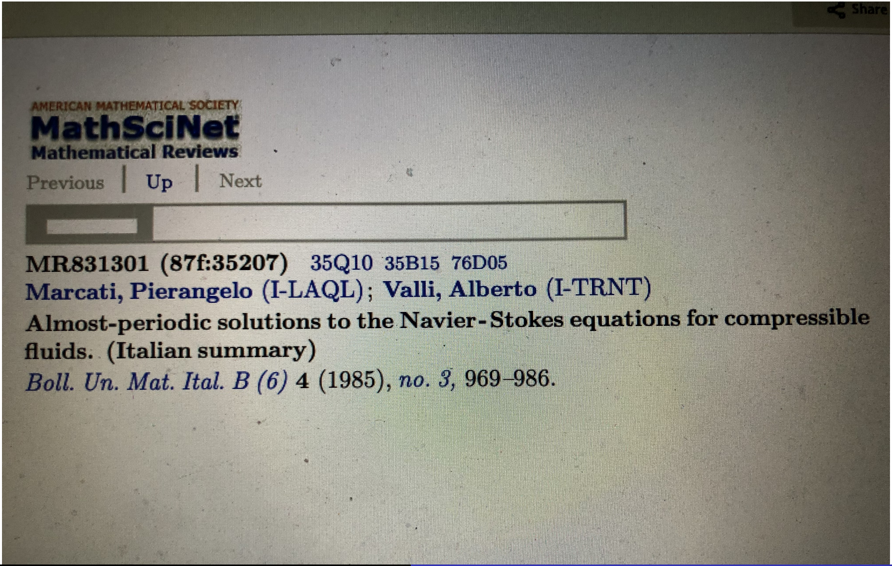
On the proof of Taylor's conjecture: helicity is conserved for a magnetically-closed plasma

Alberto Valli

Dipartimento di Matematica, Università di Trento, Italy



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Marcati, Pierangelo (I-LAQL); **Valli, Alberto** (I-TRNT)
Almost-periodic solutions to the Navier-Stokes equations for compressible fluids. (Italian summary)
Boll. Un. Mat. Ital. B (6) **4** (1985), *no. 3,* 969–986.

Our presentation is related to the papers

- A. V., *A variational interpretation of the Biot–Savart operator and the helicity of a bounded domain*, J. Math. Phys., 60 (2019), No. 021503, 7 pp.
- D. MacTaggart and A. V., *Magnetic helicity in multiply connected domains*, J. Plasma Phys., 85 (2019), No. 775850501, 15 pp.
- D. Faraco and S. Lindberg, *Proof of Taylor's conjecture on magnetic helicity conservation*, Comm. Math. Phys., 373 (2020), pp. 707–738.
- D. Faraco, S. Lindberg, D. MacTaggart and A. V., *On the proof of Taylor's conjecture in multiply connected domains*, Appl. Math. Letters, 124 (2022), No. 107654, 7 pp.

Outline

Introduction and physical remarks

The problem

The Taylor conjecture is a classical problem in plasma physics and has been formulated years ago in a couple of papers (Taylor (1974) and (1986)). It can be expressed as follows:

- in ideal MagnetoHydroDynamics (MHD) the helicity of the magnetic flux density $\mathbf{B}$ is conserved in time, no matter the topological shape of the domain containing the fluid.

Here "ideal" means that the kinematic viscosity ν and the magnetic resistivity η are equal to 0; what is the helicity will be clarified in the next slides.

Basic notations and geometry

We assume that Ω is a bounded connected open set in $\mathbb{R}^3$, with a sufficiently smooth boundary Γ (say, $\Gamma \in C^{1,1}$).

The unit outward normal vector on Γ will be denoted by $\mathbf{n}$.

Basic notations and geometry (cont'd)

We recall some geometrical results (see, e.g., Cantarella et al. (2002); see also Benedetti et al. (2012)).

Suppose that the first Betti number of $\overline{\Omega}$ is positive, say, $g > 0$ (for $g = 0$, namely, a simply-connected domain, what we are going to explain is not needed); then the first Betti number of Γ is equal to $2g$ and it is possible to consider $2g$ non-bounding cycles on Γ, $\{\gamma_j\}_{j=1}^g \cup \{\gamma'_j\}_{j=1}^g$, that are (representative of) the generators of the first homology group of Γ.

They are such that $\{\gamma_j\}_{j=1}^g$ are (representative of) the generators of the first homology group of $\overline{\Omega}$ (the tangent vector on γ_j is denoted by $\mathbf{t}_j$), while $\{\gamma'_j\}_{j=1}^g$ are (representative of) the generators of the first homology group of $\overline{\Omega'}$, where $\Omega' = B \setminus \overline{\Omega}$, B being an open ball containing $\overline{\Omega}$ (the tangent vector on γ'_j is denoted by $\mathbf{t}'_j$).

Basic notations and geometry (cont'd)

It is also known that

- in Ω there exist g 'cutting' surfaces $\{\Sigma_j\}_{j=1}^g$, that are connected orientable Lipschitz surfaces with $\partial\Sigma_j \subset \Gamma$, such that every curl-free vector in Ω has a global potential in the 'cut' domain $\Omega^0 := \Omega \setminus \bigcup_{j=1}^g \Sigma_j$; each surface Σ_j satisfies $\partial\Sigma_j = \gamma'_j$, 'cuts' the corresponding cycle γ_j and does not intersect the other cycles γ_i for $i \neq j$;
- in Ω' there exist g 'cutting' surfaces $\{\Sigma'_j\}_{j=1}^g$, that are connected orientable Lipschitz surfaces with $\partial\Sigma'_j \subset \Gamma$, such that every curl-free vector in Ω' has a global potential in the 'cut' domain $(\Omega')^0 := \Omega' \setminus \bigcup_{j=1}^g \Sigma'_j$; each surface Σ'_j satisfies $\partial\Sigma'_j = \gamma_j$, 'cuts' the corresponding cycle γ'_j, and does not intersect the other cycles γ'_i for $i \neq j$.

Basic notations and geometry (cont'd)

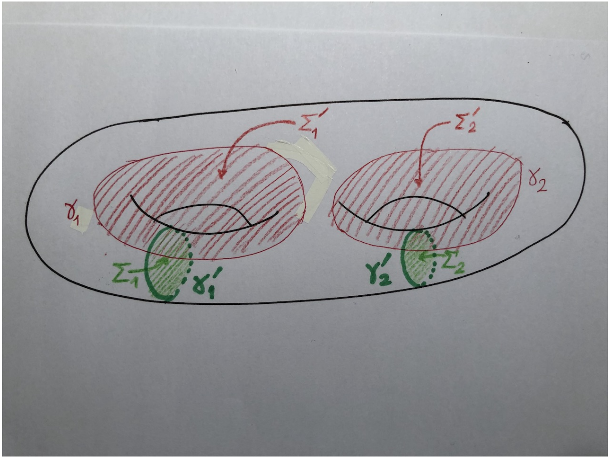


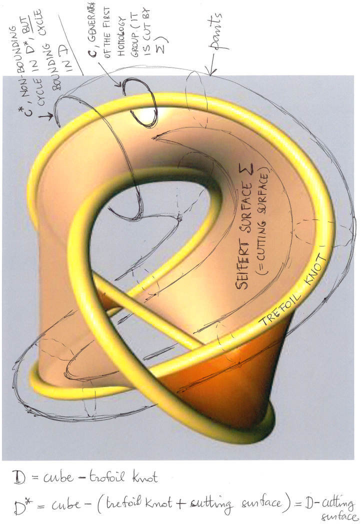
Figure: Ω is the two-fold torus.

Basic notations and geometry (cont'd)

[Looking back at the literature on this topic, where some misunderstanding can be noticed, it is interesting to make clear that:

- the statement concerning the 'cutting' surfaces Σ_j does not mean that the 'cut' domain Ω^0 is simply-connected nor that it is homologically trivial: an example in this sense is furnished by $\Omega = Q \setminus K$, where Q is a cube and K is the trefoil knot.]

The trefoil knot and its Seifert surface



Functional spaces

We define the Hilbert spaces

- $H(\mathrm{curl};\Omega) = \{\mathbf{w} \in (L^2(\Omega))^3 \,|\mathrm{curl}\,\mathbf{w} \in (L^2(\Omega))^3\}\,,$
  endowed with the norm

$$\|\mathbf{w}\|_{\mathrm{curl};\Omega} = \{\|\mathbf{w}\|^2_{L^2(\Omega)} + \|\mathrm{curl}\,\mathbf{w}\|^2_{L^2(\Omega)}\}^{1/2}\,;$$

- $\mathcal{V} = \{\mathbf{w} \in (L^2(\Omega))^3 \,|\, \mathrm{div}\,\mathbf{w} = 0 \text{ in } \Omega, \mathbf{w}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}\,.$
  endowed with the norm

$$\|\mathbf{w}\|_{\mathcal{V}} = \|\mathbf{w}\|_{L^2(\Omega)}\,.$$

The magnetic flux density **B** for a confined (magnetically-closed) plasma typically belongs to $\mathcal{V}$.

The space of harmonic fields

We also need to introduce the space of harmonic Neumann vector fields

$$\mathcal{H}(m) = \{\boldsymbol{\rho} \in (L^2(\Omega))^3 \,|\, \mathrm{curl}\, \boldsymbol{\rho} = \mathbf{0} \text{ in } \Omega, \\ \mathrm{div}\, \boldsymbol{\rho} = 0 \text{ in } \Omega, \boldsymbol{\rho} \cdot \mathbf{n} = 0 \text{ on } \Gamma\} \, .$$

This space has dimension g, the first Betti number of $\overline{\Omega}$; in particular, it is trivial for a simply-connected domain Ω. A basis for it will be denoted by $\{\boldsymbol{\rho}_j\}_{j=1}^{g}$, where $\boldsymbol{\rho}_j$ satisfies $\oint_{\gamma_k} \boldsymbol{\rho}_j \cdot \mathbf{t}_k = \delta_{jk}$ (see, e.g., Cantarella et al. (2002), Alonso Rodríguez et al. (2018)).

The helicity and the Biot–Savart operator

The helicity of a vector field

What is the helicity? Let us give a precise definition (and we will return on this later on).

The helicity of a vector field $\mathbf{v}$, a concept introduced by Woltjer (1958) and named by Moffatt (1969), is given by

$$H(\mathbf{v}) = \frac{1}{4\pi} \int_\Omega \int_\Omega \mathbf{v}(\mathbf{x}) \times \mathbf{v}(\mathbf{y}) \cdot \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \, d\mathbf{x} \, d\mathbf{y} \, .$$

It is a "measure of the extent to which the field lines wrap and coil around one another" [Cantarella et al. (2000a), Cantarella et al. (2001)]. Focusing on the physical meaning, "it is widely recognized that the key property of turbulence that is most conducive to dynamo action is its helicity" [Moffatt (2016)].[1]

[1] Dynamo action is the physical mechanism through which a rotating, convecting, and electrically conducting fluid is able to maintain a magnetic field.

The Biot–Savart operator

The Biot–Savart operator BS is defined in $\boldsymbol{\mathcal{V}}$ as

$$BS(\mathbf{v})(\mathbf{x}) = \frac{1}{4\pi} \int_\Omega \mathbf{v}(\mathbf{y}) \times \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\, d\mathbf{y} \quad , \quad \mathbf{x} \in \mathbb{R}^3 . \qquad (1)$$

The relation between helicity and Biot–Savart operator is clearly expressed by

$$H(\mathbf{v}) = \int_\Omega \mathbf{v} \cdot BS(\mathbf{v})_{|\Omega} .$$

Therefore for having a better understanding of helicity it is interesting to analyze the Biot–Savart operator more in depth.

Introducing the vector field

$$\widetilde{\mathbf{v}} = \begin{cases} \mathbf{v} & \text{in } \Omega \\ \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \overline{\Omega} \end{cases}$$

we see that $BS(\mathbf{v})$ can be clearly rewritten as

$$BS(\mathbf{v})(\mathbf{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \widetilde{\mathbf{v}}(\mathbf{y}) \times \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \, d\mathbf{y} \, .$$

The Biot–Savart operator (cont'd)

Since $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$ and $\operatorname{div} \mathbf{v} = 0$ in Ω, it follows that $\widetilde{\mathbf{v}}$ satisfies $\operatorname{div} \widetilde{\mathbf{v}} = 0$ in $\mathbb{R}^3$.

Therefore it is well-known that $BS(\mathbf{v}) \in (H^1(\mathbb{R}^3))^3$ and satisfies the relations $\operatorname{curl} BS(\mathbf{v}) = \widetilde{\mathbf{v}}$ and $\operatorname{div} BS(\mathbf{v}) = 0$ in $\mathbb{R}^3$.

Hence the restriction of $BS(\mathbf{v})$ to Ω, denoted by $BS_\Omega(\mathbf{v})$, satisfies $BS_\Omega(\mathbf{v}) \in (H^1(\Omega))^3$ and

$$\begin{cases} \operatorname{curl} BS_\Omega(\mathbf{v}) = \mathbf{v} & \text{in } \Omega \\ \operatorname{div} BS_\Omega(\mathbf{v}) = 0 & \text{in } \Omega\,. \end{cases}$$

The projected Biot–Savart operator

Let us introduce the scalar function $\phi_{\mathbf{v}} \in H^1(\Omega)$, solution to the Neumann problem

$$\begin{cases} \Delta\phi_{\mathbf{v}} = 0 & \text{in } \Omega \\ \operatorname{grad}\phi_{\mathbf{v}} \cdot \mathbf{n} = BS_\Omega(\mathbf{v}) \cdot \mathbf{n} & \text{on } \partial\Omega \\ \int_\Omega \phi_{\mathbf{v}} = 0\,, & \end{cases}$$

whose existence is guaranteed by the fact that

$$\int_{\partial\Omega} BS_\Omega(\mathbf{v}) \cdot \mathbf{n} = \int_\Omega \operatorname{div} BS_\Omega(\mathbf{v}) = 0\,.$$

The projected Biot–Savart operator (cont'd)

The projected Biot–Savart operator is defined in $\boldsymbol{\mathcal{V}}$ as follows:

$$\widehat{BS}(\mathbf{v}) = BS_{\Omega}(\mathbf{v}) - \operatorname{grad} \phi_{\mathbf{v}} \,. \tag{2}$$

Clearly, $\widehat{BS}(\mathbf{v})$ is the $(L^2(\Omega))^3$-orthogonal projection of $BS_{\Omega}(\mathbf{v})$ over $\boldsymbol{\mathcal{V}}$, and satisfies

$$\begin{cases} \operatorname{curl} \widehat{BS}(\mathbf{v}) = \mathbf{v} & \text{in } \Omega \\ \operatorname{div} \widehat{BS}(\mathbf{v}) = 0 & \text{in } \Omega \\ \widehat{BS}(\mathbf{v}) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \,. \end{cases} \tag{3}$$

Vanishing line integrals

Another important property of both standard and projected Biot–Savart field is the following:

Proposition (1)

It holds

$$\oint_{\gamma_j} BS_\Omega(\mathbf{v}) \cdot \mathbf{t}_j = 0 \textit{ and } \oint_{\gamma_j} \widehat{BS}(\mathbf{v}) \cdot \mathbf{t}_j = 0 \quad \forall\, j = 1, \ldots g\,.$$

Vanishing line integrals (cont'd)

Proof. Let us recall that $BS_\Omega(\mathbf{v})$ is the restriction to Ω of $BS(\mathbf{v})$ defined in $\mathbb{R}^3$: hence we can apply the Stokes theorem on the surface $\Sigma'_j \subset \Omega'$, which satisfies $\partial\Sigma'_j = \gamma_j$. We have

$$\oint_{\gamma_j} BS_\Omega(\mathbf{v}) \cdot \mathbf{t}_j = \int_{\Sigma'_j} \mathrm{curl}\, BS(\mathbf{v}) \cdot \mathbf{n} = 0\,,$$

as $\mathrm{curl}\, BS(\mathbf{v}) = \widetilde{\mathbf{v}}$ in $\mathbb{R}^3$, hence $\mathrm{curl}\, BS(\mathbf{v}) = \mathbf{0}$ in Ω'.
The same result holds for $\widehat{BS}(\mathbf{v})$, as it differs from $BS_\Omega(\mathbf{v})$ by $\mathrm{grad}\, \phi_{\mathbf{v}}$. □

A characterization of the projected Biot–Savart operator

In conclusion, the projected Biot–Savart field $\widehat{BS}(\mathbf{v})$ satisfies

$$\begin{cases} \operatorname{curl} \widehat{BS}(\mathbf{v}) = \mathbf{v} & \text{in } \Omega \\ \operatorname{div} \widehat{BS}(\mathbf{v}) = 0 & \text{in } \Omega \\ \widehat{BS}(\mathbf{v}) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \\ \oint_{\gamma_j} \widehat{BS}(\mathbf{v}) \cdot \mathbf{t}_j = 0 & \forall\, j = 1, \dots g\,. \end{cases} \tag{4}$$

It is well-known that this problem has a unique solution (here we will show that problem (4) is equivalent to a well-posed saddle-point variational problem).

A consequence is that the projected Biot–Savart operator is completely characterized as the solution operator to problem (4).

A variational formulation

Define now

$$\boldsymbol{\mathcal{X}} = \{\mathbf{w} \in H(\mathrm{curl};\Omega) \,|\, \mathrm{curl}\ \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } \Gamma\}$$

$$\boldsymbol{\mathcal{Z}} = \{\mathbf{w} \in \boldsymbol{\mathcal{X}} \mid \oint_{\gamma_j} \mathbf{w} \cdot \mathbf{t}_j = 0 \text{ for } j = 1, \ldots, g\}$$

$$\boldsymbol{\mathcal{H}} = \mathrm{grad}\, H^1(\Omega)\,.$$

Note also that $\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{H}}^{\perp}$.

A variational formulation (cont'd)

A suitable variational formulation of problem (4) is the following constrained least-square formulation.

For $\mathbf{v} \in \boldsymbol{\mathcal{V}}$, the couple $(\widehat{BS}(\mathbf{v}), \mathbf{0})$ is the solution $(\mathbf{u}, \mathbf{q}) \in \boldsymbol{\mathcal{Z}} \times \boldsymbol{\mathcal{H}}$ of the problem

$$\int_\Omega \operatorname{curl} \mathbf{u} \cdot \operatorname{curl} \mathbf{w} + \int_\Omega \mathbf{q} \cdot \mathbf{w} = \int_\Omega \mathbf{v} \cdot \operatorname{curl} \mathbf{w} \tag{5a}$$

$$\int_\Omega \mathbf{u} \cdot \mathbf{p} = 0 \tag{5b}$$

for each $(\mathbf{w}, \mathbf{p}) \in \boldsymbol{\mathcal{Z}} \times \boldsymbol{\mathcal{H}}$.

Note that equation (5b) says that $\mathbf{u} \in \boldsymbol{\mathcal{H}}^\perp = \boldsymbol{\mathcal{V}}$.

A variational formulation (cont'd)

The existence and uniqueness theory for problem (5) is based on classical results for saddle-point problems. It can be proved that it has a solution and that the solution is unique.

The projected Biot–Savart operator revisited

We have thus characterized the projected Biot–Savart operator $\widehat{BS}$ in the following way.

Theorem (2)

Let $\mathbf{T} : \boldsymbol{\mathcal{V}} \to \boldsymbol{\mathcal{Z}} \cap \boldsymbol{\mathcal{V}}$ be the solution operator $\mathbf{T}\mathbf{v} = \mathbf{u}$, where $(\mathbf{u}, \mathbf{q}) \in \boldsymbol{\mathcal{Z}} \times \boldsymbol{\mathcal{H}}$ is the solution to problem (5). Then $\mathbf{T}$ is the projected Biot–Savart operator $\widehat{BS}$.

This characterization opens the way to a complete spectral analysis. In fact, the projected Biot–Savart operator is self-adjoint and compact in $\boldsymbol{\mathcal{V}}$ (see, e.g., Cantarella et al. (2001), Alonso Rodríguez et al. (2018)), therefore its spectrum is discrete.

Symmetry of the curl operator

For the analysis of the spectral problem associated to the Biot–Savart operator it is fundamental to prove the symmetry of the curl operator in $\mathcal{Z}$. It can be shown that:

Theorem (3)

For all $\mathbf{v}$, $\mathbf{w} \in \mathcal{Z}$,

$$\int_\Omega (\operatorname{curl} \mathbf{w} \cdot \overline{\mathbf{v}} - \mathbf{w} \cdot \operatorname{curl} \overline{\mathbf{v}}) = 0.$$

In other words, it is possible to see that for $\mathbf{v}$, $\mathbf{w} \in \mathcal{Z}$ it holds

$$\int_\Gamma \overline{\mathbf{v}} \times \mathbf{n} \cdot \mathbf{w} = 0\,.$$

In the sequel we will be back to this result.

The spectrum of the curl operator

It is easily checked that the eigenvalues of the curl operator are the reciprocals of the eigenvalues of $\mathbf{T}$.

Note also that the curl operator is an unbounded operator from $\mathcal{Z} \subset (L^2(\Omega))^3$ in $(L^2(\Omega))^3$. Having seen that it is symmetric, the proof that it is self-adjoint requires some additional work. Indeed, the following sufficient conditions hold: first, its spectrum consists only of eigenvalues; second, the range of $\mathrm{curl} \pm i\mathbf{I}$ is the whole space $(L^2(\Omega))^3$.

Back to the helicity

Let us go back to the helicity of a vector field $\mathbf{v} \in (L^2(\Omega))^3$, defined as

$$H(\mathbf{v}) = \frac{1}{4\pi} \int_\Omega \int_\Omega (\mathbf{v}(\mathbf{x}) \times \mathbf{v}(\mathbf{y})) \cdot \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^3} \, d\mathbf{x} \, d\mathbf{y} \, .$$

We have already seen that

$$H(\mathbf{v}) = \int_\Omega \mathbf{v} \cdot BS_\Omega(\mathbf{v}) \, .$$

If the vector field $\mathbf{v}$ satisfies the additional assumption $\mathbf{v} \in \boldsymbol{\mathcal{V}}$, an easy consequence of the fact that $\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{H}}^\perp$ is that

$$H(\mathbf{v}) = \int_\Omega \mathbf{v} \cdot \widehat{BS}(\mathbf{v}) \, . \qquad (6)$$

Back to the helicity (cont'd)

This last relation says that for a vector field $\mathbf{v} \in \boldsymbol{\mathcal{V}} \cap \mathcal{H}(m)^{\perp}$ the helicity could be also defined as

$$H(\mathbf{v}) = \int_{\Omega} \mathbf{v} \cdot \mathbf{A}\,, \tag{7}$$

where $\operatorname{curl} \mathbf{A} = \mathbf{v}$, namely, $\mathbf{A}$ is a vector potential of $\mathbf{v}$ (see Moffatt (1969)). In fact, for any other vector field $\mathbf{A}_\sharp$ with $\operatorname{curl} \mathbf{A}_\sharp = \mathbf{v}$ it holds $\operatorname{curl}(\mathbf{A} - \mathbf{A}_\sharp) = \mathbf{0}$ in Ω, thus $(\mathbf{A} - \mathbf{A}_\sharp) \in \boldsymbol{\mathcal{H}} \oplus \mathcal{H}(m)$. Therefore $\mathbf{v}$ is orthogonal to $\mathbf{A} - \mathbf{A}_\sharp$, and the helicity turns out to be the same for any vector potential of $\mathbf{v}$ (namely, the definition (7) is gauge invariant, and for $\mathbf{A}$ we can take $\widehat{BS}(\mathbf{v})$).

- For a simply-connected domain one has $\mathcal{H}(m) = \{\mathbf{0}\}$: thus in this case definition (7) is equivalent to (6).

Back to the helicity (cont'd)

However, this is not true if Ω is not a simply-connected domain. A gauge invariant definition of the helicity of $\mathbf{v} \in \boldsymbol{\mathcal{V}}$ in terms of a vector potential has been devised by MacTaggart and V. (2019). It reads as follows

$$H(\mathbf{v}) = \int_\Omega \mathbf{v} \cdot \mathbf{A} - \sum_{j=1}^{g} \Big(\oint_{\gamma_j} \mathbf{A} \cdot \mathbf{t}_j \Big) \Big(\int_{\Sigma_j} \mathbf{v} \cdot \mathbf{n}_j \Big) , \qquad (8)$$

where, as before, $\mathrm{curl}\, \mathbf{A} = \mathbf{v}$ (and $\mathbf{t}_j$ and $\mathbf{n}_j$ are oriented so that their scalar product at the point $\gamma_j \cap \Sigma_j$ is positive).

Note that the two definitions (6) and (8) are the same, as the projected Biot–Savart vector field $\widehat{BS}(\mathbf{v})$ satisfies Proposition (1).

Back to the helicity (cont'd)

The proof of (8) is based on the fact that a vector field $\mathbf{Q}$ with $\operatorname{curl} \mathbf{Q} = \mathbf{0}$ in Ω belongs to $\boldsymbol{\mathcal{H}} \oplus \mathcal{H}(m)$, thus it can be written as

$$\mathbf{Q} = \operatorname{grad} \eta + \sum_{j=1}^{g} \alpha_j \boldsymbol{\rho}_j \, .$$

Here, as already indicated, $\boldsymbol{\rho}_j$ are the basis elements of the space $\mathcal{H}(m)$ of harmonic Neumann vector fields; in particular, the coefficients α_j are given by $\alpha_j = \oint_{\gamma_j} \mathbf{Q} \cdot \mathbf{t}_j$.

Back to the helicity (cont'd)

Assuming that $\mathbf{v} \in \boldsymbol{\mathcal{V}}$, some computations lead to

$$\begin{aligned}\int_\Omega \mathbf{v} \cdot \mathbf{Q} &= \int_\Omega \mathbf{v} \cdot \left(\operatorname{grad} \eta + \sum_{j=1}^g \alpha_j \boldsymbol{\rho}_j\right) = \sum_{j=1}^g \alpha_j \int_\Omega \mathbf{v} \cdot \boldsymbol{\rho}_j \\ &= \sum_{j=1}^g \alpha_j \int_{\Sigma_j} \mathbf{v} \cdot \mathbf{n}_j = \sum_{j=1}^g \left(\oint_{\gamma_j} \mathbf{Q} \cdot \mathbf{t}_j \right) \left(\int_{\Sigma_j} \mathbf{v} \cdot \mathbf{n}_j \right) .\end{aligned}$$

This shows that for $\mathbf{v} \in \boldsymbol{\mathcal{V}}$ the quantity

$$\int_\Omega \mathbf{v} \cdot \mathbf{A} - \sum_{j=1}^g \left(\oint_{\gamma_j} \mathbf{A} \cdot \mathbf{t}_j \right) \left(\int_{\Sigma_j} \mathbf{v} \cdot \mathbf{n}_j \right)$$

is invariant for any $\mathbf{A}$ for which $\operatorname{curl} \mathbf{A}$ is the same.

MHD equations

The MagnetoHydroDynamics equations

Let us finally consider the MHD equations

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \frac{1}{\mu_0 \rho_0}(\mathbf{B} \cdot \nabla)\mathbf{B} - \nu \, \Delta \mathbf{u} + \nabla \pi = \mathbf{0} \\ \partial_t \mathbf{B} + \operatorname{curl} (\mathbf{B} \times \mathbf{u}) + \frac{\eta}{\mu_0} \operatorname{curl} \operatorname{curl} \mathbf{B} = \mathbf{0} \\ \operatorname{div} \mathbf{u} = 0 \\ \operatorname{div} \mathbf{B} = 0 \, , \end{cases} \tag{9}$$

with the initial conditions

$$\mathbf{u}_{|t=0} = \mathbf{u}_0 \quad , \quad \mathbf{B}_{|t=0} = \mathbf{B}_0$$

and the boundary conditions

$$\mathbf{u}_{|\Gamma} = \mathbf{0} \quad , \quad (\mathbf{B} \cdot \mathbf{n})_{|\Gamma} = 0 \quad , \quad (\operatorname{curl} \mathbf{B} \times \mathbf{n})_{|\Gamma} = \mathbf{0} \, .$$

Here $\nu > 0$ (kinematic viscosity), $\eta > 0$ (magnetic resistivity), $\mu_0 > 0$ (magnetic permeability) and $\rho_0 > 0$ (density) are physical constants.

A first conservation result

Faraco and Lindberg (2020) proved that there exists a Leray–Hopf solution of this problem.

Moreover, they proved that if a weak solution **u** and **B** of ideal MHD (namely, when $\nu = 0$ and $\eta = 0$ and the boundary conditions reduce to $(\mathbf{u} \cdot \mathbf{n})_{|\Gamma} = 0$ and $(\mathbf{B} \cdot \mathbf{n})_{|\Gamma} = 0$ on Γ) exists and is the weak limit of Leray–Hopf solutions of (9) as $\nu \to 0$ and $\eta \to 0$, then the following quantity

$$Z(\mathbf{B}(t)) = \int_{\Omega} \mathbf{B}(t) \cdot \mathbf{A}(t) - \int_{\partial\Omega} (\mathbf{A}^{\Sigma}(t) \times \mathbf{n}) \cdot \mathbf{A}^{\mathcal{H}}(t) \qquad (10)$$

is conserved in time.

Orthogonal decomposition

To understand the result we need some notation.
The space $\boldsymbol{\mathcal{V}}$ can be decomposed in two orthogonal subspaces

$$\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}_\Sigma \oplus \mathcal{H} \,,$$

where

$$\boldsymbol{\mathcal{V}}_\Sigma = \left\{ \mathbf{w} \in \boldsymbol{\mathcal{V}} \,\Big|\, \int_{\Sigma_j} \mathbf{w} \cdot \mathbf{n}_j = 0 \text{ for } j = 1, \cdots, g \right\}.$$

Thus $\mathbf{B}(t) \in \boldsymbol{\mathcal{V}}$ can be written as

$$\mathbf{B}(t) = \mathbf{B}^\Sigma(t) + \mathbf{B}^\mathcal{H}(t) \;,\; \mathbf{B}^\Sigma(t) \in \boldsymbol{\mathcal{V}}_\Sigma \;,\; \mathbf{B}^\mathcal{H}(t) \in \mathcal{H} \,.$$

Orthogonal decomposition (cont'd)

We denote by $\mathbf{A}(t)$ any vector potential of $\mathbf{B}(t)$ and by $\mathbf{A}^{\mathcal{H}}(t)$ the unique vector potential of $\mathbf{B}^{\mathcal{H}}(t)$ belonging to $\mathcal{V}_{\Sigma} \cap H(\mathrm{curl};\Omega)$.
It can be proved that

- $\mathbf{B}^{\mathcal{H}}(t) = \mathbf{B}_0^{\mathcal{H}}$ [conservation of the harmonic component]

[A consequence of this result is the conservation of the fluxes $\int_{\Sigma_j} \mathbf{B}(t) \cdot \mathbf{n}_j$ for each $j = 1, \dots, g$.]

Therefore $\mathbf{A}^{\mathcal{H}}(t) = \mathbf{A}_0^{\mathcal{H}}$ and we set

$$\mathbf{A}^{\Sigma}(t) = \mathbf{A}(t) - \mathbf{A}_0^{\mathcal{H}} .$$

so that $\mathbf{A}(t) = \mathbf{A}^{\Sigma}(t) + \mathbf{A}_0^{\mathcal{H}}$.

Helicity conservation (Ω simply-connected)

A couple of remarks: if Ω is simply-connected, and therefore $\mathcal{H}(m) = \{\mathbf{0}\}$,

- from (7) we know that the helicity of $\mathbf{B}$ is given by $H(\mathbf{B}(t)) = \int_\Omega \mathbf{B}(t) \cdot \mathbf{A}(t)$;
- we have $\mathbf{B}^{\mathcal{H}}(t) = \mathbf{0}$ and $\mathbf{A}^{\mathcal{H}}(t) = \mathbf{0}$, therefore from (10) it follows that $\int_\Omega \mathbf{B}(t) \cdot \mathbf{A}(t)$ is conserved.

In other words, if Ω is simply-connected the helicity of $\mathbf{B}$ is conserved.

What about the multiply-connected case?

Helicity conservation

Starting from (10), to prove that the Taylor conjecture is true it is enough to show that

$$Z(\mathbf{B}(t)) - Z(\mathbf{B}_0) = H(\mathbf{B}(t)) - H(\mathbf{B}_0)\,.$$

Therefore we have to show that

$$-\int_{\partial\Omega} (\mathbf{A}^{\Sigma}(t) \times \mathbf{n}) \cdot \mathbf{A}^{\mathcal{H}}(t) + \int_{\partial\Omega} (\mathbf{A}_0^{\Sigma} \times \mathbf{n}) \cdot \mathbf{A}_0^{\mathcal{H}}$$

is equal to

$$-\sum_{j=1}^{g} \Big(\oint_{\gamma_j} \mathbf{A}(t) \cdot \mathbf{t}_j \Big) \Big(\int_{\Sigma_j} \mathbf{B}(t) \cdot \mathbf{n}_j \Big) + \sum_{j=1}^{g} \Big(\oint_{\gamma_j} \mathbf{A}_0 \cdot \mathbf{t}_j \Big) \Big(\int_{\Sigma_j} \mathbf{B}_0 \cdot \mathbf{n}_j \Big)\,.$$

Helicity conservation (cont'd)

The crucial point is the representation formula

$$\int_{\partial\Omega}(\mathbf{q}\times\mathbf{n})\cdot\mathbf{p} = \sum_{j=1}^{g}\Big(\oint_{\gamma_j}\mathbf{q}\cdot\mathbf{t}_j\Big)\Big(\oint_{\gamma'_j}\mathbf{p}\cdot\mathbf{t}'_j\Big) - \sum_{j=1}^{g}\Big(\oint_{\gamma'_j}\mathbf{q}\cdot\mathbf{t}'_j\Big)\Big(\oint_{\gamma_j}\mathbf{p}\cdot\mathbf{t}_j\Big),$$

which is valid for vector fields $\mathbf{w} \in \boldsymbol{\mathcal{X}}$, namely, belonging to $H(\mathrm{curl};\Omega)$ and satisfying $(\mathrm{curl}\,\mathbf{w}\cdot\mathbf{n})_{|\Gamma} = 0$ on Γ.

The following results are easily obtained (just remember that $\gamma'_j = \partial\Sigma_j$ and use the Stokes theorem):

- $\oint_{\gamma'_j}\mathbf{A}^{\mathcal{H}}(t)\cdot\mathbf{t}'_j = \int_{\Sigma_j}\mathbf{B}^{\mathcal{H}}(t)\cdot\mathbf{n}_j = \int_{\Sigma_j}\mathbf{B}_0^{\mathcal{H}}\cdot\mathbf{n}_j$
- $\oint_{\gamma'_j}\mathbf{A}^{\Sigma}(t)\cdot\mathbf{t}'_j = \int_{\Sigma_j}\mathbf{B}^{\Sigma}(t)\cdot\mathbf{n}_j = 0$.

Helicity conservation (cont'd)

Thus:

$$
\begin{aligned}
&- \int_{\partial\Omega} (\mathbf{A}^{\Sigma}(t) \times \mathbf{n}) \cdot \mathbf{A}^{\mathcal{H}}(t) + \int_{\partial\Omega} (\mathbf{A}_0^{\Sigma} \times \mathbf{n}) \cdot \mathbf{A}_0^{\mathcal{H}} \\
&\quad = - \sum_{j=1}^{g} \Big(\oint_{\gamma_j} \mathbf{A}^{\Sigma}(t) \cdot \mathbf{t}_j \Big) \Big(\int_{\Sigma_j} \mathbf{B}_0^{\mathcal{H}} \cdot \mathbf{n}_j \Big) \\
&\qquad + \sum_{j=1}^{g} \Big(\oint_{\gamma_j} \mathbf{A}_0^{\Sigma} \cdot \mathbf{t}_j \Big) \Big(\int_{\Sigma_j} \mathbf{B}_0^{\mathcal{H}} \cdot \mathbf{n}_j \Big) \\
&\quad = \sum_{j=1}^{g} \Big(\oint_{\gamma_j} (\mathbf{A}_0^{\Sigma} - \mathbf{A}^{\Sigma}(t)) \cdot \mathbf{t}_j \Big) \Big(\int_{\Sigma_j} \mathbf{B}_0^{\mathcal{H}} \cdot \mathbf{n}_j \Big) \\
&\quad = \sum_{j=1}^{g} \Big(\oint_{\gamma_j} (\mathbf{A}_0 - \mathbf{A}(t)) \cdot \mathbf{t}_j \Big) \Big(\int_{\Sigma_j} \mathbf{B}_0^{\mathcal{H}} \cdot \mathbf{n}_j \Big) ,
\end{aligned}
$$

as $\mathbf{A}^{\mathcal{H}}(t) = \mathbf{A}_0^{\mathcal{H}}$.

Helicity conservation (cont'd)

A remark: we have proved that

$$\int_{\partial\Omega} (\mathbf{A}^{\Sigma}(t) \times \mathbf{n}) \cdot \mathbf{A}^{\mathcal{H}}(t) = \sum_{j=1}^{g} \Big(\oint_{\gamma_j} \mathbf{A}^{\Sigma}(t) \cdot \mathbf{t}_j \Big) \Big(\int_{\Sigma_j} \mathbf{B}_0^{\mathcal{H}} \cdot \mathbf{n}_j \Big) .$$

Therefore if $\mathbf{B} \in \boldsymbol{\mathcal{V}} \cap \mathcal{H}(m)^{\perp}$, so that $\int_{\Sigma_j} \mathbf{B}(t) \cdot \mathbf{n}_j = 0$ for each $j = 1, \dots, g$, we have $\int_{\Sigma_j} \mathbf{B}_0^{\mathcal{H}} \cdot \mathbf{n}_j = \int_{\Sigma_j} \mathbf{B}_0 \cdot \mathbf{n}_j = 0$ and thus $\int_{\partial\Omega} (\mathbf{A}^{\Sigma}(t) \times \mathbf{n}) \cdot \mathbf{A}^{\mathcal{H}}(t) = 0$. Hence in (10) the conserved quantity is $\int_{\Omega} \mathbf{B}(t) \cdot \mathbf{A}(t)$, that is the helicity of $\mathbf{B}$ when $\mathbf{B} \in \boldsymbol{\mathcal{V}} \cap \mathcal{H}(m)^{\perp}$ (see (7)).

However, the assumption $\mathbf{B} \in \boldsymbol{\mathcal{V}} \cap \mathcal{H}(m)^{\perp}$ is not physically justified (just think at the flux of $\mathbf{B}$ across the section of a tokamak).

The correct physical assumption on $\mathbf{B}$ must be simply $\mathbf{B} \in \boldsymbol{\mathcal{V}}$.

Helicity conservation (cont'd)

Thus let us go on with our proof. We easily have

$$\begin{aligned} -\sum_{j=1}^{g} \Big(\oint_{\gamma_j} \mathbf{A}(t) \cdot \mathbf{t}_j \Big) \Big(\int_{\Sigma_j} \mathbf{B}(t) \cdot \mathbf{n}_j \Big) \\ + \sum_{j=1}^{g} \Big(\oint_{\gamma_j} \mathbf{A}_0 \cdot \mathbf{t}_j \Big) \Big(\int_{\Sigma_j} \mathbf{B}_0 \cdot \mathbf{n}_j \Big) \\ = \sum_{j=1}^{g} \Big(\oint_{\gamma_j} (\mathbf{A}_0 - \mathbf{A}(t)) \cdot \mathbf{t}_j \Big) \Big(\int_{\Sigma_j} \mathbf{B}_0^{\mathcal{H}} \cdot \mathbf{n}_j \Big) \,, \end{aligned}$$

and we have finally shown that

$$0 = Z(\mathbf{B}(t)) - Z(\mathbf{B}_0) = H(\mathbf{B}(t)) - H(\mathbf{B}_0) \,,$$

therefore the conservation of the helicity is proved.

Helicity conservation (cont'd)

Note that, as a by-product, we have proved that

$$Z(\mathbf{B}(t)) = H(\mathbf{B}(t)) + \sum_{j=1}^{g} \Big(\oint_{\gamma_j} \mathbf{A}_0^{\mathcal{H}} \cdot \mathbf{t}_j \Big) \Big(\int_{\Sigma_j} \mathbf{B}_0^{\mathcal{H}} \cdot \mathbf{n}_j \Big) ,$$

and this clearly says that the conservation of $Z(\mathbf{B}(t))$ is equivalent to the conservation of the helicity $H(\mathbf{B}(t))$.

[To tell all the truth, this says that Faraco and Lindberg (2020) did not realize to have proved the conservation result also without requiring $\int_{\Sigma_j} \mathbf{B}(t) \cdot \mathbf{n}_j = 0$ for each $j = 1, \dots, g$...]

The helicity of a domain

The helicity of a domain

The helicity of a domain Ω is defined as

$$H_\Omega = \sup_{\mathbf{v}\in\boldsymbol{\mathcal{V}},\|\mathbf{v}\|_{L^2(\Omega)}=1} |H(\mathbf{v})| . \qquad (11)$$

As a consequence of the fact that the projected Biot–Savart operator $\widehat{BS}$ is self-adjoint and compact, the helicity of Ω can be represented as

$$H_\Omega = |\sigma^\Omega_{\max}| ,$$

where $\sigma^\Omega_{\max}$ is the eigenvalue of $\widehat{BS}$ in Ω of maximum absolute value.

Explicit value of the helicity

The geometrical domains for which the eigenvalue of maximum absolute value of the projected Biot–Savart operator $\widehat{BS}$ is known are quite a few: to our knowledge, only the ball and the spherical shell (see Cantarella et al. (2000a)).

We remind that for the ball of radius b the result is $|\sigma^{\Omega}_{\max}| \approx \frac{b}{4.49341}$ (the approximation is due to the fact that the correct denominator is the first positive solution of the equation $x = \tan x$, that approximately is 4.49341).

Numerical calculation of the helicity

Due to this lack of explicit results, it is important that an efficient approximation method for the computation of the eigenvalues is available.

Starting from the variational formulation (5), in Alonso Rodríguez et al. (2018) edge finite elements are used for the approximation of the spectrum of the operator $\widehat{BS}$, for any type of bounded domain Ω.

The "isoperimetric" problem

A geometrical question now arises:

- for which bounded domain the helicity is the maximum among all the bounded domains with the same volume?

This is an open problem. We have not a theoretical answer, but we can present some numerical computations.

The "isoperimetric" problem (cont'd)

- If Ω is a torus of radii $r_1 = 1$ and $r_2 = 0.5$ one has $|\sigma^{\Omega}_{\max}| \approx \frac{1}{4.89561} \approx 0.20426$. The helicity of a ball B having the same volume of this torus is $H_B \approx 0.23505$, a larger value.
- If Ω is a perforated cylinder (topologically, a torus) with rectangular cross section given by $[0.005, 1] \times [-0.5, 0.5]$ one has $H_\Omega \approx 0.20175$, while for the ball B with the same volume it holds $H_B \approx 0.20219$, a larger but very close value.
- If Ω is a torus of radii $r_1 = 0.505$ and $r_2 = 0.5$ one has $H_\Omega \approx 0.19073$, a larger value than that of the helicity of the ball B with the same volume, given by $H_B \approx 0.18718$.

The "isoperimetric" problem (cont'd)

This goes in the direction of confirming a conjecture in Cantarella et al. (2000b), who suggested that the domain with maximum helicity among all the domains with the same volume is not the sphere, but a sort of "extreme solid torus, in which the hole has been shrunk to a point".

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https://www.youtube.com/watch?v=iH3oOVKt0WI

The helicity of a domain (cont'd)

The proof of this result follows a well-known argument. Being self-adjoint, the projected Biot–Savart operator has a complete system of eigenfunctions $\{\boldsymbol{\omega}_k\}_{k=1}^{\infty}$, which are orthonormal in $\boldsymbol{\mathcal{V}}$ (or, equivalently, in $(L^2(\Omega)^3)$. Associated to these eigenfunctions there is a sequence of (real) eigenvalues $\{\sigma_k\}_{k=1}^{\infty}$. Therefore, writing $\mathbf{v} = \sum_{k=1}^{\infty} v_k \boldsymbol{\omega}_k$, it follows that $\|\mathbf{v}\|^2_{L^2(\Omega)} = \sum_{k=1}^{\infty} v_k^2$ and

$$\begin{aligned} H(\mathbf{v}) &= \sum_{k,j=1}^{\infty} \int_\Omega v_k \boldsymbol{\omega}_k \cdot v_j \widehat{BS}(\boldsymbol{\omega}_j) = \sum_{k,j=1}^{\infty} \int_\Omega v_k \boldsymbol{\omega}_k \cdot v_j \sigma_j \boldsymbol{\omega}_j \\ &= \sum_{k=1}^{\infty} v_k^2 \sigma_k \,. \end{aligned}$$

The helicity of a domain (cont'd)

Moreover, for $\|\mathbf{v}\|_{L^2(\Omega)} = 1$, we have

$$|H(\mathbf{v})| = \left| \sum_{k=1}^{\infty} v_k^2 \sigma_k \right| \leq |\sigma_{\max}^{\Omega}| \sum_{k=1}^{\infty} v_k^2 = |\sigma_{\max}^{\Omega}| \, ,$$

and also, being $\boldsymbol{\omega}_{\max}$ an eigenfunction associated to $\sigma_{\max}^{\Omega}$,

$$|H(\boldsymbol{\omega}_{\max})| = \left| \int_{\Omega} \boldsymbol{\omega}_{\max} \cdot \widehat{BS}(\boldsymbol{\omega}_{\max}) \right| = |\sigma_{\max}^{\Omega}| \int_{\Omega} |\boldsymbol{\omega}_{\max}|^2 = |\sigma_{\max}^{\Omega}| \, ,$$

hence $H_{\Omega} = |\sigma_{\max}^{\Omega}|$.