Hopf, Caccioppoli and Schauder, reloaded

Giuseppe Mingione

Marcati's 70th
GSSI, 06/20/2023

Five reasons why I like (and everybody should) Piero

- He is a top-notch mathematician, of course.
- He is a self-made man and mathematician. He's not coming from any school. He found his own way out of very little (this is not very common in Italy).
- He is a top-notch advisor. He likes youngsters (this is not very common in Italy).
- He's a builder, not an extractor. He built GSSI and this fixes for him a neat place in the history of Italian math.
- He likes Jazz music. I would compare him to

(pic from Wikipedia).

Functionals and equations

- We consider both integral functionals

$$v \mapsto \int_\Omega F(x, Dv)\, dx$$

- and equations of the type

$$-\mathrm{div}\, A(x, Du) = 0\,.$$

- The catch is of course given by the Euler-Lagrange equation

$$-\mathrm{div}\, \partial_z F(x, Du) = 0\,.$$

Ellipticity

- Here by ellipticity mean that a conditions of the type

$$g_1(x, |z|)\mathbb{I}_d \leq \partial_z A(x, z) \leq g_2(x, |z|)\mathbb{I}_d$$

is verified for non-negative functions $g_1, g_2 \colon [0, \infty) \to \mathbb{R}$.

- That is, in the case of functionals

$$g_1(x, |z|)\mathbb{I}_d \leq \partial_{zz} F(x, z) \leq g_2(x, |z|)\mathbb{I}_d \,.$$

- Autonomous case

$$g_1(|z|)\mathbb{I}_d \leq \partial_z A(z) \leq g_2(|z|)\mathbb{I}_d \,.$$

Uniform Ellipticity

- Uniform ellipticity means

$$\limsup_{|z|\to\infty} \frac{g_2(x,|z|)}{g_1(x,|z|)} < \infty.$$

uniformly with respect to x.

- When $\partial_z A(\cdot)$ is symmetric (for instance in the variational case)

$$\sup_{|z|\geq 1} \frac{\text{highest eigenvalue of } \partial_z A(x,z)}{\text{lowest eigenvalue of } \partial_z A(x,z)} \leq c\,.$$

Nonuniform ellipticity

- It is a classical topic.
- When dealing with equations of the type

$$-\text{div}\, A(x, Du) = [\text{right-hand side}]$$

nonuniform ellipticity means that

$$\limsup_{|z|\to\infty} \frac{\text{highest eigenvalue of } \partial_z A(x_0, z)}{\text{lowest eigenvalue of } \partial_z A(x_0, z)} = \infty$$

holds for at least one point x_0 [needless to say, $\partial_z A(\cdot)$ is symmetric here].

Nonuniformly elliptic classics $\leq 70s$

- Ladyzhenskaya & Uraltseva (Book + CPAM 1970)
- Gilbarg (1963)
- Stampacchia (CPAM 1963)
- Hartman & Stampacchia (Acta Math. 1965)
- Ivočkina & Oskolkov (Zap. LOMI 1967)
- Oskolkov (Trudy Mat. Inst. Steklov 1967)
- Serrin (Philos. Trans. Roy. Soc. London Ser. A 1969)
- Ivanov (Proc. Steklov Inst. Math. 1970)
- Trudinger (Thesis, Bull. AMS 1967, ARMA 1971)
- Leon Simon (Indiana Univ. Math. J. 1976)

Nonuniformly elliptic classics $\geq 80s$

- Trudinger (Invent. Math. 1981)
- Zhikov (papers and books from the 80/90s)
- N.N. Ural'tseva & A. B. Urdaletova (Vestnik Leningrad Univ. Math. 1984)
- Lieberman (Indiana Univ. Math. J. 1983)
- Ivanov (Proc. Steklov Inst. Math. Book 1984)
- Marcellini (ARMA 1987, JDE 1991, Ann. Pisa 1996)

THE DIRICHLET PROBLEM FOR NONUNIFORMLY ELLIPTIC EQUATIONS[1]

BY NEIL S. TRUDINGER

Communicated by F. John, January 23, 1967

Introduction. Let Ω be a bounded domain in E^n. The operator

$$Qu = a^{ij}(x, u, u_x)u_{x_i x_j} + a(x, u, u_x)$$

acting on functions $u(x) \in C^2(\Omega)$ is *elliptic* in Ω if the minimum eigenvalue $\lambda(x, u, p)$ of the matrix $[a^{ij}(x, u, p)]$ is positive in $\Omega \times E^{n+1}$. Here

$$u_x = (u_{x_1}, \cdots u_{x_n}), \qquad p = (p_1, \cdots p_n)$$

and repeated indices indicate summation from 1 to n. The functions $a^{ij}(x, u, p)$, $a(x, u, p)$ are defined in $\Omega \times E^{n+1}$. If furthermore for any $M > 0$, the ratio of the maximum to minimum eigenvalues of $[a^{ij}(x, u, p)]$ is bounded in $\Omega \times (-M, M) \times E^n$, Qu is called *uniformly elliptic*. A solution of the *Dirichlet problem* $Qu = 0$, $u = \phi(x)$ on $\partial\Omega$ is a $C^0(\bar{\Omega}) \cap C^2(\Omega)$ function $u(x)$ satisfying $Qu = 0$ in Ω and agreeing with $\phi(x)$ on $\partial\Omega$.

When Qu is elliptic, but not necessarily uniformly elliptic, it is referred to as *nonuniformly elliptic*. In this case it is well known from two dimensional considerations, that in addition to smoothness of the boundary data $\partial\Omega$, $\phi(x)$ and growth restrictions on the coefficients of Qu, geometric conditions on $\partial\Omega$ may play a role in the solvability of the Dirichlet problem. A striking example of this in higher dimensions is the recent work of Jenkins and Serrin [**4**] on the minimal surface equation, mentioned below.

On the Regularity of Generalized Solutions of Linear, Non-Uniformly Elliptic Equations

NEIL S. TRUDINGER

Communicated by J. C. C. NITSCHE

1. Introduction

We consider in this paper the simplest form of a second order, linear, divergence structure equation in n variables, namely

$$\frac{\partial}{\partial x_i}\left(a^{ij}(x)\frac{\partial u}{\partial x_j}\right) = 0 \tag{1.1}$$

where the coefficients a^{ij}, $1 \leq i, j \leq n$, are measurable functions on a domain Ω in Euclidean n space E^n. Following the usual summation convention, repeated indices will indicate summation from 1 to n. We assume always that $n \geq 2$.

Equation (1.1) is *elliptic* in Ω if the coefficient matrix $\mathscr{A}(x) = [a^{ij}(x)]$ is positive almost everywhere in Ω. Let $\lambda(x)$ denote the minimum eigenvalue of $\mathscr{A}(x)$ and set

$$\mu(x) = \sup_{1 \leq i, j \leq n} |a^{ij}(x)| \tag{1.2}$$

so that

$$\lambda(x)|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq n^2\mu(x)|\xi|^2 \tag{1.3}$$

for all $\xi \in E^n$, $x \in \Omega$. We will say that equation (1.1) is *uniformly elliptic* in Ω if the function $\gamma(x) = \mu(x)/\lambda(x)$ is essentially bounded in Ω. If γ is not necessarily bounded, then equation (1.1) is referred to as *non-uniformly elliptic*. We note here that uniformly elliptic equations for which λ^{-1} is unbounded have sometimes been referred to as *degenerate elliptic* [9].

Uniformly elliptic equations of the form (1.1), with bounded λ^{-1} and μ, have been extensively studied in the literature, two of the major results being a *Hölder estimate* for generalized solutions, due to DEGIORGI [1] and NASH [11], and a *Harnack inequality*, due to MOSER [7]. The purpose of this paper is to extend these results to a class of non-uniformly elliptic equations. In order to accomplish this, our methods differ substantially from those previously proposed and hence may be considered as new proofs of the original results. Various features of our proofs do coincide, however, with techniques in MOSER's two papers [6], [7]. An essential difference is that in order to obtain the stronger results we need to extract more information from the equation.

Classics from Leon Simon

INTERIOR GRADIENT BOUNDS FOR NON-UNIFORMLY

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

OF DIVERGENCE FORM

by

L.M. SIMON

A Thesis submitted for the Degree of

Doctor of Philosophy

in the University of Adelaide,

Department of Pure Mathematics,

December, 1971.

Interior Gradient Bounds for Non-uniformly Elliptic Equations

LEON SIMON

In [1] Bombieri, De Giorgi and Miranda were able to derive a local interior gradient bound for solutions of the minimal surface equation with n independent variables, $n \geqq 2$, thus extending the result previously established by Finn [2] for the case $n = 2$. Their method was to use test function arguments together with a Sobolev inequality on the graph of the solution (Lemma 1 of [1]). A much simplified proof of their result was later given by Trudinger in [12].

Since the essential features of the test function arguments given in [1] generalized without much difficulty to many other non-uniformly elliptic equations, it was apparent that interior gradient bounds could be obtained for these other equations provided appropriate analogues for the Sobolev inequality of [1] could be established. Ladyzhenskaya and Ural'tseva obtained such inequalities ([4], Lemma 1) for a rather large class of equations, including the minimal surface equation as a special case. They were thus able to obtain gradient bounds for this class of equations.

In §2 of [5] a general Sobolev inequality was established on certain generalized submanifolds of Euclidean space. In the special case of nonparametric hypersurfaces in $\mathbf{R}^{n+1}$ of the form $x_{n+1} = u(x)$, where u is a C^2 function defined on an open subset $\Omega \subset \mathbf{R}^n$, the inequality of [5] implies

$$(1)\qquad \left\{\int_\Omega h^{n/(n-1)} v\, dx\right\}^{(n-1)/n} \leqq c \int_\Omega \left[\left\{\sum_{i,j=1}^n g^{ij} h_{x_i} h_{x_j}\right\}^{1/2} + h\,|H|\right] v\, dx$$

for each non-negative C^1 function h with compact support in Ω, where

$$v = (1 + |Du|^2)^{1/2}$$

$$g^{ij} = \delta_{ij} - \nu_i \nu_j\,, \qquad \nu_i = u_{x_i}/v, \qquad i, j = 1, \cdots, n$$

$$H = \frac{1}{n} v^{-1} \sum_{i,j=1}^n g^{ij} u_{x_i x_j}\,,$$

and where c is a constant depending only on n. (See the discussion in §2 below.) The quantity H appearing in this inequality is in fact the mean curvature of the hypersurface $x_{n+1} = u(x)$ and in the special case when $H \equiv 0$ (*i.e.* when u

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The variational setting

- For local estimates the variational setting

$$v \mapsto \int_\Omega F(x, Dv)\, dx$$

turns out to be the most appropriate one.

- The Euler-Lagrange reads as

$$-\operatorname{div} \partial_z F(x, Du) = 0\,.$$

- Nonuniform ellipticity reads as

$$\lim_{|z|\to\infty} \mathcal{R}_{\partial_z F(x,\cdot)}(z) = \lim_{|z|\to\infty} \frac{\text{highest eigenvalue of } \partial_{zz} F(x,z)}{\text{lowest eigenvalue of } \partial_{zz} F(x,z)} = \infty\,.$$

Polynomial Nonuniform Ellipticity

- This happens, when, for $|z|$ is large,

$$\mathcal{R}_{\partial_z F(x,\cdot)}(z) \approx |z|^{\delta} \qquad \text{for some } \delta \geq 0$$

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- These are usually formulated prescribing

$$|z|^{p-2}\mathbb{I}_d \lesssim \partial_{zz}F(x,z) \lesssim |z|^{q-2}\mathbb{I}_d$$

so that

$$\mathcal{R}_{\partial_z F(x,\cdot)}(z) \lesssim |z|^{q-p}, \;\; \text{for } |z| \text{ large}\,, \quad 1 < p \leq q\,.$$

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- These are called (p,q)-growth conditions in Marcellini's terminology. They describe a general situation in which polynomial nonuniform ellipticity occurs. They usually couple with growth conditions on $F(\cdot)$

$$|z|^p \lesssim F(z) \lesssim |z|^q + 1 \qquad \text{and} \qquad 1 < p \leq q\,.$$

Recent gap bounds miscellanea

- Bella & Schäffner, CPAM 2021 – Analysis & PDE 2021

$$\frac{q}{p} < 1 + \frac{2}{n-1} \Longrightarrow Du \in L^{\infty}_{\rm loc} \quad [n > 2, \text{scalar case}]\,.$$

- Schäffner, Calc. Var. & PDE 2021

$$\frac{q}{p} < 1 + \frac{2}{n-1} \Longrightarrow Du \in L^{q}_{\rm loc} \quad [n > 2, \text{vectorial case}]\,.$$

- Hirsch & Schäffner, Comm. Cont. Math. 2020

$$\frac{1}{p} - \frac{1}{q} < \frac{1}{n-1} \Longrightarrow u \in L^{\infty}_{\rm loc}\,.$$

Two different ellipticity ratios

For nonautonomous functionals of the type

$$v \mapsto \int_{\Omega} F(x, Dv)\, dx\,.$$

We have the pointwise ellipticity ratio

$$\mathcal{R}_{\partial_z F(x_0,\cdot)}(z) = \frac{\text{highest eigenvalue of } \partial_{zz} F(x_0, z)}{\text{lowest eigenvalue of } \partial_{zz} F(x_0, z)}\,.$$

The nonlocal ellipticity ratio

- The nonlocal ellipticity ratio is defined by

$$\mathcal{R}_{\partial_z F}(z, B) = \frac{\sup_{x \in B} \ \text{highest eigenvalue of } \partial_{zz} F(x, z)}{\inf_{x \in B} \ \text{lowest eigenvalue of } \partial_{zz} F(x, z)}$$

  where B is a ball.
- In general it is

$$\mathcal{R}_{\partial_z F(x_0, \cdot)}(z) \lesssim \mathcal{R}_{\partial_z F}(z, B) \qquad \forall\, x_0 \in B\,.$$

- The second ratio usual detects milder, non traditional forms of nonuniform ellipticity.

Two different notions

- The integrand $F(\cdot)$ is nonunifomly elliptic if

$$\sup_{x_0,|z|\geq 1} \mathcal{R}_{\partial_z F(x_0,\cdot)}(z) = \infty\,.$$

- We call it softly nonuniformly elliptic if

$$\sup_{x_0,|z|\geq 1} \mathcal{R}_{\partial_z F(x_0,\cdot)}(z) < \infty \quad \text{but} \quad \lim_{|z|\to\infty} \mathcal{R}_{\partial_z F}(z,B) = \infty$$

  for at least one ball B.

- We call it uniformly elliptic if

$$\sup_{B,|z|\geq 1} \mathcal{R}_{\partial_z F}(z,B) < c\,.$$

- Discussion in De Filippis & Min. ARMA 2021.
- See also Beck & Min. CPAM 2020.

A functional of Zhikov

Zhikov considered, between the 80s and the 90s, the following functional

$$v \mapsto \int_{\Omega} (|Dv|^p + a(x)|Dv|^q)\, dx \qquad a(x) \geq 0$$

motivations: modelling of strongly anisotropic materials, Elasticity, Homogenization, Lavrentiev phenomenon etc.

The double phase functional

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx \equiv \int_\Omega F(x, Dv)\, dx$$

allows for treating Hölder coefficients but is pointwise uniformly elliptic, in the sense that, whenever we freeze x_0

$$\mathcal{R}_{\partial_z F(x_0,\cdot)}(z) = \frac{\text{highest eigenvalue of } \partial_{zz} F(x_0, z)}{\text{lowest eigenvalue of } \partial_{zz} F(x_0, z)} < \infty\,.$$

Soft nonuniform ellipticity

In the double phase case

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx \equiv \int_\Omega F(x, Dv)\, dx$$

we have (with $B \cap \{a(x) = 0\} \neq \emptyset$)

$$\begin{aligned}\mathcal{R}_{\partial_z F}(z, B) &= \frac{\sup_{x\in B} \ \text{highest eigenvalue of } \partial_{zz} F(x,z)}{\inf_{x\in B} \ \text{lowest eigenvalue of } \partial_{zz} F(x,z)} \\ &\approx 1 + \|a\|_{L^\infty(B)}|z|^{q-p} \to \infty\end{aligned}$$

vs

$$\mathcal{R}_{\partial_z F(x_0,\cdot)}(z) = \frac{\text{highest eigenvalue of } \partial_{zz} F(x_0,z)}{\text{lowest eigenvalue of } \partial_{zz} F(x_0,z)} < \infty\,.$$

A counterexample

Theorem (Fonseca, Malý & Min. ARMA 2004)

Take $n \geq 2$, $B \subset \mathbb{R}^n$ *and* $\varepsilon, \sigma > 0$, $0 < \alpha \leq 1$. *There exists* $a(\cdot) \in C^{0,\alpha}$, *a boundary datum* $u_0 \in W^{1,\infty}(B)$ *and exponents* p, q *satisfying*

$$n - \varepsilon < p < n < n + \alpha < q < n + \alpha + \varepsilon$$

such that the solution to the Dirichlet problem

$$\begin{cases} u \mapsto \min\limits_{v} \displaystyle\int_B (|Dv|^p + a(x)|Dv|^q)\, dx \\ v \in u_0 + W_0^{1,p}(B) \end{cases}$$

has a singular set of essential discontinuity points of Hausdorff dimension larger than $n - p - \sigma$.

One point singularity: Esposito, Leonetti & Min. JDE 2004.

Many years later

Theorem

Let $u \in W^{1,1}(\Omega)$, $\Omega \subset \mathbb{R}^n$, be a local minimiser of the functional

$$v \mapsto \int_\Omega (|Dv|^p + a(x)|Dv|^q)\, dx \qquad 0 \le a(\cdot) \in C^{0,\alpha}(\Omega)$$

with

$$\frac{q}{p} \le 1 + \frac{\alpha}{n}\,.$$

Then

Du is locally Hölder continuous.

- Colombo & Min. ARMA (2 papers) 2015
- Baroni, Colombo & Min. Calc. Var. 2018.

Soft Nonuniform ellipticity, special structures

$$w \mapsto \int_\Omega F(x, Dw)\, dx$$

$$\mathcal{R}(x,z) \lesssim 1, \quad \mathcal{R}(z,B) \to \infty.$$

Based on the methods on the papers with Baroni and Colombo:

- De Filippis & Oh *JDE* 2019.
- Byun & Oh *Analysis & PDE* 2020.
- Karppinen & Lee *IMRN* 2021.
- Hästo & Ok *JEMS* 2022; *ARMA* 2022.
- Baroni *JDE* 2023.
- Baasandorj & Byun *Memoirs AMS*, to appear.

The **frozen integrand** $z \mapsto F(x_0, z)$ is **uniformly elliptic**.

Pointwise nonuniform ellipticity

- This does not happen for basic model examples as

$$v \mapsto \int_\Omega c(x) F(Dv)\, dx \equiv \int_\Omega \bar{F}(x, Dv)\, dx$$

under genuine non-uniform ellipticity

$$|z|^{p-2}\mathbb{I}_d \leq \partial_{zz} F(z) \leq |z|^{q-2}\mathbb{I}_d$$

- Freezing yields

$$\mathcal{R}_{\partial_z \bar{F}(x_0,\cdot)} \approx \frac{\text{highest eigenvalue of } \partial_{zz}\bar{F}(x_0, z)}{\text{lowest eigenvalue of } \partial_{zz}\bar{F}(x_0, z)} \approx \frac{\text{highest eigenvalue of } \partial_{zz}F(z)}{\text{lowest eigenvalue of } \partial_{zz}F(z)} \approx |z|^{q-p}\,.$$

Classical Schauder estimates

- Solutions to

$$-\triangle u = -\mathrm{div}\,(Du) = 0$$

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- How much of such regularity is preserved when adding coefficients (ingredients)?

$$-\mathrm{div}\,(A(x)Du) = -(A^{ij}(x)D_j u)_{x_i} = 0\,.$$

Classical Schauder estimates

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$$-\mathrm{div}\,(A(x)Du) = -(A^{ij}(x)D_j u)_{x_i} = 0\,.$$

- The matrix $A(\cdot)$ is bounded and elliptic

$$\nu\mathbb{I}_d \leq A(x) \leq L\mathbb{I}_d\,.$$

Classical Schauder estimates

- As Du and $A(x)$ stick together we have

$$A(\cdot) \in C^{0,\alpha} \Longrightarrow Du \in C^{0,\alpha} \qquad 0 < \alpha < 1\,.$$

- Similar results hold for equations not in divegence form
- This kind of results where first obtained by Hopf (1929), Caccioppoli (1934) and Schauder (1934), in various forms, and are today known as Schauder estimates (see also some contributions of Giraud). They have also parabolic analogs.
- This is a basic tool in elliptic and parabolic PDES and in the Calculus of Variations.

- The original proofs involve heavy potential theory, as many others early elliptic results.
- Similar results hold for equations not in divergence form.
- Modern proofs have been given by Campanato, Trudinger and Leon Simon. All these proof rely, in a way or in another, on perturbation/comparison methods.
- In turn, all these proofs require that the estimates involved are homogeneous. This is the crucial point.

Nonlinear Schauder estimates

- Nonlinear theory is a more recent story, dating back to the 80s, by Manfredi (see also papers by Giaquinta & Giusti, DiBenedetto, where first results are obtained under additional assumptions). Based on Uraltseva & Uhlenbeck theory.
- A model example is given by the p-Laplacean equation with coefficients

$$\begin{cases} -\operatorname{div}\left(\mathrm{c}(x)|Du|^{p-2}Du\right) = 0 \\ \mathrm{c}(\cdot) \in C^{0,\alpha} \\ 0 < \nu \leq \mathrm{c}(x) \leq L\,. \end{cases}$$

- Extends to general equations of the type

$$-\operatorname{div} A(x, Du) = 0$$

modelled on the p-Laplacean operator according to the classical assumptions of Ladyzhenskaya & Uraltseva.

Non-differentiable functionals

- What happens when dealing for instance with classical model functionals as

$$v \mapsto \int_{\Omega} [F(Dv) + \mathtt{h}(x, v)] \, dx$$

when $y \mapsto \mathtt{h}(\cdot, y)$ is not differentiable, but only Hölder?

- As $\mathtt{h}(\cdot)$ is not differentiable, the Euler-Lagrange equation

$$-\mathrm{div}\, \partial_z F(Du) + \partial_u \mathtt{h}(x, u) = 0$$

does not exists.

- This is done in classic papers by Giaquinta & Giusti (Acta Math. 1981, Invent. Math. 1983) from the beginning of the 80s.

- Classical fact 1. For solutions to

$$-\mathrm{div}\,(\mathrm{c}(x)|Du|^{p-2}Du) = 0$$

  and, more in general, uniformly elliptic equations with p-growth we have

$$\mathrm{c}(\cdot) \text{ is Hölder} \Longrightarrow Du \text{ is Hölder.}$$

- Classical fact 2. For minima of non-differentiable functionals

$$v \mapsto \int_\Omega [\mathrm{c}(x)|Dv|^p + \mathtt{h}(x,v)]\,dx$$

  and, more in general, uniformly elliptic integrals with p-growth, we have

$$\mathrm{c}(\cdot), \mathtt{h}(x,\cdot) \text{ are Hölder} \Longrightarrow Du \text{ is Hölder.}$$

Two classical issues

- Open problem 1. Schauder for nonuniformlly elliptic. For solutions to

$$-\text{div}\,(c(x)A(Du)) = 0\,, \qquad \mathcal{R}_A(z) \approx |z|^{q-p}$$

and more general, equations with polynomial nonuniform ellipticity

coefficients (like $c(\cdot)$) are Hölder $\Longrightarrow$ Du is Hölder.

- Open problem 2. Nondifferentiable functionals. For minima of non-differentiable functionals

$$v \mapsto \int_\Omega [F(Dv) + \mathtt{h}(x, v)]\, dx$$

and, more in general, of integrals with polynomial nonuniform ellipticity, it holds that

coefficients (like $\mathtt{h}(\cdot,\cdot)$) are Hölder $\Longrightarrow$ Du is Hölder.

Publications results for "Author=(giaquinta) AND Reviewer=(lieberman)"

MR0749677 (85k:35077) Reviewed

Giaquinta, M. (I-FRNZ); Giusti, E. (I-FRNZ)

Global $C^{1,\alpha}$-regularity for second order quasilinear elliptic equations in divergence form.

J. Reine Angew. Math. 351 (1984), 55–65.
35J60 (35B65 49A22)

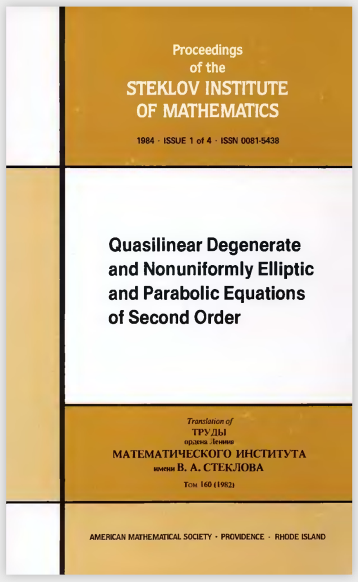
It is by now classical that solutions of the Dirichlet problem for a divergence form elliptic equation: $\operatorname{div} A(x,u,Du) = B(x,u,Du)$ in Ω, $u = \varphi$ on $\partial\Omega$, are $C^{k,\alpha}$ if $\varphi \in C^{k,\alpha}$ for any nonnegative integer $k \neq 1$, under suitable hypotheses on the coefficients A and B. Moreover, the reviewer has proved this result for $k = 1$ [Comm. Partial Differential Equations 6 (1981), no. 4, 437–497; MR0612553] assuming, among other things, that A has Holder continuous first derivatives and that B is Holder continuous. The present paper provides an alternative proof of this regularity result for $k = 1$ by means of some interesting techniques developed by the authors to study the regularity of minima of functionals [Invent. Math. 72 (1983), no. 2, 285–298; MR0700772]. Basically, they freeze the coefficient vector A at a point and then use a perturbation argument.

As well as being applicable to minimization problems, their method allows weaker smoothness hypotheses, namely, A is C^1 in Du and $C^{0,\alpha}$ in x and u, and B is bounded and measurable. In addition, bounded weak solutions of the Dirichlet problem can be studied directly when certain growth properties are imposed on the coefficients for large Du.

A comment needs to be made concerning their brief application to equations when their growth properties fail. As they point out, such equations fall under their considerations provided a global gradient bound has been established; however, this gradient bound has only been proved when A is differentiable with respect to all its arguments, and in many cases more smoothness of the coefficients is needed. The results of this paper are thus much more striking when applied to uniformly elliptic equations than to nonuniformly elliptic ones.

Reviewed by Gary M. Lieberman

Discussion. Ivanov's book.



THEOREM OF LADYZHENSKAYA AND URAL'TSEVA [83]. *Suppose that a function* $u \in C^2(\bar{\Omega})$ *satisfying the condition*

$$\max_{\bar{\Omega}}|u| \leqslant m, \quad \max_{\bar{\Omega}}|\nabla u| \leqslant M, \tag{2.5}$$

is a solution of (1.1) *in a bounded domain* $\Omega \subset \mathbf{R}^n$, $n \geqslant 2$, *and that equation* (1.1) *is elliptic at this solution in the sense that*

$$a^{ij}(x, u(x), \nabla u(x))\xi_i\xi_j \geqslant \nu\xi^2, \qquad \nu = \text{const} > 0, \xi \in \mathbf{R}^n, x \in \bar{\Omega}. \tag{2.6}$$

Suppose that on the set $\mathcal{F}_{\Omega,m,M} \equiv \bar{\Omega} \times \{|u| \leqslant m\} \times \{|p| \leqslant M\}$ *the functions* $a^{ij}(x, u, p)$, $i, j = 1, \dots, n$, *and* $a(x, u, p)$ *satisfy the condition*

$$|a^{ij}| + \left|\frac{\partial a^{ij}}{\partial x}\right| + \left|\frac{\partial a^{ij}}{\partial u}\right| + \left|\frac{\partial a^{ij}}{\partial p}\right| + |a| \leqslant M_1 \equiv \text{const} > 0 \quad on\ \mathcal{F}_{\Omega,m,M}. \tag{2.7}$$

Then there exists a number $\gamma \in (0, 1)$, *depending only on* n, ν, M *and* M_1, *such that for any subdomain* Ω', $\bar{\Omega}' \subset \Omega$,

$$\|\nabla u\|_{C^\gamma(\bar{\Omega}')} \leqslant c_1, \tag{2.8}$$

where c_1 *depends only on* n, ν, M, M_1, *and the distance from* Ω' *to* $\partial\Omega$. *If the domain* Ω *belongs to the class* C^2 *and* $u = \varphi(x)$ *on* $\partial\Omega$, *where* $\varphi \in C^2(\bar{\Omega})$, *then*

$$\|\nabla u\|_{C^\gamma(\bar{\Omega})} \leqslant c_2, \tag{2.9}$$

terms of the majorants $\mathscr{E}_1$ and $\mathscr{E}_2$. Here it is also important to note that the structure of these conditions and the character of the basic a priori estimates established for solutions of (2) do not depend on the "parabolicity constant" of the equation. This determines at the outset the possibility of using the results obtained to study in addition boundary value problems for quasilinear degenerate parabolic equations. In view of the results of Ladyzhenskaya and Ural'tseva (see [80]), the proof of classical solvability of the first boundary value problem for equations of the form (2) can be reduced to establishing an a priori estimate of $\max_Q |\nabla u|$, where ∇u is the spatial gradient, for solutions of a one-parameter family of equations (2) having the same structure as the original equation (see §2.1).

Our technique is a nonlinear version of the well-known method of freezing the coefficients A^j at a point x_0, and then using a perturbation argument. A special form of De Giorgi's theorem is needed that requires linear growth for the A^i and at most a quadratic growth for B. However, the general case of coefficients A^i and B of arbitrary growth can easily be reduced to this once a gradient bound has been proved. This happens for instance for the minimal surface equation

$$\operatorname{div}\left\{\frac{Du}{\sqrt{1+|Du|^2}}\right\}=0$$

for which a gradient estimate for $C^{1,\alpha}$ boundary values has been proved in [10] (see also [13]).

- Solutions in a paper by Cristiana De Filippis (Parma) & Min. (Arxiv 2022).
- Catches both cases of non-differentiable functionals and equations with Hölder continuous coefficients.
- Introduces a direct, non-perturbation approach to Schauder theory.
- Crucial point in the proof is to get L^∞-bounds for the gradient.
- I will present a sample of the results.

Nondifferentiable functionals

Theorem (De Filippis & Min. #1)

Let $u \in W^{1,1}$ be a minimizer of the functional

$$v \mapsto \int_\Omega [F(Dv) + \mathtt{h}(x, v)]\, dx$$

where

- $|z|^{p-2}\mathbb{I}_d \lesssim \partial_{zz}F(z) \lesssim |z|^{q-2}\mathbb{I}_d$
- $|\mathtt{h}(x, y_1) - \mathtt{h}(x, y_2)| \lesssim |y_1 - y_2|^\alpha\,, \quad \alpha \in (0, 1]$
- *and*

$$\frac{q}{p} \leq 1 + \frac{1}{5}\left(1 - \frac{\alpha}{p}\right)\frac{\alpha}{n}\,.$$

Then Du is locally Hölder continuous in Ω.

Theorem (De Filippis & Min. #2)

Let $u \in W^{1,1}$ be a minimizer of the functional

$$v \mapsto \int_\Omega c(x) F(Dv)\, dx$$

where

- $|z|^{p-2}\mathbb{I}_d \lesssim \partial_{zz}F(z) \lesssim |z|^{q-2}\mathbb{I}_d$
- $0 < c(\cdot) \in C^{0,\alpha}(\Omega)$
- *and*

$$\frac{q}{p} \leq 1 + \frac{1}{5}\left(\frac{\alpha}{n}\right)^2 .$$

Then Du is locally Hölder continuous in Ω.

Theorem (De Filippis & Min. #3)

If, in addition, we have $p \geq 2$,

- $(|z|+1)^{p-2}\mathbb{I}_d \lesssim \partial_{zz}F(z) \lesssim (|z|+1)^{q-2}\mathbb{I}_d$
- *and*

$$\frac{q}{p} \leq 1 + \frac{1}{5}\left(\frac{\alpha}{n}\right)^2 .$$

Then

$$c(\cdot) \in C^{0,\alpha}(\Omega) \Longrightarrow u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega).$$

Equations: Two possible approaches

- Start with energy solutions $u \in W^{1,q}$.
- Prove existence of more regular solutions, selected via approximation methods.

Schauder estimates for equations

- We consider Dirichlet problems of the type

$$\begin{cases} -\operatorname{div} A(x, Du) = 0 & \text{in } \Omega \\ u \equiv u_0 & \text{on } \partial\Omega \,, \end{cases} \qquad u_0 \in W^{1,\frac{p(q-1)}{p-1}}(\Omega) \,,$$

- Under the assumptions

$$\begin{cases} |z|^{p-2}\mathbb{I}_d \lesssim \partial_z A(x,z) \lesssim |z|^{q-2}\mathbb{I}_d \\ |A(x_1,z) - A(x_2,z)| \lesssim |x_1 - x_2|^{\alpha}|z|^{q-1} \,. \end{cases}$$

- For nonuniformly elliptic problems, the concept of energy solutions is not well defined. This leads to prove existence-and-regularity theorems.

Theorem (De Filippis & Min. #4)

If

$$\frac{q}{p} \leq 1 + \frac{p-1}{10p}\left(\frac{\alpha}{n}\right)^2 ,$$

then there exists a solution u to the above Dirichlet problem such that Du is locally Hölder continuous in Ω.

Theorem (De Filippis & Min. #5)

If in addition, $p \geq 2$ and the problem is non-degenerate, i.e.,

$$(|z|+1)^{p-2}\mathbb{I}_d \lesssim \partial_z A(x,z) \lesssim (|z|+1)^{q-2}\mathbb{I}_d$$

we have

$$u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)\,.$$

Shot 1: Bernstein for degenerate problems

Take a (uniformly) elliptic equation like for instan

$$-\mathrm{div}\,(|Du|^{p-2}Du) = 0\,.$$

Crucial remark:

$$v = |Du|^p \text{ is a subsolution to a linear elliptic equation}$$
$$\Longrightarrow v \in L^\infty_{\mathrm{loc}} \Longrightarrow Du \in L^\infty_{\mathrm{loc}}\,.$$

De Giorgi's nonlinear mechanism

- We have that

$$\int_{B_\sigma} |D(v-k)_+|^2\,dx \lesssim \frac{1}{(\tau-\sigma)^2}\int_{B_\tau} (v-k)_+^2\,dx$$

holds for every level $\kappa \geq 0$ and concentric $B_\sigma \subset B_\tau$.

- This implies the following reverse Hölder inequality:

$$\left(\fint_{B_{r/2}} (v-\kappa)_+^{2\chi}\,dx\right)^{\frac{1}{2\chi}} \lesssim \left(\fint_{B_r} (v-\kappa)_+^{2}\,dx\right)^{\frac{1}{2}}, \qquad \chi > 1\,.$$

- De Giorgi's iteration leads to the classical $L^2 - L^\infty$-estimate:

$$\|v\|_{L^\infty(B/2)} \lesssim \left(\fint_B v^2\,dx\right)^{\frac{1}{2}}.$$

Shot 2: Almost Bernstein

- Take now

$$-\mathrm{div}\,(\mathrm{c}(x)|Du|^{p-2}Du) = 0 \qquad \mathrm{c}(x) \approx 1\,.$$

- When $\mathrm{c}(x)$ is differentiable, then

$$\int_{B_\sigma} |D(v-k)_+|^2\,dx \lesssim \frac{1}{(\tau-\sigma)^2}\int_{B_\tau}(v-k)_+^2\,dx + \text{terms}\,.$$

- When $\mathrm{c}(x)$ is not differentiable, then you use perturbation methods.
- None of the above methods work in the nonuniformly elliptic regime and nondifferentiable coefficients.

We prove renormalized fractional Caccioppoli inequalities

$$\int_{B_{r/2}}\int_{B_{r/2}} \frac{(v(x)-\kappa)_+ - (v(y)-\kappa)_+|^2}{|x-y|^{n+2\sigma}}\,dx\,dy \lesssim \frac{H(\|v\|_{L^\infty})}{r^{2\sigma}}\int_{B_r}(v-\kappa)_+^2\,dx + \frac{1}{r^{2\sigma}}\sum_i K_i(\|v\|_{L^\infty})r^{\alpha_i}\int_{B_r}|Du|^{p_i}\,dx$$

for every $\kappa \geq 0$, for some $0<\sigma\equiv\sigma(n,p,q,\alpha)$ and for suitable parameters (α_i,p_i) that can be chosen in a appropriate way as functions of p,q,α,μ, and power type functions $H(\cdot)$ and $K_i(\cdot)$.

A nonlinear atomic type decoposition

The previous inequality follows from a very delicate local decomposition argument that resembles the Littlewood-Paley type constructions made in the dyadic decomposition of Besov spaces (Triebel). Here the usual atoms made of harmonic type functions are replaced by solutions to nonlinear equations.

Sketch

- From Besov $B^{\sigma}_{2,2} \equiv W^{\sigma,2}$ to Besov $B^{\sigma+}_{2,\infty}$

$$\sup_{|h|} \int \frac{|v(x+h) - v(x)|^2}{|h|^{2\sigma}}\, dx < \text{right-hand side terms}\,.$$

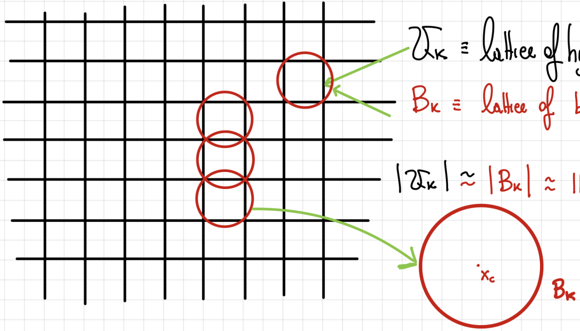
- Fix a scale

$$\frac{1}{2^{m+1}} \leq |h|^{\beta_0} \leq \frac{1}{2^m} \qquad 0 < \beta_0 < 1$$

  that corresponds to an annulus in Littlewood-Paley frequency partition.

- Then we take a dyadic partition of the space in cubes with mesh comparable to $|h|^{\beta_0}$.

Nonlinear atomic decomposition



$Q_K \equiv$ lattice of hypercubes

$B_K \equiv$ lattice of balls.

$|Q_K| \approx |B_K| \approx |h|^{\beta_0 n}$

$V \equiv V_{h,K} \equiv V(B_K)$.

$$\begin{cases} -\operatorname{div} A(x_c, Dv) = 0 & \text{in } B_K \\ v = u & \text{on } \partial B_K \end{cases}$$

$$u \approx \sum_K V_K \mathbb{1}_{B_K} + o(|h|^{\beta_0}).$$

Classical Atoms.

→ Take a dyadic decomposition with mesh $|h|^{\beta_0}$, $\{\mathcal{Q}_k\}$.

→ Take a family of smooth functions $\{a_k\}$ such that:

$\text{supp } a_k \subset \mathcal{Q}_k$

$$|D^s a_k| \lesssim |h|^{-\beta_0 s}$$

Nonlinear Atoms.

→ v is defined on B_k (actually a slight enlargement).

→ v satisfies:

$$\|Dv\|_{L^2} \lesssim |h|^{-\beta_0} \|v\|_{L^2} \quad (s=1).$$

Second main tool: Nonlinear potential theory

With $t, \delta > 0$, $m, \theta \geq 0$

$$\mathbf{P}^{m,\theta}_{t,\delta}(f; x_0, r) := \int_0^r \varrho^{\delta} \left(\fint_{B_\varrho(x_0)} |f|^m \, dx \right)^{\theta/t} \frac{d\varrho}{\varrho} \, .$$

Don't be scared by the parameters. It is what it is.

Extends well-known and used potentials

- Riesz potentials

$$\int_{B_{r/2}(x_0)} \frac{|f(x)|}{|x-x_0|^{n-1}}\,dx \lesssim \mathbf{I}_1(f;x_0,r)$$
$$:= \int_0^r \frac{1}{\varrho^{n-1}} \int_{B_{\mathbb{R}}(x_0)} |f|\,dx\,\frac{d\varrho}{\varrho} \approx \mathbf{P}^{1,1}_{1,1}(f;x_0,r)\,.$$

- Havin-Maz'ya-Wolff potentials

$$\mathbf{W}^{f}_{\beta,p}(x_0,r) := \int_0^r \left(\frac{|f|(B_\varrho(x_0))}{\varrho^{n-\beta p}}\right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho}$$
$$\approx \mathbf{P}^{1,1}_{p-1,\frac{\beta p}{p-1}}(f;x_0,r)\,.$$

Nonlinear potential theoretic De Giorgi

Lemma (after De Giorgi and Kilpeläinen & Malý)

Take $v \in L^t$, $|f_i|^{m_i} \in L^1(B_{2r_0}(x_0))$, *and constants* $\chi > 1$, $t \geq 1$, $\delta, m, \theta > 0$. *Assume that for every ball* $B_r(x_0) \subset B_{r_0}(x_0)$, *the inequality*

$$\left(\fint_{B_{r/2}(x_0)} (v-\kappa)_+^{t\chi} \, dx \right)^{1/\chi} \lesssim \fint_{B_r(x_0)} (v-\kappa)_+^t \, dx + \sum_i r^{t\delta_i} \left(\fint_{B_r(x_0)} |f_i|^{m_i} \, dx \right)^{\theta_i}$$

holds for every $\kappa \geq 0$. *If* x_0 *is a Lebesgue point of* v, *then*

$$v(x_0) \lesssim \kappa_0 + \left(\fint_{B_{r_0}(x_0)} (v-\kappa_0)_+^t \, dx \right)^{1/t} + \sum_i \mathbb{P}_{t,\delta_i}^{m_i,\theta_i}(f_i; x_0, 2r_0) \, .$$

Quantitative De Giorgi (a subtle and overlooked fact)

Lemma

Assume that for every ball $B_r(x_0) \subset B_{r_0}(x_0)$, the inequality

$$\left(\fint_{B_{r/2}(x_0)} (v-\kappa)_+^{t\chi} \, dx \right)^{1/\chi} \lesssim \mathfrak{M}_0^t \fint_{B_r(x_0)} (v-\kappa)_+^t \, dx + \sum_i \mathfrak{M}_i^t r^{t\delta_i} \left(\fint_{B_r(x_0)} |f_i|^{m_i} \, dx \right)^{\theta_i}$$

holds. If x_0 is a Lebesgue point of v, then

$$v(x_0) \lesssim \kappa_0 + \mathfrak{M}_0^{\frac{\chi}{\chi-1}} \left(\fint_{B_{r_0}(x_0)} (v-\kappa_0)_+^t \, dx \right)^{1/t} + \sum_i \mathfrak{M}_0^{\frac{1}{\chi-1}} \mathfrak{M}_i \mathbb{P}_{t,\delta_i}^{m_i,\theta_i}(f_i; x_0, 2r_0) \, .$$

Very hidden facts

- Many experts know that in De Giorgi's theory during the iteration the constants deteriorate exponentially (dyadic slicing, log inequalities). Constant essentially depend on the ellipticity ratio.
- Almost no expert notes that in De Giorgi's theory (as well as in Moser's), during the L^∞-iteration the constants deteriorate polynomially (since everything keeps geometric). The exponential worsening only occurs when proving Hölder continuity.
- This makes a crucial difference in this setting.

The pointwise hybrid bound

Using the lemma we gain

$$v(x_0) \lesssim \|v\|_{L^\infty(B_r(x_0))}^{\lambda_0} \left(\fint_{B_r(x_0)} v^2 \, dx \right)^{1/2} + \sum_i \|v\|_{L^\infty(B_r(x_0))}^{\tilde{\gamma}_i} [\mathbb{P}_i(|Du|; x_0, 2r)]^{t_i}$$

for almost every x_0.

Covering

Take a reference ball B_R and concentric balls

$$B_{R/2} \Subset B_{\tau_1} \Subset B_{\tau_2} \subset B_R \,.$$

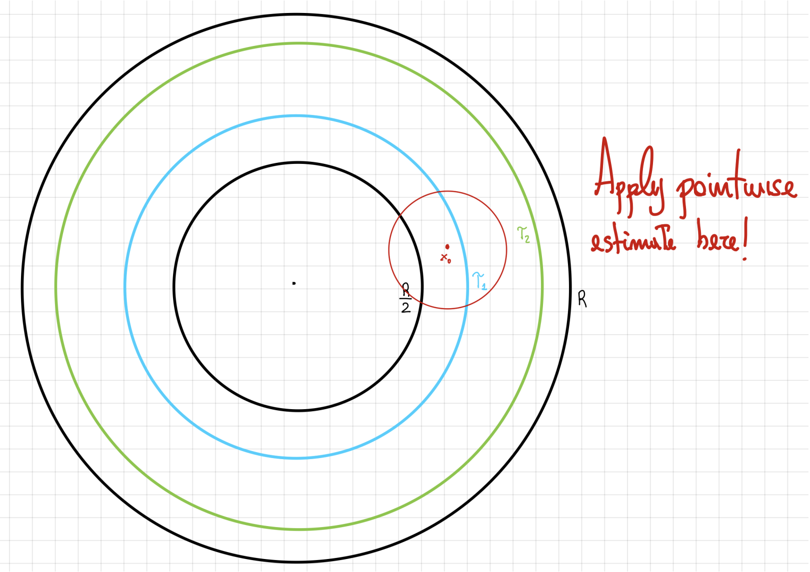
Applying the previous estimate on

$$B_r(x_0) \equiv B_{(\tau_2-\tau_1)/2}(x_0), \qquad \forall\ x_0 \in B_{\tau_1}\,, \qquad M \equiv \|Du\|_{L^\infty(B_{\tau_2})}$$

we gain

$$\|v\|_{L^\infty(B_{\tau_1})} \lesssim \frac{\|v\|_{L^\infty(B_{\tau_2})}^{\lambda_0+1/2}}{(\tau_2-\tau_1)^{n/2}} \left(\int_{B_R} F(x,Du)\,dx\right)^{1/2} + \sum_i \|v\|_{L^\infty}^{\gamma_i} \|\mathbb{P}_i(|Du|)\|_{L^\infty}^{t_i}$$

Pointwise-to-global trick



Algebra of exponents

- The assumed bound

$$\frac{q}{p} < 1 + \mathrm{o}\left(\frac{\alpha}{n}\right)$$

allows to conclude, for suitable choice of the parameters, that

$$\begin{cases} \lambda_0 + 1/2 < 1 \quad \text{and} \quad \gamma_i < 1 \\ \|\mathbb{P}(|Du|)\|_{L^\infty} \lesssim \|v\|_{L^1}^{\gamma} + 1\,. \end{cases}$$

The last point goes via analysis of mapping properties of nonlinear potentials in Lorentz spaces.

- Applying Young's inequality yields

$$\|v\|_{L^\infty(B_{\tau_1})} \leq \frac{1}{2}\|v\|_{L^\infty(B_{\tau_2})} + \frac{c}{(\tau_2 - \tau_1)^{\theta n}} \left(\int_{B_R} F(x, Du)\, dx\right)^{\theta} + c\,.$$

Iteration lemma

Lemma (Giaquinta & Giusti, Acta Math. 82)

Let $\mathcal{Z}\colon [r,R) \to [0,\infty)$ *be a function which is bounded on every interval* $[r,R_*]$ *with* $R_* < R$*. Let* $\varepsilon \in (0,1)$*,* $a,\gamma \geq 0$ *be numbers. If*

$$\mathcal{Z}(\tau_1) \leq \varepsilon \mathcal{Z}(\tau_2) + \frac{a}{(\tau_2-\tau_1)^\gamma}\ ,$$

for all $r \leq \tau_1 < \tau_2 < R$*, then*

$$\mathcal{Z}(r) \leq \frac{c(\varepsilon,\gamma)a}{(R-r)^\gamma}\,.$$

leads to

$$\|v\|_{L^\infty(B_{R/2})} \leq c\left(\fint_{B_R} F(x,Du)\,dx\right)^\theta + c\,.$$

Time to rest: Thanks a lot!