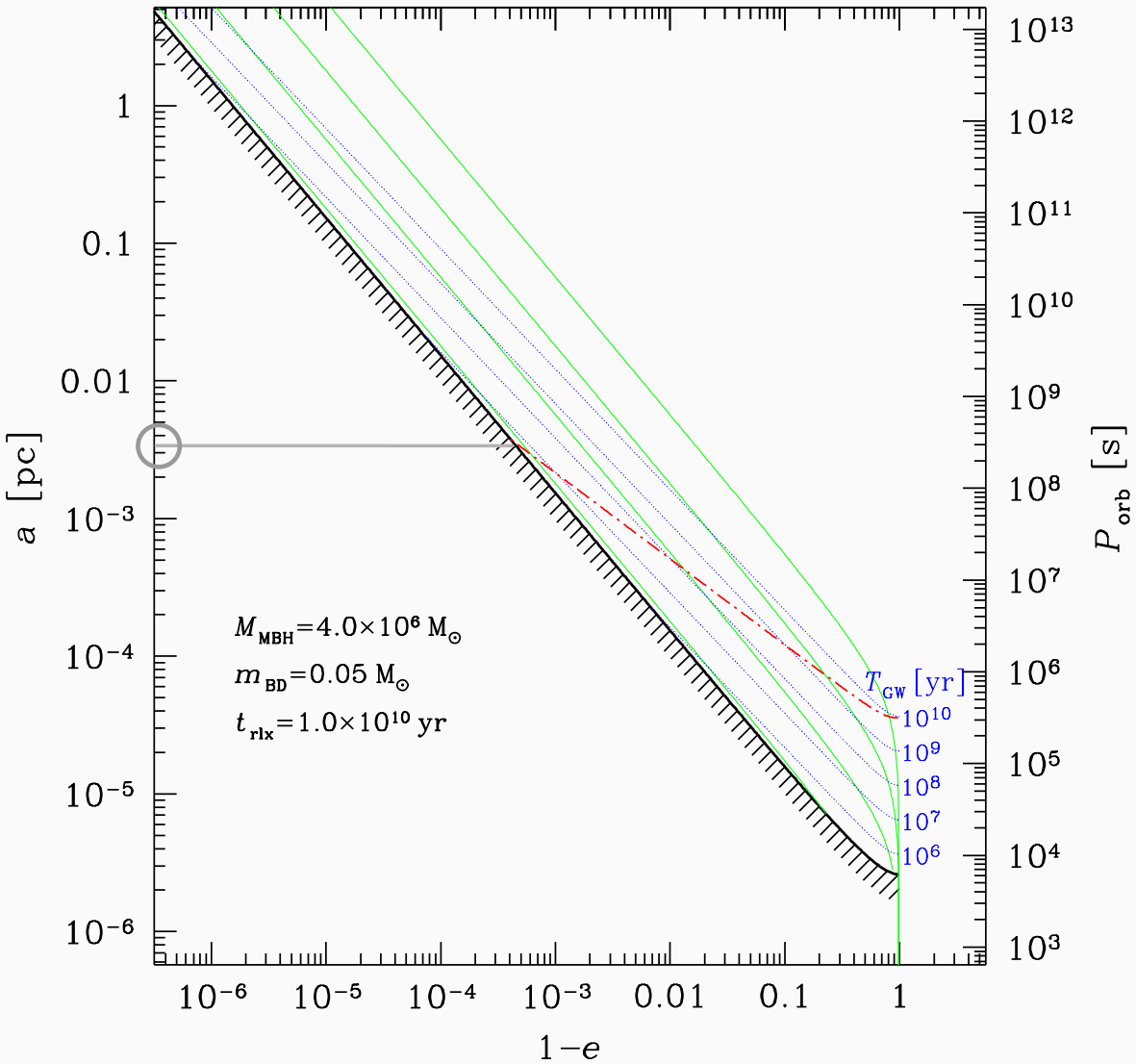
- EMRI orbits have $e \sim 1$, **hence**

$$T_{\mathrm{GW}}(a,\, e) \sim \sqrt{2}\, \frac{24}{85}\, \frac{c^5}{G^3}\, \frac{a^4\, (1-e)^{7/2}}{m_{\mathrm{bh}}\, M_{\mathrm{BH}}^2}$$

- So that we obtain

$$a_{\mathrm{crit}} = R_0 \left[\frac{20480}{1207} (3-\gamma)(1+\gamma)^{3/2} C\, \mathcal{W}(\iota,\, \mathrm{s})^{5/2}\, N_0 \ln(\Lambda) \left(\frac{M_{\mathrm{BH}}}{m_{\mathrm{bh}}} \right)^{-1} \right]^{\frac{1}{\gamma-3}}$$

The critical radius



Definition of $a_{\rm crit}$, at a fixed $t_{\rm rlx}$ for illustration.

- The integral can be solved analytically (*)

$$\dot{\Gamma} \sim 1.92 \times 10^{-6}\,\mathrm{yrs}^{-1}\tilde{N}_0\,\tilde{\Lambda}\,\tilde{R}_0^{-2}\,\tilde{m}^2 \times$$
$$\Big\{1.6 \times 10^{-1}\tilde{R}_0^{1/2}\tilde{N}_0^{-1/2}\tilde{\Lambda}^{-1/2}\tilde{m}^{1/2}\,\mathcal{W}(\iota,\,\mathrm{s})^{-5/4} \times$$
$$\left[\ln\left(9138\,\tilde{R}_0\,\tilde{N}_0^{-1}\tilde{\Lambda}^{-1}\tilde{m}\,\mathcal{W}(\iota,\,\mathrm{s})^{-5/2}\right) - 2\right] -$$
$$4 \times 10^{-2}\tilde{R}_0^{1/2} \times \left[\ln\left(618\,\tilde{R}_0\right) - 2\right]\Big\},$$

(*) If you distrust computer algebra systems.

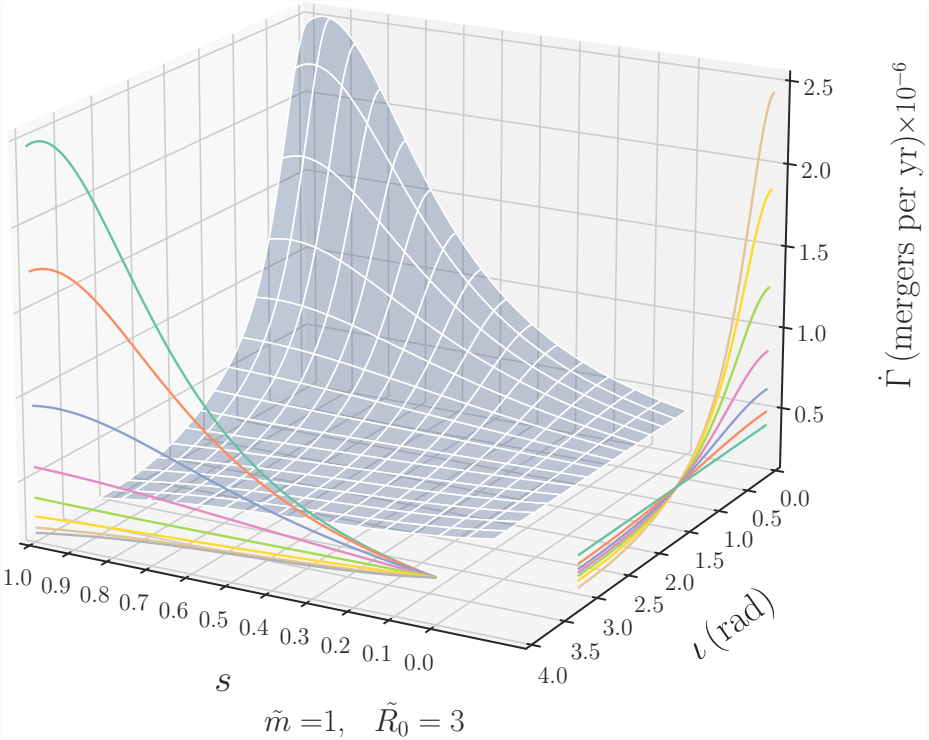
The rates

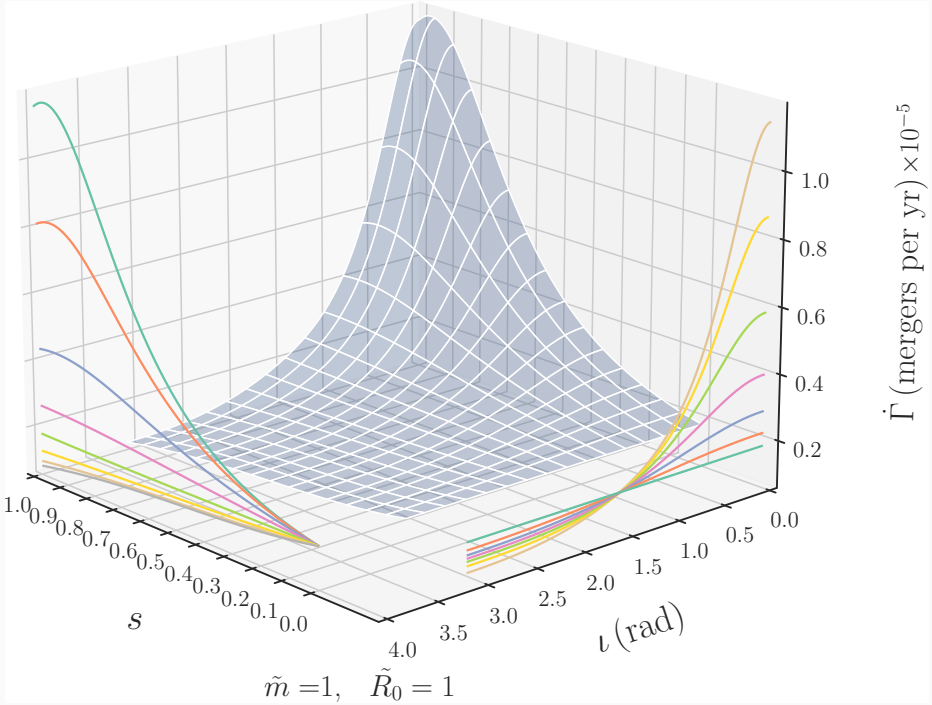
- with the following notation,

$$\tilde{\Lambda} := \left(\frac{\ln(\Lambda)}{13}\right), \; \tilde{N}_0 := \left(\frac{N_0}{12000}\right)$$
$$\tilde{R}_0 := \left(\frac{R_{\mathrm{h}}}{1\mathrm{pc}}\right), \; \tilde{m} := \left(\frac{m}{10\, M_\odot}\right).$$

- The advantage is that $\dot{\Gamma}$ contains **all physical information, including the relaxation time and critical radius, embedded**

EMRI event rate at the GC for $\tilde{m} = 1$



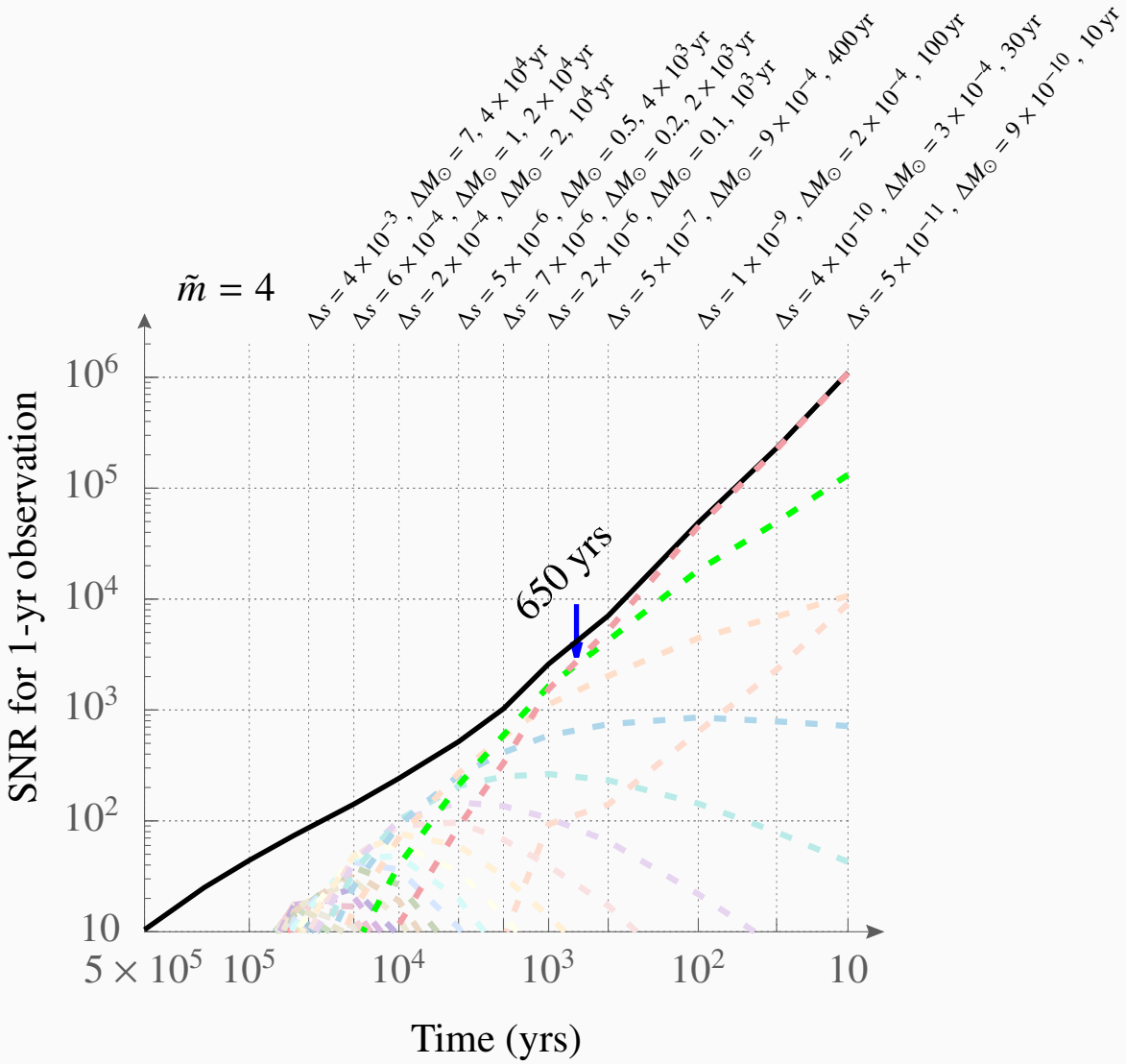


Assume a *Alexander &Hopman 2009, Preto & AS 2009* exponent based on *Peebles 1972* power-law solution, $m_{\rm bh} = 10\ M_\odot$. The event rate depends on the inclination of the orbit (ι) and the spin of the MBH (s).

The values for $\tilde{m} = 4$ are somewhat larger.

So much fuss for this? We knew it.

Again.



500000 yr before plunge, E-EMRI on band with SNR$>$ 10

[PAS, Lin & Tzanavaris (TBS)]

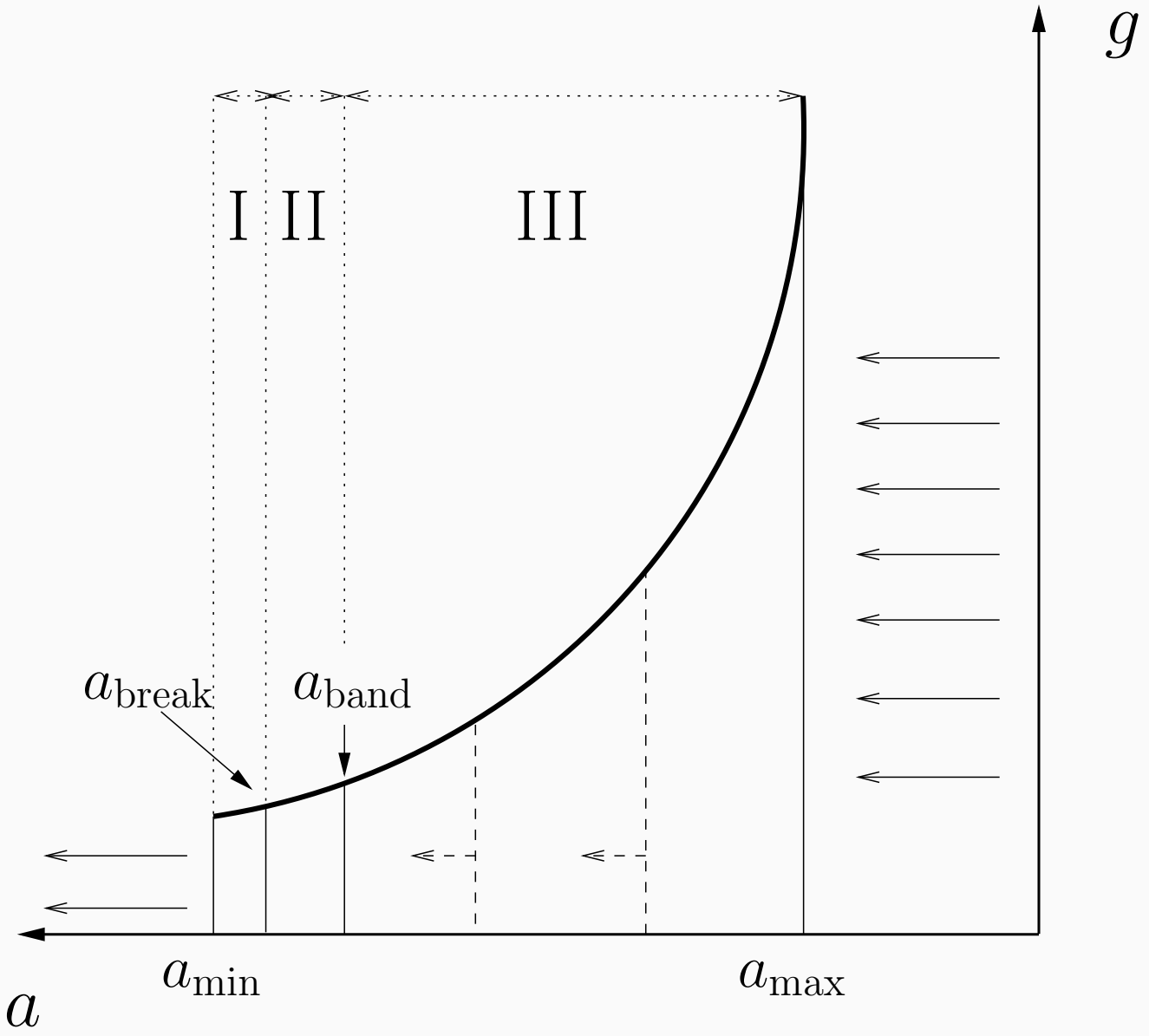
Time is of the essence

- E-EMRIs **spend a long time on band**
- The lifetime with SNR>10 in LISA is of $T \sim 10^5\,\mathrm{yr}^{-1}$ **and the event rate** $\dot{\Gamma} \cong 10^{-6}\,\mathrm{yr}^{-1}$
- Therefore... **How many of these in band??**

Final numbers

- From the continuity equation of the events **we can derive the relative occupation fractions of the line density** $g = dN/da$
- Taking into account the eccentricity of the sources when integrating N, the inclinations and spins **we find the final numbers**

Number of sources in band, at any given moment



[PAS, Yiren & Tzanavaris TBS]

Three equations, three unknowns

$$\frac{N_{\mathrm{II}}}{N_{\mathrm{III}}} = \frac{a_{\mathrm{band}}^{1/2} - a_{\mathrm{thr}}^{1/2}}{a_{\mathrm{crit}}^{1/2} - a_{\mathrm{band}}^{1/2}}$$

$$\frac{N_{\mathrm{I}}}{N_{\mathrm{II}} + N_{\mathrm{III}}} = \frac{1}{8} \times \frac{1 - \left(a_{\mathrm{min}}/a_{\mathrm{thr}}\right)^4}{\left(a_{\mathrm{crit}}/a_{\mathrm{thr}}\right)^{1/2} - 1}$$

$$N_{\mathrm{I}} + N_{\mathrm{II}} = \dot{\Gamma} \times T\left(a_{\mathrm{crit}},\ e\right)$$

Relative occupation fractions

$$
\begin{aligned}
N_{\mathrm{I}} &= \dot{\Gamma} \times T(a_{\mathrm{crit}},\, e) \times \Omega_1 \\
N_{\mathrm{II}} &= \dot{\Gamma} \times T(a_{\mathrm{crit}},\, e) \times \Omega_2 \\
N_{\mathrm{III}} &= \dot{\Gamma} \times T(a_{\mathrm{crit}},\, e) \times \Omega_3,
\end{aligned}
\tag{1}
$$

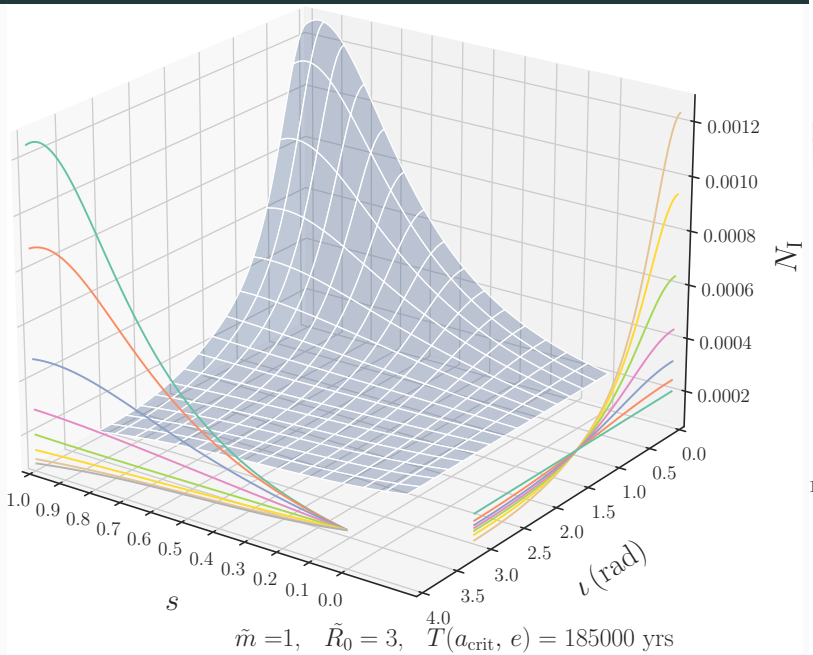
where we have introduced the weighting functions Ω_1, Ω_2 and Ω_3

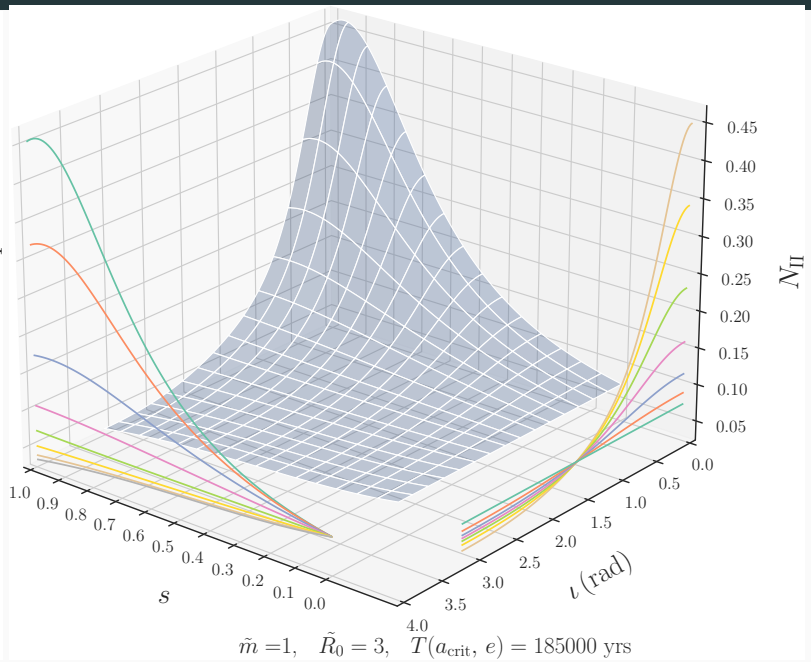
$$
\Omega_1 \equiv \frac{(\sqrt{a_{\mathrm{thr}}} - \sqrt{a_{\mathrm{crit}}})(a_{\mathrm{min}}^4 - a_{\mathrm{thr}}^4)}{(a_{\mathrm{thr}}^4(8\sqrt{a}(\sqrt{a_{\mathrm{crit}}/a_{\mathrm{thr}}} - 1) + \sqrt{a_{\mathrm{crit}}}) + a_{\mathrm{thr}}^{9/2}(7 - 8\sqrt{a_{\mathrm{crit}}/a_{\mathrm{thr}}}) + a_{\mathrm{min}}^4(\sqrt{a_{\mathrm{thr}}} - \sqrt{a_{\mathrm{crit}}})}
$$

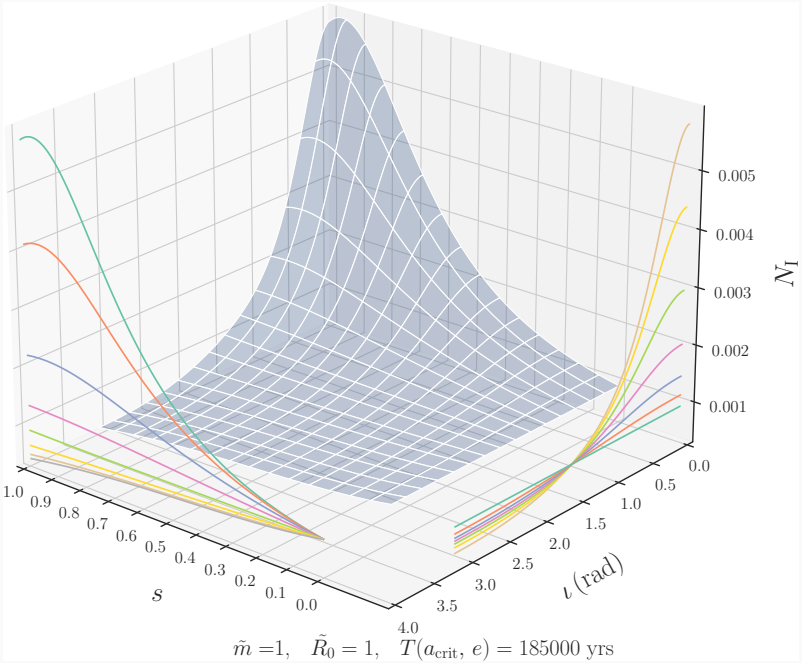
$$
\Omega_2 \equiv \frac{8a_{\mathrm{thr}}^4(\sqrt{a} - \sqrt{a_{\mathrm{thr}}})(\sqrt{a_{\mathrm{crit}}/a_{\mathrm{thr}}} - 1)}{a_{\mathrm{thr}}^4(8\sqrt{a}(\sqrt{a_{\mathrm{crit}}/a_{\mathrm{thr}}} - 1) + \sqrt{a_{\mathrm{crit}}}) + a_{\mathrm{thr}}^{9/2}(7 - 8\sqrt{a_{\mathrm{crit}}/a_{\mathrm{thr}}}) + a_{\mathrm{min}}^4(\sqrt{a_{\mathrm{thr}}} - \sqrt{a_{\mathrm{crit}}})}
$$

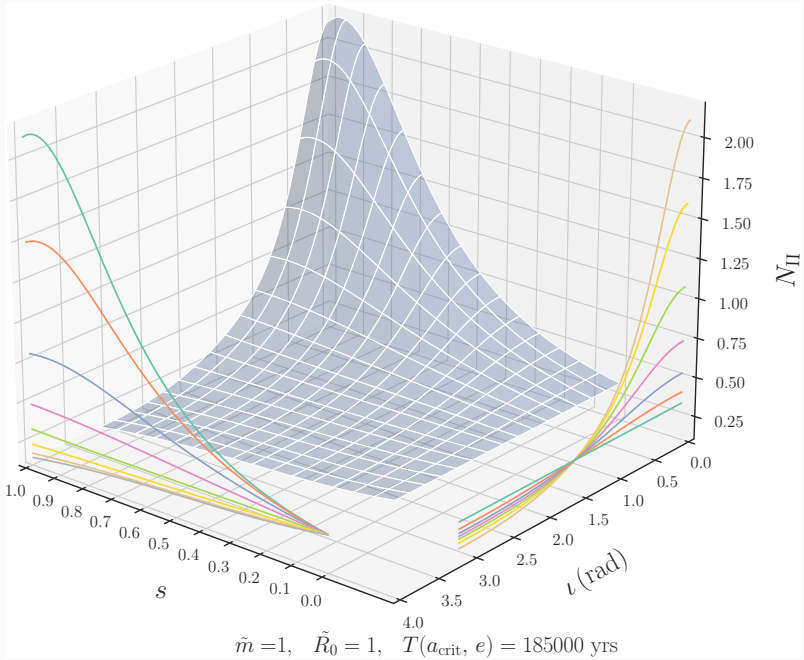
$$
\Omega_3 \equiv \frac{8a_{\mathrm{thr}}^4(\sqrt{a} - \sqrt{a_{\mathrm{crit}}})(1 - \sqrt{a_{\mathrm{crit}}/a_{\mathrm{thr}}})}{(a_{\mathrm{thr}}^4(8\sqrt{a}(\sqrt{a_{\mathrm{crit}}/a_{\mathrm{thr}}} - 1) + \sqrt{a_{\mathrm{crit}}}) + a_{\mathrm{thr}}^{9/2}(7 - 8\sqrt{a_{\mathrm{crit}}/a_{\mathrm{thr}}}) + a_{\mathrm{min}}^4(\sqrt{a_{\mathrm{thr}}} - \sqrt{a_{\mathrm{crit}}})}
$$

Total number for $\tilde{m} = 1$ at any given time(*)

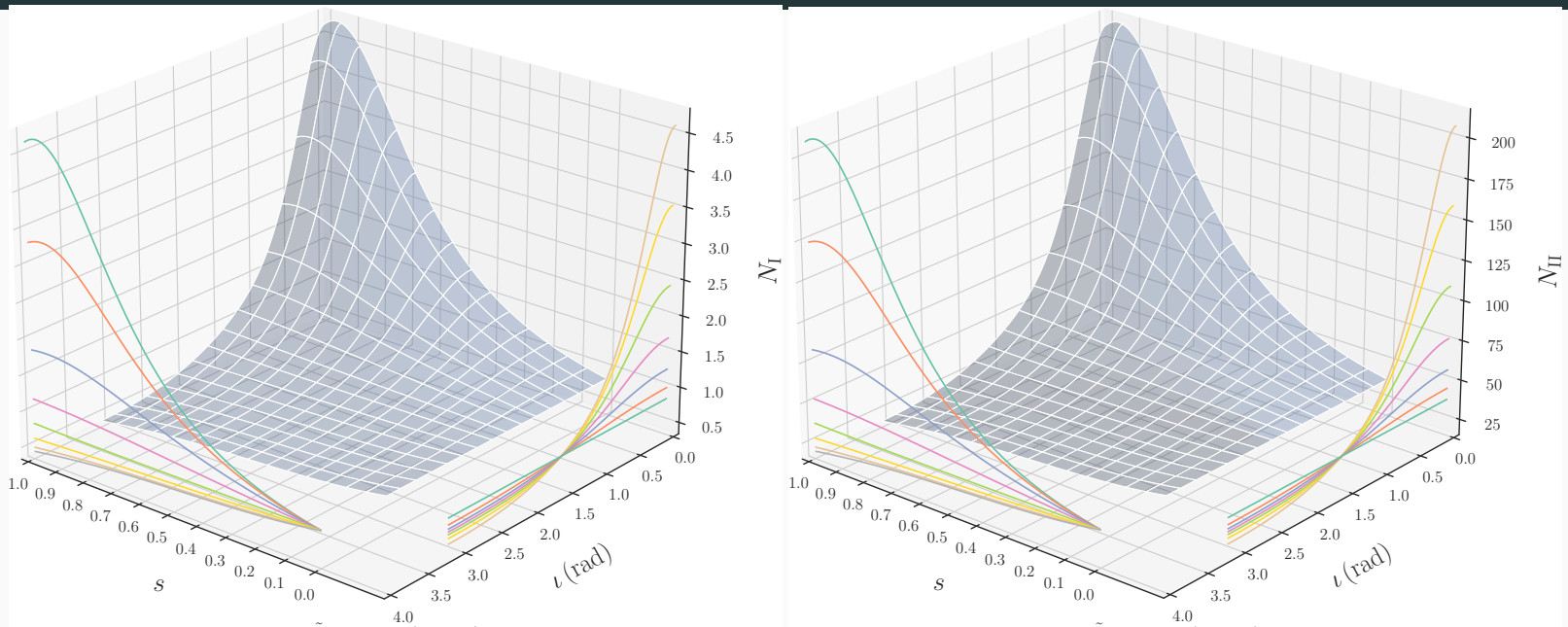


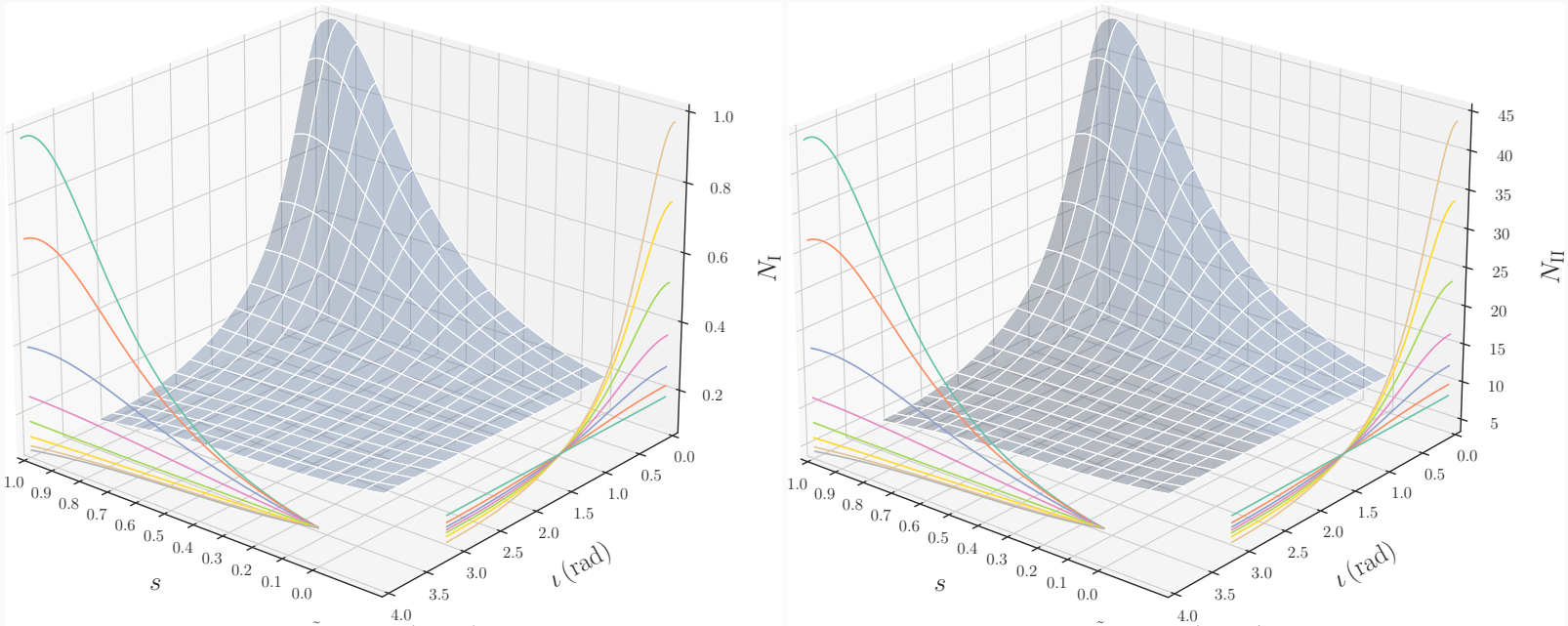






Total number for $\tilde{m} = 4$ at any given time(*)

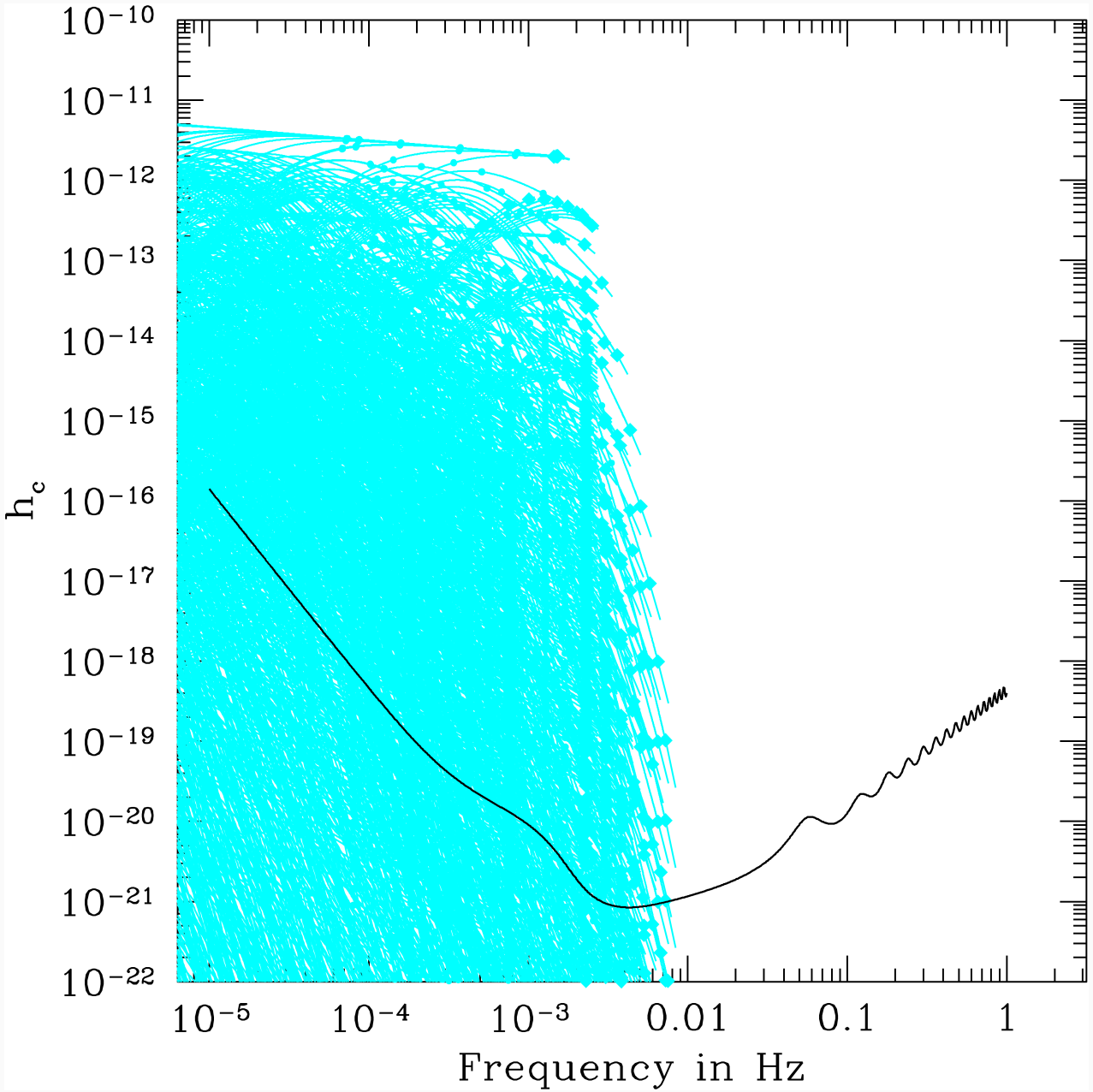




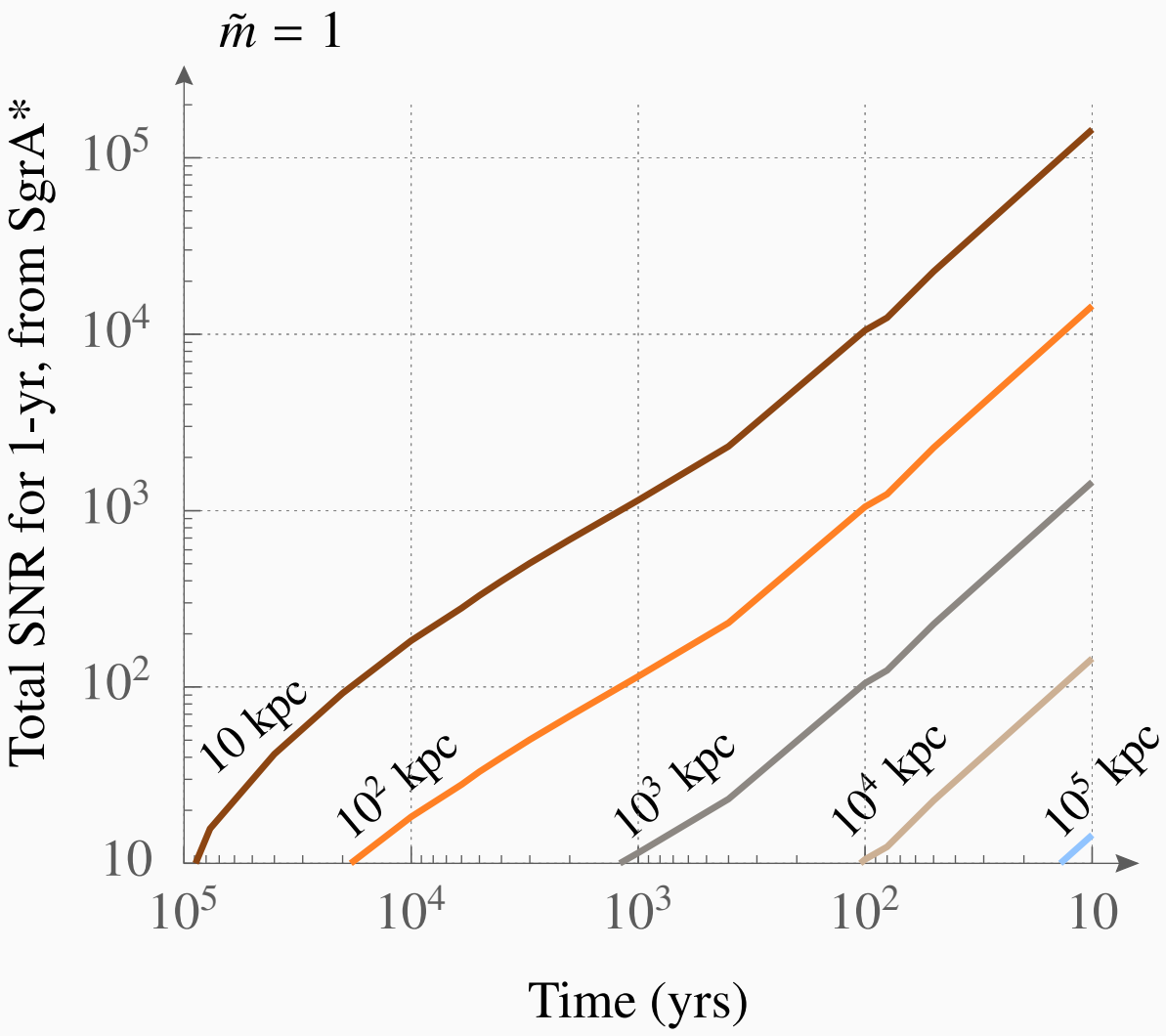
A forest of E-EMRIs:

fore- and background population

A forest of E-EMRIs: T_{OBS} removed (naïve picture)



In band out to 0.1 Gpc (naïve picture)



[PAS, Yiren & TzanavarisTBS]

Total strain

- The combined amplitude **is an incoherent sum pondered by different weights** ξ

Total strain

- The combined amplitude **is an incoherent sum pondered by different weights** ξ
- The weight represents **the occupation fraction probability**

Total strain

- The combined amplitude **is an incoherent sum pondered by different weights** ξ
- The weight represents **the occupation fraction probability**
- Even if one particular source is not in band **the addition (multiplication) of that particular population fraction might be**

Total strain

- The combined amplitude **is an incoherent sum pondered by different weights** ξ
- The weight represents **the occupation fraction probability**
- Even if one particular source is not in band **the addition (multiplication) of that particular population fraction might be**
- A bit like Kosmas Indikopleustes' paradox[1] (if you allow me the comparison) **Only that this does not decay as** $1/r^2$ **because in GWs we capture the "full wave", contrary to electromagnetism**

Total strain

- The combined amplitude **is an incoherent sum pondered by different weights** ξ
- The weight represents **the occupation fraction probability**
- Even if one particular source is not in band **the addition (multiplication) of that particular population fraction might be**
- A bit like Kosmas Indikopleustes' paradox[1] (if you allow me the comparison) **Only that this does not decay as** $1/r^2$ **because in GWs we capture the "full wave", contrary to electromagnetism**
- The forest hence **extends beyond 0.1 Gpc**

So, how does the fore- and background look like?

I don’t know (yet).

So...

E-EMRIs - Early-inspiral EMRIs

- EMRIs can be divided into three categories

E-EMRIs - Early-inspiral EMRIs

- EMRIs can be divided into three categories
  - **Monochromatic**

E-EMRIs - Early-inspiral EMRIs

- EMRIs can be divided into three categories
  - **Monochromatic**
  - **Oligochromatic**

  } Early-inspiral EMRIs

E-EMRIs - Early-inspiral EMRIs

- EMRIs can be divided into three categories
  - **Monochromatic** } Early-inspiral EMRIs
  - **Oligochromatic** } Early-inspiral EMRIs
  - **Polychromatic** } Late-type; The usual ones.

E-EMRIs - Early-inspiral EMRIs

- EMRIs can be divided into three categories
  - **Monochromatic** } Early-inspiral EMRIs
  - **Oligochromatic** } Early-inspiral EMRIs
  - **Polychromatic** } Late-type; The usual ones.
- If we consider the GC, then we can map these categories with SNR

E-EMRIs - Early-inspiral EMRIs

- EMRIs can be divided into three categories
  - **Monochromatic** } Early-inspiral EMRIs
  - **Oligochromatic** } Early-inspiral EMRIs
  - **Polychromatic** } Late-type; The usual ones.
- If we consider the GC, then we can map these categories with SNR
  - **Mono:** $\sim 10^{2-3}$

E-EMRIs - Early-inspiral EMRIs

- EMRIs can be divided into three categories
  - **Monochromatic** } Early-inspiral EMRIs
  - **Oligochromatic** } Early-inspiral EMRIs
  - **Polychromatic** } Late-type; The usual ones.
- If we consider the GC, then we can map these categories with SNR
  - **Mono:** $\sim 10^{2-3}$
  - **Oligo:** $\sim 10^{3-5}$

E-EMRIs - Early-inspiral EMRIs

- EMRIs can be divided into three categories
  - **Monochromatic** } Early-inspiral EMRIs
  - **Oligochromatic** } Early-inspiral EMRIs
  - **Polychromatic** } Late-type; The usual ones.
- If we consider the GC, then we can map these categories with SNR
  - **Mono:** $\sim 10^{2-3}$
  - **Oligo:** $\sim 10^{3-5}$
- For mono- and oligo-EMRIs it is not the rate that matters

E-EMRIs - Early-inspiral EMRIs

- EMRIs can be divided into three categories
  - **Monochromatic** } Early-inspiral EMRIs
  - **Oligochromatic** } Early-inspiral EMRIs
  - **Polychromatic** } Late-type; The usual ones.
- If we consider the GC, then we can map these categories with SNR
  - **Mono:** $\sim 10^{2-3}$
  - **Oligo:** $\sim 10^{3-5}$
- For mono- and oligo-EMRIs it is not the rate that matters
  - **It is the rate and the time spent on band being detectable, plus their cosmical number**

E-EMRIs - Early-inspiral EMRIs

- EMRIs can be divided into three categories
  - **Monochromatic** } Early-inspiral EMRIs
  - **Oligochromatic** } Early-inspiral EMRIs
  - **Polychromatic** } Late-type; The usual ones.
- If we consider the GC, then we can map these categories with SNR
  - **Mono:** $\sim 10^{2-3}$
  - **Oligo:** $\sim 10^{3-5}$
- For mono- and oligo-EMRIs it is not the rate that matters
  - **It is the rate and the time spent on band being detectable, plus their cosmical number**
- Because they have large SNRs, and so:

E-EMRIs - Early-inspiral EMRIs

- EMRIs can be divided into three categories
  - **Monochromatic** } Early-inspiral EMRIs
  - **Oligochromatic** } Early-inspiral EMRIs
  - **Polychromatic** } Late-type; The usual ones.
- If we consider the GC, then we can map these categories with SNR
  - **Mono:** $\sim 10^{2-3}$
  - **Oligo:** $\sim 10^{3-5}$
- For mono- and oligo-EMRIs it is not the rate that matters
  - **It is the rate and the time spent on band being detectable, plus their cosmical number**
- Because they have large SNRs, and so:
  - **E-EMRIs add up to create a fore- and background signal**

E-EMRIs - Early-inspiral EMRIs

- EMRIs can be divided into three categories
  - **Monochromatic** } Early-inspiral EMRIs
  - **Oligochromatic** } Early-inspiral EMRIs
  - **Polychromatic** } Late-type; The usual ones.
- If we consider the GC, then we can map these categories with SNR
  - **Mono:** $\sim 10^{2-3}$
  - **Oligo:** $\sim 10^{3-5}$
- For mono- and oligo-EMRIs it is not the rate that matters
  - **It is the rate and the time spent on band being detectable, plus their cosmical number**
- Because they have large SNRs, and so:
  - **E-EMRIs add up to create a fore- and background signal**
  - **This "forest" will cause trouble (up to 0.01 Hz affected, huge SNRs)**

E-EMRIs - Early-inspiral EMRIs

- EMRIs can be divided into three categories
  - **Monochromatic** } Early-inspiral EMRIs
  - **Oligochromatic** } Early-inspiral EMRIs
  - **Polychromatic** } Late-type; The usual ones.
- If we consider the GC, then we can map these categories with SNR
  - **Mono:** $\sim 10^{2-3}$
  - **Oligo:** $\sim 10^{3-5}$
- For mono- and oligo-EMRIs it is not the rate that matters
  - **It is the rate and the time spent on band being detectable, plus their cosmical number**
- Because they have large SNRs, and so:
  - **E-EMRIs add up to create a fore- and background signal**
  - **This "forest" will cause trouble (up to 0.01 Hz affected, huge SNRs)**
- What is the shape of the forest?

E-EMRIs - Early-inspiral EMRIs

- EMRIs can be divided into three categories
  - **Monochromatic** } Early-inspiral EMRIs
  - **Oligochromatic** } Early-inspiral EMRIs
  - **Polychromatic** } Late-type; The usual ones.
- If we consider the GC, then we can map these categories with SNR
  - **Mono:** $\sim 10^{2-3}$
  - **Oligo:** $\sim 10^{3-5}$
- For mono- and oligo-EMRIs it is not the rate that matters
  - **It is the rate and the time spent on band being detectable, plus their cosmical number**
- Because they have large SNRs, and so:
  - **E-EMRIs add up to create a fore- and background signal**
  - **This "forest" will cause trouble (up to 0.01 Hz affected, huge SNRs)**
- What is the shape of the forest?
  - **I am working on it...**

Mono- and oligochromatic EMRIs

Pau Amaro Seoane

amaro@riseup.net

astro-gr.org

Extra material

An implicit assumption

- I am assuming that different mass "species" contribute to relaxation individually...
  ... and that the total amount of relaxation in the system can be added up linearly from them.
  Can I do that?
  "Yes."
- The distribution function of mass and velocity is $f(m,\ v)$,
  and a moment of the change of velocities is of the form

$$< dv^2 >= \int dv^2 f(m,\ v)\, dm\, dv.$$

- And this can be envisaged as

$$< dv^2 >= \sum_m n(m) \left(\int dv^2 f(v) dv \right),$$

with $n(m)$ the density of stars of mass m.

Harmonics

The strain amplitude in the n-th harmonic at a given distance D, normalized to the typical values of this work is

$$
\begin{aligned}
h_n &= g(n,e)\frac{G^2\, M_{\rm BH} m_{\rm CO}}{D\, a\, c^4} \\
&\simeq 8\times 10^{-23} g(n,e)\left(\frac{D}{500\,{\rm Mpc}}\right)^{-1}\left(\frac{a}{10^{-5}\,{\rm pc}}\right)^{-1} \\
&\left(\frac{M_{\rm BH}}{10^3\, M_\odot}\right)\left(\frac{m_{\rm CO}}{10\, M_\odot}\right).
\end{aligned}
$$

In this expression $M_{\rm BH}$ is the mass of the IMBH, $m_{\rm CO}$ is the mass of the compact object (CO), and $g(n,\ e)$ is a function of the harmonic number n and the eccentricity e *[Peters & Matthews 1963]*. We consider the RMS amplitude averaged over the two GW polarizations and all directions.