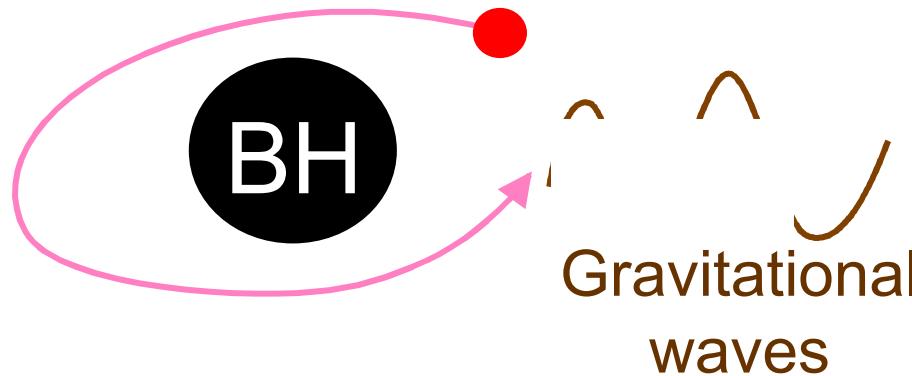




GRADUATE SCHOOL OF
FACULTY OF SCIENCE
KYOTO UNIVERSITY



Hamiltonian formulation for extreme mass-ratio inspirals(EMRIs)



Takahiro Tanaka
(Kyoto University)

Mostly based on
arXiv:1809.11118
(arXiv:1612.02504)¹

Our Target

Giving a prescription how to predict
the gravitational waveform from EMRIs,
using black hole perturbation.

Accurate and fast evaluation of
GW phase evolution is especially important to
compare the template with observations.

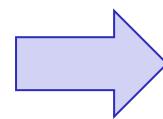
So, we want to establish
an economical way to compute
the orbital phase evolution
taking into account the self-force.

Gauge invariance



Particle's trajectory

Perturbation is everywhere small outside the world tube
“tube radius” $\gg \mu$ (mass of satellite)



Unavoidable ambiguity in the perturbed trajectory of $O(\mu)$

“Self-force is *gauge dependent*”

$F_{\text{self}}^{\mu}(\tau, \gamma)$ has unnecessary information.
Source trajectory

Whilst “long term orbital evolution is *gauge invariant*”

There must be a concise description that keeps only the gauge invariant information

Leading order wave form

Energy balance argument is sufficient.

$$\boxed{\frac{dE_{GW}}{dt} = \frac{dE_{orbit}}{dt}}$$

Waveform $\equiv \frac{df}{dt}$ for quasi-circular orbits, for example.

$$\frac{df}{dt} = \frac{dE_{GW}}{dt} \Bigg/ \frac{dE_{orbit}}{df}$$

leading order

$$\frac{dE_{GW}}{dt} = 0 + O(\mu) + O(\mu^2)$$

$$\frac{dE_{orbit}}{df} = (\text{geodesic}) + O(\mu) + O(\mu^2)$$

Radiation reaction for general orbits

● Radiation reaction to the Carter constant

Schwarzschild “constants of motion” $E, L_i \Leftrightarrow$ Killing vector ξ

$$E = -\xi_{(t)}^\mu u_\mu$$

Conserved current for the field corresponding to Killing vector exists.

$$E_{GW} = \int d\Sigma^\mu t_{\mu\nu}^{(GW)} \xi^\nu$$

$$\dot{E} = -\dot{E}_{GW} \quad \text{In total, conservation law holds.}$$

Kerr conserved quantities $E, L_z \Leftrightarrow$ Killing vector

$Q \not\Leftrightarrow$ Killing vector

$$Q \equiv \underline{K^{\mu\nu}} u_\mu u_\nu \quad K_{(\mu\nu;\rho)} = 0$$

Killing tensor

→ We need to directly evaluate the self-force acting on the particle.

To obtain the self-forced motion, we just need to solve the geodesic equation on an appropriately regularized perturbed spacetime.

$$S = \frac{1}{2} \int g^{\mu\nu} u_\mu u_\nu d\tau = \underbrace{\frac{1}{2} \int g_{(0)}^{\mu\nu} u_\mu u_\nu d\tau}_{H_0} - \underbrace{\frac{1}{2} \int h_{(ret-S)}^{\mu\nu} u_\mu u_\nu d\tau}_{\text{interaction Hamiltonian } H_{\text{int}}}$$

Canonical transformation to the action angle variables with the aid of constants of motion in the background:

$$P_\alpha \equiv \{ H_0, -E, L_z, Q \}$$

Generating function:

$$W(x, J) = J_t t + J_\phi \phi + \int^r \tilde{u}_r(r', J) dr' + \int^\theta \tilde{u}_\theta(\theta', J) d\theta'$$



 well-known for Kerr geodesic motion

$$J_r = \oint \tilde{u}_r(r', P) dr' \quad J_\theta = \oint \tilde{u}_\theta(\theta', P) d\theta'$$

$$u_\mu = \frac{\partial W}{\partial x^\mu} \quad w^\alpha = \frac{\partial W}{\partial J_\alpha}$$

Gauge invariance of the angular velocity

Angle variables $w^a = O(\eta^{-1})$ (\propto radiation reaction time) are gauge invariant in the context of long term evolution.

$$\eta := \mu/M$$

allowing $O(\eta)$ gauge ambiguity, if not $O(\eta \Delta T)$.

→ $\omega^a = \langle \dot{w}^a \rangle = O(\eta^0)$ should be invariant up to $O(\eta)$



$$W(x, J) = J_t t + J_\phi \phi + \int^r \tilde{u}_r(r', J) dr' + \int^\theta \tilde{u}_\theta(\theta', J) d\theta'$$

$$J_r = \oint \tilde{u}_r(r', J) dr', \quad J_\theta = \oint \tilde{u}_\theta(\theta', J) d\theta'$$

$$W(x, J) = \tilde{W}(x, J) + n_r J_r + n_\theta J_\theta$$

where we introduce a single valued function with respect to x :

$$\tilde{W}(x, J) = J_t t + J_\phi \phi + \int_{r_0}^r \tilde{u}_r(r', J) dr' + \int_{\theta_0}^\theta \tilde{u}_\theta(\theta', J) d\theta'$$

$$w^I = \frac{\partial W(x, J)}{\partial J_I} = \frac{\partial \tilde{W}(x, J)}{\partial J_I} + n_I \quad (I = r, \theta) \quad w^i = \frac{\partial W(x, J)}{\partial J_i} = \frac{\partial \tilde{W}(x, J)}{\partial J_i} \quad (i = t, \phi)$$

Small variations of x and J are not amplified in w .

Radiation reaction to the action variables (constants of motion)

“retarded” field= “**radiative**” + “**symmetric**”

$$\frac{\text{"ret"- "adv"}}{2} \quad \frac{\text{"ret"+ "adv"}}{2}$$

regularization is unnecessary

$$\left\langle \frac{dJ_\alpha}{d\tau} \right\rangle = \left\langle \frac{\partial H_{\text{int}}}{\partial w^\alpha} \right\rangle = \left[\int d\tau \int d\tau' \frac{\partial}{\partial w^\alpha} G^{(\text{ret}-S)}(w, J; \gamma') \right]_{(w, J) = \gamma' = \gamma}$$

$\partial/\partial w^\alpha$ can be replaced with $\underline{\partial/\partial w_{\text{ini}}^\alpha}$.
initial value

$$\begin{aligned} \left\langle \frac{dJ_\alpha}{d\tau} \right\rangle &= \left[\int d\tau \int d\tau' \frac{\partial}{\partial w_{\text{ini}}^\alpha} G^{(\text{ret}-S)}(\gamma, \gamma') \right]_{\gamma'=\gamma} \\ &= \left[\int d\tau \int d\tau' \frac{\partial}{\partial w_{\text{ini}}^\alpha} G^{(\text{rad})}(\gamma, \gamma') \right]_{\gamma'=\gamma} + \left[\int d\tau \int d\tau' \frac{\partial}{\partial w_{\text{ini}}^\alpha} G^{(\text{sym}-S)}(\gamma, \gamma') \right]_{\gamma'=\gamma} \end{aligned}$$

Simplification

Symmetric part

$$\left[\int d\tau \int d\tau' \frac{\partial}{\partial w_{\text{ini}}^\alpha} G^{(\text{sym}-S)}(\gamma, \gamma') \right]_{\gamma'=\gamma} = \frac{1}{2} \frac{\partial}{\partial w_{\text{ini}}^\alpha} \left[\int d\tau \int d\tau' G^{(\text{sym}-S)}(\gamma, \gamma') \right]_{\gamma'=\gamma} = \frac{1}{2} \frac{\partial}{\partial w_{\text{ini}}^\alpha} \langle H_{\text{sym}} \rangle$$

H_{sym} after substitution $\gamma'=\gamma$ is independent of w_{ini}^a for non-resonant case.

At the leading order in $\eta = \mu/M$, only the radiative part determines the change of “constants of motion”, except for resonance orbits.

(Mino (2003))

radiative part

$$H_{rad}(x, u) = - \int d\omega \sum_{l,m} \frac{\mu}{4i\omega^4} \left\{ Z_{\omega lm}^{out} \Phi_{\omega lm}^{out}(x, u) + \frac{\omega}{p_{\omega lm}} Z_{\omega lm}^{down} \Phi_{\omega lm}^{down}(x, u) \right\} + (c.c.)$$

$$Z_{\omega lm}^{out/down} \equiv \int d\tau' \overline{\Phi_{\omega lm}^{out/down}}(\tau')$$

$$\langle j_\alpha \rangle = - \int d\omega \sum_{l,m} \frac{\mu}{4i\omega^4} \left\{ Z_{\omega lm}^{out} \left\langle \frac{\partial \Phi_{\omega lm}^{out}(x, u)}{\partial w^\alpha} \right\rangle + \frac{\omega}{p_{\omega lm}} Z_{\omega lm}^{down} \left\langle \frac{\partial \Phi_{\omega lm}^{down}(x, u)}{\partial w^\alpha} \right\rangle \right\} + (c.c.)$$

Fourier exp. w.r.t. w

$$\boxed{\Phi_{\omega lm}^{out/down}(w, J) \equiv \sum_{k,n} \phi_{\omega lmkn}^{out/down}(J) e^{-i(\omega w^t - mw^\varphi - kw^\theta - nw^r)}}$$

$$\begin{aligned} \rightarrow Z_{\omega lm}^{out/down} &\equiv \sum_{k,n} \int d\tau \sum_{k,n} \overline{\phi_{\omega lmkn}^{out/down}(J)} e^{i(\omega w^t - mw^\varphi - kw^\theta - nw^r)} \Big|_{\gamma} \\ &= \sum_{k,n} \tilde{Z}_{lmkn}^{out/down} e^{i\chi_{mkn}} \delta(\omega - \omega_{mkn}) \end{aligned}$$

where $\omega_{mkn} \equiv m\Omega^\varphi + k\Omega^\theta + n\Omega^r$, $\chi_{mkn} \equiv \omega_{mkn} w_0^t - (mw_0^\varphi + kw_0^\theta + nw_0^r)$,

$$\tilde{Z}_{lmkn}^{out} \equiv 2\pi z \overline{\phi_{\omega_{mkn} lmkn}^{out}(J)} \Big|_{\gamma}$$

$$\begin{aligned}
& \xrightarrow{\text{Red Arrow}} \int d\omega Z_{\omega lm}^{out} \frac{\partial \Phi_{\omega lm}^{out}(x, u)}{\partial w^\alpha} \\
&= \frac{\partial}{\partial w^\alpha} \sum_{k,n} \sum_{k',n'} \tilde{Z}_{lmkn}^{out} e^{i\chi_{mkn}} \phi_{\omega_{mkn} lmk'n'}^{out}(J) e^{-i(\omega_{mkn} w^t - mw^\varphi - k'w^\theta - n'w^r)} \Big|_\gamma \\
&= i \sum_{k,n} \sum_{k',n'} \tilde{Z}_{lmkn}^{out} \left(\varepsilon_\alpha \phi_{\omega_{mkn} lmk'n'}^{out}(J) \right) e^{i((k'-k)w^\theta + (n'-n)w^r)} \Big|_\gamma \\
&\quad \varepsilon_\alpha \equiv (-\omega_{mkn}, n', k', m),
\end{aligned}$$

Orbital average implies that only contributions with $k=k'$ and $n=n'$ remain.

$$\xrightarrow{\text{Red Arrow}} \int d\omega Z_{\omega lm}^{out} \left\langle \frac{\partial \Phi_{\omega lm}^{out}(x, u)}{\partial w^\alpha} \right\rangle = \frac{i}{2\pi z} \sum_{k',n'} \varepsilon_\alpha |\tilde{Z}_{lmk'n'}^{out}|^2$$

$$\tilde{Z}_{lmkn}^{out} \equiv 2\pi z \overline{\phi_{\omega_{mkn} lmk'n'}^{out}(J)} \Big|_\gamma$$

The formulas for $\langle j_\theta \rangle$ and $\langle j_r \rangle$ are analogous to those for $\langle j_t \rangle$ and $\langle j_\varphi \rangle$.

Next Leading Order of η in waveform

Orbital frequencies:

$$\omega^\alpha \equiv \frac{dw^\alpha}{d\tau}$$

Wave form is specified by $\frac{d\omega^\alpha}{d\tau}(\omega)$

$$\frac{d\omega^\alpha}{dt} = \frac{\partial\omega^\alpha}{\partial J_\beta} \frac{dJ_\beta}{d\tau}$$

$$\frac{dJ_\beta}{dt} = 0$$

$$\frac{\partial\omega^\alpha}{\partial J_\beta} = (\text{geodesic}) + O(\eta) + O(\eta^2)$$

linear perturbation

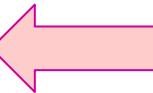
leading order ($O(\eta^{-1})$ phase)

next leading order
($O(1)$ phase)

Long term evolution

EOM)

$$\frac{dJ_\alpha}{d\tau} = -\frac{\partial H_{\text{int}}}{\partial w^\alpha}$$



We need to solve this equation to the second order

$$\frac{dw^\alpha}{d\tau} = \Omega_{(0)}^\alpha(J) + \frac{\partial H_{\text{int}}}{\partial J^\alpha}$$

We separate the variables

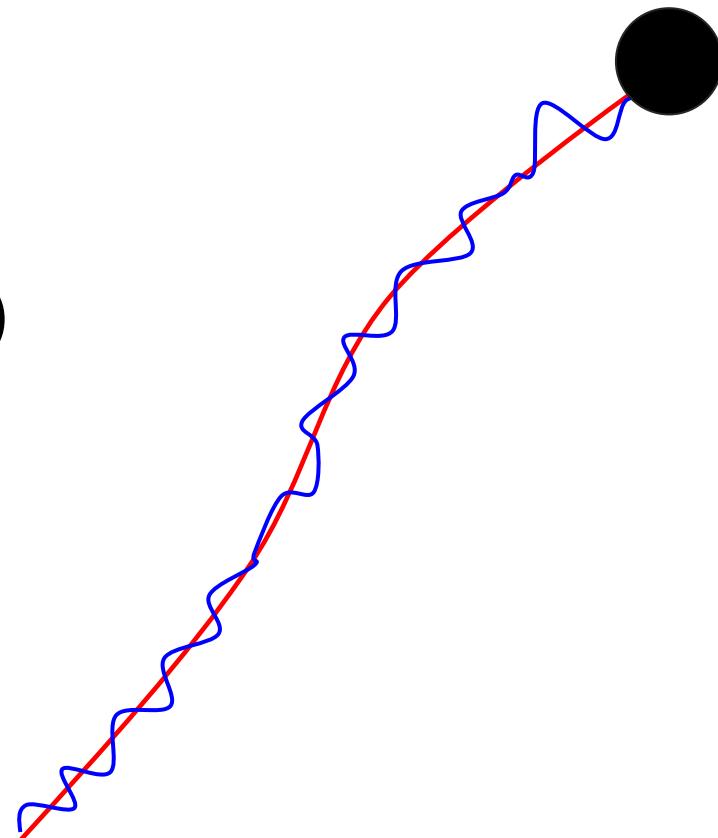
$$J_\alpha = \bar{J}_\alpha(\tilde{\tau}) + \delta J_\alpha(\tilde{\tau}, w^I)$$

$$w^\alpha = \bar{w}^\alpha(\tilde{\tau}) + \delta w^\alpha(\tilde{\tau}, w^I) \quad (I = r, \theta)$$

so that

(\bar{J}, \bar{w}) :depends only on slow time

$(\delta J, \delta w)$:can rapidly oscillate
but always remains small



Second order perturbation)

$$H_{\text{int}} = H_{\text{int}}^{(1)} + H_{\text{int}}^{(2)} + \dots$$

Source for $H_{\text{int}}^{(2)} = -h_{(2)}^{\mu\nu} u_\mu u_\nu$ has roughly two different types:

- 1) Quadratic term of the first order metric perturbation “ $\partial h \partial h$ ”
- 2) First order deviation of the source orbit from the osculating orbit

$$\Delta J_\alpha(\tau_0; \tau) := J_\alpha(\tilde{\tau}) - \bar{J}_\alpha^{osc}(\tau_0; \tau) = \delta J_\alpha^{(1)} + \left\langle \frac{dJ_\alpha}{d\tau} \right\rangle (\tau - \tau_0) + \dots$$

$$\Delta w^\alpha(\tau_0; \tau) := w^\alpha(\tau) - \bar{w}_{osc}^\alpha(\tau_0; \tau) = \delta w^\alpha(\tau) + \frac{1}{2} \frac{\partial \Omega_{(0)}^\alpha}{\partial J_\beta} \left\langle \frac{dJ_\beta}{d\tau} \right\rangle_{osc} (\tau - \tau_0)^2 + \dots$$

$$\frac{dJ_a}{d\tau} = -\frac{\partial H_{\text{int}}^{(1)}}{\partial w^a} - \frac{\partial H_{\text{int}}^{(2,hh)}}{\partial w^a} - \underbrace{\left(\Delta J_\beta \frac{\partial}{\partial J_\beta^{(s)}} + \Delta w^\beta \frac{\partial}{\partial w_{(s)}^\beta} \right) \frac{\partial H_{\text{int}}^{(1)}}{\partial w^\alpha}}_{\text{differentiation with respect to the source orbit}}$$

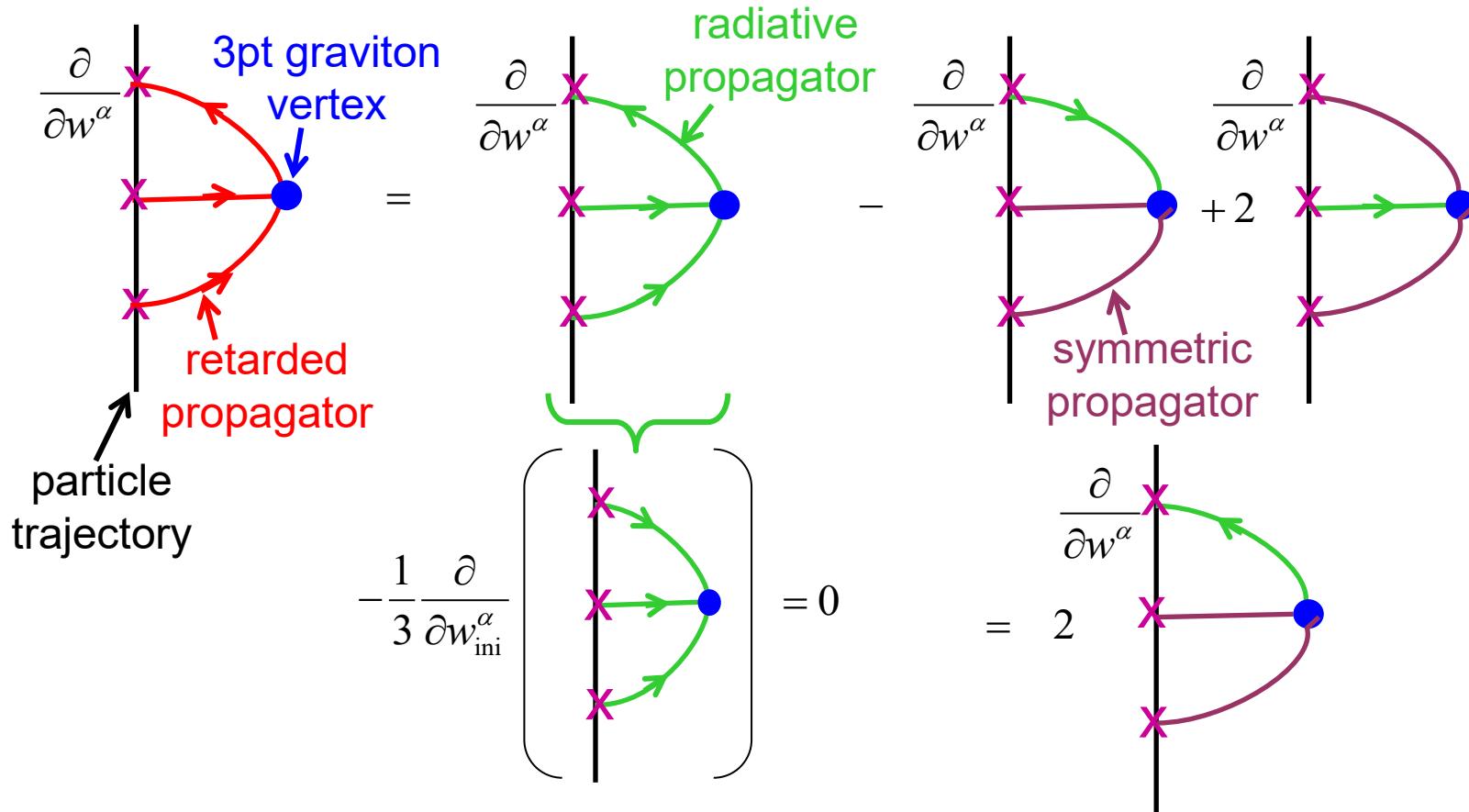
$$\frac{d\bar{J}_\alpha}{d\tau} = - \left\langle \frac{\partial H_{\text{int}}^{(1)}}{\partial w^\alpha} \right\rangle_{osc} - \left\langle \frac{\partial H_{\text{int}}^{(2,hh)}}{\partial w^\alpha} \right\rangle_{osc} - \left\langle \left(\Delta J_\beta \frac{\partial}{\partial J_\beta^{(s)}} + \Delta w^\beta \frac{\partial}{\partial w_{(s)}^\beta} \right) \frac{\partial H_{\text{int}}^{(1)}}{\partial w^\alpha} \right\rangle_{osc}$$

Drastic simplification may occur

From here on I just give a highly speculative argument.

$$\frac{d\bar{J}_\alpha}{d\tau} \supset -\frac{\partial H_{\text{int}}^{(2)}}{\partial w^\alpha}$$

Second order
dissipative part = Graviton 2-loop



$$\begin{aligned}
 \frac{\partial}{\partial w^\alpha} &= \int d\tau \int d\tau' \int d\tau'' \int d^4x' \\
 &\times \frac{\partial}{\partial w^\alpha} \hat{V}_{x'_1, x'_2, x'_3} G_{(rad)}(x, x'_1) \Big|_{x=\gamma(\tau)} G_{(sym)}(x'_2, \gamma(\tau')) G_{(sym)}(x'_3, \gamma(\tau'')) \Big|_{x'_1=x'_2=x'_3=x} \\
 \int d^4x' G_{(rad)}(x, x') S(x') &= \int d^4x' G_{(rad)}(x, x') [S(x') - \Delta h]
 \end{aligned}$$

We can rather freely subtract counter term.

Second order source must satisfy conservation $T_{\mu\nu}^{;\nu=0}$ as a whole

➡ $\nabla^\nu T_{\mu\nu}^{(\text{particle})} = \nabla^\nu G_{\mu\nu}^{[2]}(h_{(sym)}, h_{(sym)})$

necessary deviation from geodesic comes from symmetric contribution only.

$$\begin{aligned}
 \frac{\partial}{\partial w^\alpha} &= \int d\tau \frac{\partial}{\partial w^\alpha} \int d\tau' \delta J_\beta^{(1)}(\tau') \frac{\partial}{\partial J_\beta(\tau')} G_{(rad)}(x, \gamma(\tau'))
 \end{aligned}$$

Summary

The "flux-balance formulae" that determine the averaged evolution of energy, angular momentum and Carter constant in terms of the averaged asymptotic partial wave fluxes for EMRIs in Kerr spacetime were first derived in Sago et al. 2005.

The new derivation of the flux formulae based on Hamiltonian dynamics of a self-forced particle motion using action-angle variables is much simpler than the previous one.

(The conservative effect of the first order perturbation can be encapsulated in the effective Hamiltonian.)

Formal discussion based on Hamiltonian dynamics strongly suggests that the next leading order calculation can be simplified a lot.