On The Existence of Multi-dimensional Compressible MHD Contact Discontinuities

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International Conference on Partial Differential Equations and Applications
In honor of the 70th birthday of Professor Pierangelo Marcati
L'Aquila, Italy
June 19-24, 2023

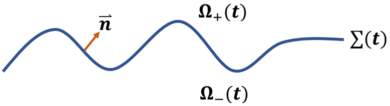
Happy 70th Birthday to my friend Piero!

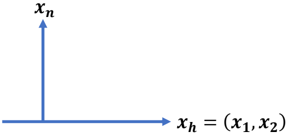
1. Introduction

Contact discontinuities, together with shocks and rarefaction waves, are basic waves for systems of hyperbolic conservation laws:

$$\partial_t U + \mathrm{div}_x \left(F(U) \right) = 0, \qquad x \in \mathbb{R}^n \tag{1}$$

Such waves are characterized as piecewise smooth solutions with a strong characteristic discontinuity at an interface $\sum(t)$, which model many two phase flows, and are free boundary problems for (1):





Compressible Euler Equations

Compressible Euler equations of gas dynamics:

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho u) = 0 \\ \partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) + \nabla P = 0 \\ \partial_t(\varrho S) + \operatorname{div}(\varrho u S) = 0, \end{cases} \tag{2}$$

where $P = P(\varrho, S) = A\varrho^\gamma e^S$ with constants $A > 0,\ \gamma > 1.$ Note that (2) is hyperbolic if $\varrho > 0$ (The prototype systems of hyperbolic Conservation Laws).

Rankine–Hugoniot jump conditions across a discontinuity surface $\Sigma(t)$:

$$[\![\mathrm{j}]\!] = 0,\ \mathrm{j}\,[\![u_n]\!] + [\![P]\!] = 0,\ \mathrm{j}\,[\![u_\tau]\!] = 0,\ \mathrm{j}\,[\![S]\!] = 0. \tag{3}$$

Here $\mathrm{j} = \varrho(u_n - \mathcal{V})$ is the mass transfer flux, with $\mathcal{V}$ normal velocity of $\Sigma(t)$, n normal vector and $\tau = \tau_i,\ i = 1, 2,$ tangential vectors.

- $\mathrm{j} \neq 0,\ [\![\rho]\!] \neq 0 \Longrightarrow$ Shock Waves. - - - "non-characteristic"
- $\mathrm{j} = 0 \Longrightarrow$ Contact Discontinuities. - - - "characteristic"

  $u_n = \mathcal{V},\ [\![P]\!] = 0.$

  - If $[\![u_\tau]\!] \neq 0 \Longrightarrow$ Tangential Discontinuities (Vortex Sheets);
  - If $[\![u_\tau]\!] = 0 \Longrightarrow$ Contact Discontinuities (Entropy Waves).

Ideal Compressible MHD

Ideal compressible magnetohydrodynamics (MHD) of plasmas:

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho u) = 0 \\ \partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u - B \otimes B) + \nabla(P + \frac{1}{2}|B|^2) = 0 \\ \partial_t B - \operatorname{curl}(u \times B) = 0 \\ \operatorname{div} B = 0 \\ \partial_t(\varrho S) + \operatorname{div}(\varrho u S) = 0. \end{cases} \tag{4}$$

Rankine–Hugoniot jump conditions across $\Sigma(t)$:

$$\begin{gathered} [\![\mathrm{j}]\!] = 0,\ [\![B_n]\!] = 0,\ \mathrm{j}\,[\![u_n]\!] + \left[\!\!\left[P + \frac{1}{2}|B|^2\right]\!\!\right] = 0, \\ \mathrm{j}\,[\![u_\tau]\!] = B_n\,[\![B_\tau]\!]\,,\ \mathrm{j}\left[\!\!\left[\frac{B_\tau}{\rho}\right]\!\!\right] = B_n\,[\![u_\tau]\!]\,,\ \mathrm{j}\,[\![S]\!] = 0. \end{gathered} \tag{5}$$

- $\mathrm{j} \neq 0,\ [\![\rho]\!] \neq 0 \Longrightarrow$ Shock Waves.
- $\mathrm{j} \neq 0,\ [\![\rho]\!] = 0,\ B_n \neq 0 \Longrightarrow$ Rotational (Alfvén) Discontinuities.
- $\mathrm{j} = 0,\ B_n = 0 \Longrightarrow$ Tangential Discontinuities (Current-Vortex Sheets).
  $u_n = \mathcal{V},\ B_n = 0,\ [\![P + \frac{1}{2}|B|^2]\!] = 0$: Laboratory plasma;
- $\mathrm{j} = 0,\ B_n \neq 0 \Longrightarrow$ Contact Discontinuities (MHD Contact Discontinuities).
  $u_n = \mathcal{V},\ B_n \neq 0,\ [\![P]\!] = [\![u]\!] = [\![B]\!] = 0$: Astrophysical plasma.

A Brief Review for the Euler Equations

Fact: Contact discontinuities for the Euler equations are subject to both Kelvin-Helmhotz instability and Raylei-Taylor instability, which lead to the ill-posedness of the Rayleigh-Taylor and Kelvin-Helmholtz problems:

- Incompressible Euler: Ebin ('88)
- Compressible Euler: Guo-Tice ('11)

Vortex Sheets

$$\mathcal{V} = u_{\pm} \cdot n,\ [\![P]\!] = 0 \text{ on } \Sigma(t). \tag{6}$$

$[\![u]\!] \cdot n = 0$ in (6) forms an elliptic equation for the front function when $[\![u]\!] \cdot \tau \neq 0$ in 2D, and then the Rayleigh-Taylor instability is absent.

- $M > \sqrt{2}$ in 2D.
  Coulombel-Secchi ('04, '08): isentropic;
  Morando-Trebeschi (JHDE '08), Morando-Trebeschi-T. Wang (JDE '19): nonisentropic.

Linear stability:

- Supersonic 2D vortex sheets: neutrally stable
- 3D vortex sheets and subsonic 2D vortex sheets: unstable, Syrovatskij (54), Miles (58);

Fact: Surface tension has stabilizing effects on both Kelvin-Helmholtz and Rayleigh-Taylor instabilities:

- Incompressible: Cheng-Coutand-Shkoller (CPAM '08), Shatah-Zeng (CPAM '08, ARMA '11);
- Compressible: Stevens (ARMA '16).

Related Works of MHD: I

MHD Tangential Discontinuities (Current-Vortex Sheets)

$$\mathcal{V} = u_\pm \cdot n, \; B_\pm \cdot n = 0, \; \left[\!\!\left[P + \frac{1}{2}|B|^2 \right]\!\!\right] = 0 \text{ on } \Sigma(t). \tag{7}$$

$B_\pm \cdot n = 0$ in (7) forms an elliptic equation for the front function when $B_+ \nparallel B_-$ on $\Sigma(t)$, and the Rayleigh-Taylor instability is absent then.

- $|B_+ \times B_-| > 0$ on $\Sigma(t)$ + Some Sufficient Stability Condition. Chen-Y.G. Wang (ARMA '08), Trakhinin (ARMA '05, '09).
- Syrovatskij Stability Criterion for the incompressible MHD:
  $$|[\![u]\!] \times B_+|^2 + |[\![u]\!] \times B_-|^2 < 2\,|B_+ \times B_-|^2 \text{ on } \Sigma(t).$$
  Coulombel-Morando-Secchi-Trebeschi (CMP '12): A priori nonlinear estimate under a stronger condition;
  Sun-W. Wang-Zhang (CPAM '18): Well-posedness.

$\Rightarrow$ **Strong stabilizing effects of tangential magnetic fields on Kelvin-Helmholtz instability!**

Related Works of MHD: II

MHD Contact Discontinuities (Entropy Waves)

$$\mathcal{V} = u_{\pm} \cdot n, \; B_{+} \cdot n = B_{-} \cdot n \neq 0, \; [\![P]\!] = [\![u]\!] = [\![B]\!] = 0 \text{ on } \Sigma(t). \quad (8)$$

Some basic facts on Entropy Waves:

- Only neutrally linearly stable;
- Though no Kelvin-Holmotz instability, yet allow possibility of the Rayleigh-Taylor instabilitgy due to nonlinear effects;
- B.Cs (8) contain no ellipticity for the interface function, which leads to essential difficulties even for tangential derivatives estimates due to the regularity of the interface.
- Nash-Mose type linear iteration scheme may lead to loss of derivatives.

Major Goal: Can the magnetic field prevent the nonlinear Rayleigh-Taylor instability?

Known results:

- Morando-Trakhinin-Trebeschi (JDE '15, ARMA '18): Nonlinear stability in 2D under the additional Rayleigh–Taylor sign condition; see also Trakhinin-T. Wang (ARMA '22): Nonlinear stability of a two-phase MHD for which the **surface tension** is introduced in (8).

Open problems due to M-T-T:

- The existence of MHD contact discontinuities in 3D and the question whether the Rayleigh–Taylor sign condition is necessary for the existence were then left as two open problems by Morando-Trakhinin-Trebeschi.

Main result of this talk:

- Wang-Xin ('22 to appear CPAM): Well-posedness in Sobolev spaces.

In this talk, we will focus on the case:

- $\Omega = \mathbb{T}^2 \times (-1, 1)$: horizontally periodic slab;
- $\sum(t)$ (interface) extends to infinity horizontal and lies in between $\sum_{\pm} = \mathbb{T}^2 \times \{\pm 1\}$;
- $\sum_{\pm}$: the upper and lower boundaries which are assumed to be impermeable and perfectly conducting:

$$u \cdot e_3 = 0, \quad E \times e_3 = 0 \quad \text{on} \quad \sum_{\pm}$$

  with $e_3 = (0, 0, 1)$, $E = u \times B$ is the electric field;
- $\sum(0)$ (the initial contact discontinuity) is given which is assumed to be non-intersecting $\sum_{\pm}$.

2. Main Results

Lagrangian Reformulation

- Take $\Omega_\pm := \{x_3 \gtrless 0\}$ and denote $\Sigma := \{x_3 = 0\}$. Assume that there is a diffeomorphism $\eta_0 : \Omega_\pm \to \Omega_\pm(0)$ and define the flow map
$$\begin{cases} \partial_t \eta(t,x) = u(t, \eta(t,x)), \ t > 0 \\ \eta(0,x) = \eta_0(x). \end{cases} \tag{9}$$
- Assume that $\eta(t,\cdot) : \Omega_\pm \to \Omega_\pm(t)$ is invertible and define
$$(\rho, v, b, s, p)(t,x) := (\varrho, u, B, S, P)(t, \eta(t,x)). \tag{10}$$
One has $\partial_t s = 0$, which implies $s = s_0$. In Lagrangian coordinates,
$$\begin{cases} \partial_t \eta = v & \text{in } \Omega_\pm \\ \frac{1}{\gamma p}\partial_t p + \operatorname{div}_{\mathcal{A}} v = 0 & \text{in } \Omega_\pm \\ \rho \partial_t v + \nabla_{\mathcal{A}}(p + \frac{1}{2}|b|^2) = b \cdot \nabla_{\mathcal{A}} b & \text{in } \Omega_\pm \\ \partial_t b + b \operatorname{div}_{\mathcal{A}} v = b \cdot \nabla_{\mathcal{A}} v & \text{in } \Omega_\pm \\ \operatorname{div}_{\mathcal{A}} b = 0 & \text{in } \Omega_\pm \\ [\![p]\!] = 0, \ [\![v]\!] = 0, \ [\![b]\!] = 0 & \text{on } \Sigma \\ (\eta, p, v, b) \mid_{t=0} = (\eta_0, p_0, v_0, b_0), \end{cases} \tag{11}$$
where $\rho = \rho_0 p_0^{-\frac{1}{\gamma}} p^{\frac{1}{\gamma}}$. Here $\partial_i^{\mathcal{A}} := \mathcal{A}_{ij}\partial_j$ for $\mathcal{A} := (\nabla \eta)^{-T}$.

Expressions of ρ, p and b

- Denote $J := \det(\nabla\eta)$, one has

$$\partial_t J = J \operatorname{div}_{\mathcal{A}} v. \tag{12}$$

One then finds that $\partial_t(\rho J) = 0$ and hence

$$\rho = \rho_0 J_0 J^{-1} \text{ and } p = p_0 J_0^\gamma J^{-\gamma}, \tag{13}$$

and that $\partial_t(J\mathcal{A}^T b) = 0$ and hence

$$\boxed{b = J^{-1} J_0 \mathcal{A}_0^T b_0 \cdot \nabla\eta.} \tag{14}$$

We may refer to (14) as the Cauchy formula for b as its analogue to Cauchy's vorticity formula (Cauchy 1882) for the Euler equations.

Proposition

(i) $\partial_t(J \operatorname{div}_{\mathcal{A}} b) = 0$; *(ii)* $\partial_t(b \cdot \mathcal{N}) = 0$, *where* $\mathcal{N} := J\mathcal{A}e_3 = \partial_1\eta \times \partial_2\eta$.

Proposition

Assume that $[\![\eta_0]\!] = [\![\partial_3\eta_0]\!] = [\![p_0]\!] = [\![b_0]\!] = 0$ *and* $b_0 \cdot \mathcal{N}_0 \neq 0$ *on* Σ. *Then*

$$[\![\partial_3 v]\!] = [\![\eta]\!] = [\![\partial_3\eta]\!] = 0 \text{ on } \Sigma. \tag{15}$$

Main Theorem

Let $m \geq 4$ be an integer. Define the energy as

$$\mathcal{E}_m := \sum_{j=0}^{m} \left\| (\partial_t^j p, \partial_t^j v, \partial_t^j b) \right\|_{m-j}^2 + \|\eta\|_m^2 + |\eta|_m^2. \tag{16}$$

Denote

$$\mathcal{M}_0^m := P\left(\|(\eta_0, p_0, v_0, b_0, \rho_0)\|_m^2 + |\eta_0|_m^2 \right). \tag{17}$$

Theorem (Wang-Xin '22 to appear CPAM)

Assume that $\eta_0 \in H^m(\Omega_\pm) \cap H^m(\Sigma)$ *and* $p_0, v_0, b_0, \rho_0 \in H^m(\Omega_\pm)$ *are given such that* $\operatorname{div}_{\mathcal{A}_0} b_0 = 0$ *in* $\Omega_\pm$*,*

$$\begin{aligned} &[\![\eta_0]\!] = [\![\partial_3 \eta_0]\!] = 0 \text{ and } [\![b_0]\!] \cdot \mathcal{N}_0 = 0 \text{ on } \Sigma, \\ &\rho_0, p_0, |J_0| \geq c_0 > 0 \text{ in } \Omega_\pm \text{ and } |b_0 \cdot \mathcal{N}_0| \geq c_0 > 0 \text{ on } \Sigma \end{aligned} \tag{18}$$

and the necessary $(m-1)$*-th order compatibility conditions are satisfied. Then there exist a* $T_0 > 0$ *and a unique solution* (η, p, v, b) *to* (11) *on the time interval* $[0, T_0]$ *which satisfies*

$$\mathcal{E}_m(t) \leq \mathcal{M}_0^m, \ \forall t \in [0, T_0]. \tag{19}$$

Remarks

Remark

Our result in particular removes the assumption of the Rayleigh–Taylor sign condition required by Morando-Trakhinin-Trebeschi and solves the two open questions raised by them. This shows also the strong stabilizing effect of the transversal magnetic field on the Rayleigh–Taylor instability. The key point here is the new boundary regularity $|\eta|_m^2$*, which is captured from the regularizing effect of the transversal magnetic field.*

Remark

Note that there is no loss of derivatives in our well-posedness theory, which is in contrast to all the previous works on the compressible MHD where the solution is constructed by employing the Nash–Moser-type linearized iteration scheme and thus has a loss of derivatives.

Remark

The result here holds also for the cases that $\Omega = \mathbb{R}^2 \times (-1, 1)$ *or* $\Omega = \mathbb{R}^3$ *provided that we replace* (η, p, v, b, ρ) *in (16) and (17) by* $(\eta - Id, p - \bar{p}, v - \bar{v}, b - \bar{b}, \rho - \bar{\rho})$ *with* $(\bar{p}, \bar{v}, \bar{b}, \bar{\rho})$ *being a trivial contact-discontinuity state.*

Remark

Our analysis depends crucially on the following:

- *Transversality of the magnetic field across the interface;*
- *Cauchy formula for the magnetic field;*
- *an elaborate nonlinear viscous approximation.*

3. Key Ingredients
Typical Difficulties

- Denote $q = p + \frac{1}{2}|b|^2$ for the total pressure, and one has

$$\begin{cases} \frac{1}{\gamma p}\partial_t q - \frac{1}{\gamma p} b \cdot \partial_t b + \operatorname{div}_{\mathcal{A}} v = 0 & \text{in } \Omega_\pm \\ \rho \partial_t v + \nabla_{\mathcal{A}} q - b \cdot \nabla_{\mathcal{A}} b = 0 & \text{in } \Omega_\pm \\ \partial_t b - \frac{b}{\gamma p}\partial_t q + \frac{b}{\gamma p} b \cdot \partial_t b - b \cdot \nabla_{\mathcal{A}} v = 0 & \text{in } \Omega_\pm. \end{cases} \tag{20}$$

Set $Z_1 = \partial_1, Z_2 = \partial_2, Z_3 = x_3\partial_3$ and apply Z^m to (20) (Co-normal derivatives estimates).

- Typically, the estimate of $\left[Z^m, \partial_i^{\mathcal{A}}\right]$ yields a loss of one derivative (control of $||Z^m \nabla \eta||_0$). Motivated by Alinhac ('89), it is natural to introduce good unknowns (m is the highest order)

$$\begin{gathered}\mathcal{Q}^m = Z^m q - Z^m \eta \cdot \nabla_{\mathcal{A}} q, \ \mathcal{V}^m = Z^m v - Z^m \eta \cdot \nabla_{\mathcal{A}} v, \\ \mathcal{B}^m = Z^m b - Z^m \eta \cdot \nabla_{\mathcal{A}} b.\end{gathered} \tag{21}$$

This leads to that, by using $\partial_t \eta = v$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega_\pm} J\left(\frac{1}{\gamma p}\left|\mathcal{Q}^m - b\cdot\mathcal{B}^m\right|^2 + \rho\left|\mathcal{V}^m\right|^2 + \left|\mathcal{B}^m\right|^2\right) \\ &= \int_\Sigma [\![\mathcal{Q}^m]\!]\,\mathcal{V}^m\cdot\mathcal{N} - b\cdot\mathcal{N}\,[\![\mathcal{B}^m]\!]\cdot\mathcal{V}^m + \cdots \\ &= -\int_\Sigma J^{-1} Z^m \eta\cdot\mathcal{N}[\![\partial_3 q]\!]\partial_t Z^m \eta\cdot\mathcal{N} + b\cdot\mathcal{N} J^{-1} Z^m \eta\cdot\mathcal{N}[\![\partial_3 b]\!]\cdot\partial_t Z^m \eta + \cdots .\end{aligned} \tag{22}$$

New Good Unknown I (For magnetic field)

- The geometric symmetry structure of the first term in (22) is crucial:

$$-\int_\Sigma J^{-1} Z^m\eta\cdot\mathcal{N}\,[\![\partial_3 q]\!]\,\partial_t Z^m\eta\cdot\mathcal{N} = -\frac{1}{2}\frac{d}{dt}\int_\Sigma [\![\partial_3 q]\!]\,J^{-1}\left|Z^m\eta\cdot\mathcal{N}\right|^2+\cdots. \tag{23}$$

  However, there is no such symmetry for the second term in (22), which vanishes for current-vortex sheets.

- Our way to overcome this difficulty is to make use of the Cauchy formula (14) (and (13)) so that

$$b\cdot\nabla_{\mathcal{A}}b \equiv \mathcal{A}^T b\cdot\nabla b = J^{-1}J_0\mathcal{A}_0^T b_0\cdot\nabla b = \rho\rho_0^{-1}b_0\cdot\nabla_{\mathcal{A}_0}b, \tag{24}$$

  which allows one to introduce instead the **new good unknown**

$$\mathcal{B}^m = Z^m b - Z^m\eta_0\cdot\nabla_{\mathcal{A}_0}b. \tag{25}$$

Due to (25), the second term in (22) is changed to be

$$\int_\Sigma b_0 \cdot \mathcal{N}_0 J_0^{-1} Z^m \eta_0 \cdot \mathcal{N}_0 \llbracket \partial_3 b \rrbracket \cdot \partial_t Z^m \eta$$
$$= \frac{d}{dt} \int_\Sigma b_0 \cdot \mathcal{N}_0 J_0^{-1} Z^m \eta_0 \cdot \mathcal{N}_0 \llbracket \partial_3 b \rrbracket \cdot Z^m \eta + \cdots \,, \qquad (26)$$

and the integrand is linear in highest order derivatives!

New Good Unknown II (For the Interface regularity)

- By (23) and (26), one deduces from (22) that

$$\|Z^m(p,v,b)(t)\|_0^2 \lesssim \mathcal{M}_0^m + |Z^m\eta(t)|_0^2 + t^{1/2}P(\mathcal{E}_m(t)). \quad (27)$$

- Now our key point here is to use further the Cauchy formula (14) in $Z^m b = Z^m(\rho\rho_0^{-1} b_0 \cdot \nabla_{\mathcal{A}_0}\eta)$ and then introduce the **good unknown**

$$\Xi^m := Z^m\eta - Z^m\eta_0 \cdot \nabla_{\mathcal{A}_0}\eta. \quad (28)$$

These allow one to add $\left\|\mathcal{A}_0^T b_0 \cdot \nabla\Xi^m\right\|_0^2$ to LHS of (27). Recall that $(\mathcal{A}_0^T b_0)_3 = J_0^{-1} b_0 \cdot \mathcal{N}_0 \neq 0$ near Σ, and the boundary regularizing effect of the magnetic field is then captured by

$$|\Xi^m|_0^2 \lesssim \left\|\mathcal{A}_0^T b_0 \cdot \nabla\Xi^m\right\|_0 \|\Xi^m\|_0 + \|\Xi^m\|_0^2. \quad (29)$$

One can then improve (27) to be

$$\|Z^m(p,v,b)(t)\|_0^2 + |Z^m\eta(t)|_0^2 \leq \mathcal{M}_0^m + t^{1/2}P(\mathcal{E}_m(t)). \quad (30)$$

- As in Yanagisawa and Matsumura (CMP '91), due to $b_0 \cdot \mathcal{N}_0 \neq 0$ near Σ, (p, v, b_τ) are non-characteristic, and the normal derivative of the characteristic b_n is estimated through $J \operatorname{div}_{\mathcal{A}} b = J_0 \operatorname{div}_{\mathcal{A}_0} b_0$.

Nonlinear Viscous Approximation

- Our solution to (11) is constructed as the **inviscid limit** of

$$\begin{cases} \partial_t \eta = v & \text{in } \Omega_\pm \\ \frac{1}{\gamma p}\partial_t p + \operatorname{div}_{\mathcal{A}} v = 0 & \text{in } \Omega_\pm \\ \rho \partial_t v + \nabla_{\mathcal{A}}(p + \frac{1}{2}|b|^2) - \varepsilon \Delta_{\mathcal{A}} v = b \cdot \nabla_{\mathcal{A}} b + \Psi^{\varepsilon,\delta} & \text{in } \Omega_\pm \\ \partial_t b + b \operatorname{div}_{\mathcal{A}} v = b \cdot \nabla_{\mathcal{A}} v & \text{in } \Omega_\pm \\ [\![p]\!] = 0,\ [\![v]\!] = 0,\ [\![b]\!] = 0,\ [\![\partial_3 v]\!] = 0 & \text{on } \Sigma \\ (\eta, p, v, b)\mid_{t=0} = (\eta_0^\delta, p_0^\delta, v_0^\delta, b_0^\delta) \end{cases} \tag{31}$$

with $\rho = \rho_0^\delta (p_0^\delta)^{-\frac{1}{\gamma}} p^{\frac{1}{\gamma}}$, $\delta > 0$ is the smoothing parameter. Note that

$$J \operatorname{div}_{\mathcal{A}} b = J_0^\delta \operatorname{div}_{\mathcal{A}_0^\delta} b_0^\delta \text{ in } \Omega_\pm. \tag{32}$$

- Crucially, the jump conditions in (31) are essentially same as those of (11), but not standard for solving the two-phase viscous MHD. Our way of getting around this difficulty is to replace them by the following "standard" jump conditions:

$$[\![v]\!] = 0,\ [\![\nabla_{\mathcal{A}} v]\!] \mathcal{N} = 0 \text{ on } \Sigma. \tag{33}$$

See Jang-Tice-Wang ('16) for the two-phase compressible NS.

- The crucial point is then that under the initial conditions these two sets of jump conditions are indeed **equivalent**.
- The choice of corrector $\Psi^{\varepsilon,\delta}$ and jump conditions in (31) make it possible to derive the (ε, δ)-independent estimates just as our a priori estimates for (11). To this end, we need to introduce suitable anisotropic energy and associated dissipations to carry out the a priori estimates. Though the analysis is technically more involved and complicated, yet the main ideas are similar to the a priori estimates for (11) which we have outlined.

Thank You!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happy 70th birthday!
I wish you to be happier and more
energetic forever!
青春不老！幸福永隨！
Buon compleanno!