Coupling of Singular Layers for the Boltzmann Equation

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Boltzmann equation and fluid dynamics:

Harold Grad Asymptotic theory of the Boltzmann equation, *Phys. Fluids* **6** (1963), 147–181.

- *In the study of Boltzmann - fluid relationship, it is crucial to study Singular layers*
- Shock layer: Nonlinearity of fluid motion.
- Boundary layer: Boundary condition between gas and other materials.
- Initial layer: Variation of initial values.
- Physically natural setups often involve coupling of singular layers, e.g., Coupling of shock and initial layers, Coupling of shock and boundary layers, Coupling of boundary and initial layers, etc.

Boltzmann equation

$$\partial_t \mathrm{f}(\boldsymbol{x}, t, \xi) + \xi \cdot \partial_{\boldsymbol{x}} \mathrm{f}(\boldsymbol{x}, t, \xi) = \frac{1}{k} Q(\mathrm{f}, \mathrm{f})(\boldsymbol{x}, t, \xi),$$

collision operator:

$$\boldsymbol{Q}(\mathrm{g}, \mathrm{h}) \equiv \frac{1}{2} \int_{\substack{\mathbb{R}^3 \times \boldsymbol{S}^2 \\ (\xi - \xi_*) \cdot \Omega \geq 0}} \Big(- \mathrm{g}(\xi)\mathrm{h}(\xi_*) - \mathrm{h}(\xi)\mathrm{g}(\xi_*) \\ + \mathrm{g}(\xi')\mathrm{h}(\xi'_*) + \mathrm{h}(\xi')\mathrm{g}(\xi'_*) \Big) B(\xi - \xi_*, \Omega) d\xi_* d\Omega;$$

$$\begin{cases} \xi' = \xi - [(\xi - \xi_*) \cdot \Omega]\, \Omega, \\ \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \Omega]\, \Omega. \end{cases}$$

Hard sphere: $B(\xi - \xi_*, \Omega) = (\xi - \xi_*) \cdot \Omega$.

Macroscopic Variables , moments of the distribution function:

$$\rho(\boldsymbol{x},t)=\int_{\mathbb{R}^3}\mathrm{f}(\boldsymbol{x},t,\xi)d\xi \text{ density},$$

$$\rho\boldsymbol{v}(\boldsymbol{x},t)=\int_{\mathbb{R}^3}\xi\mathrm{f}(\boldsymbol{x},t,\xi)d\xi \text{ momentum},$$

$$\rho E(\boldsymbol{x},t)\equiv\int_{\mathbb{R}^3}\frac{1}{2}|\xi|^2\mathrm{f}(\boldsymbol{x},t,\xi)d\xi=\frac{1}{2}\rho|\boldsymbol{v}|^2(\boldsymbol{x},t)+\rho e(\boldsymbol{x},t),$$

$$\rho e(\boldsymbol{x},t)\equiv\int_{\mathbb{R}^3}\frac{1}{2}|\xi-\boldsymbol{v}|^2\mathrm{f}(\boldsymbol{x},t,\xi)d\xi \text{ internal energy},$$

$$P_{ij}\equiv\int_{\mathbb{R}^3}(\xi_i-v_i)(\xi_j-v_j)\mathrm{f}(\boldsymbol{x},t,\xi)d\xi,\ \mathbb{P}\equiv\left(P_{ij}\right)_{i,j=1,2,3},\text{ stress tensor},$$

$$\boldsymbol{q}\equiv\int_{\mathbb{R}^3}(\xi-\boldsymbol{v})\frac{|\xi-\boldsymbol{v}|^2}{2}\mathrm{f}(\boldsymbol{x},t,\xi)d\xi,\ \text{heat flow}.$$

Collision Invariants

The interacting law

$$\begin{cases} \xi' = \xi - [(\xi - \xi_*) \cdot \Omega]\, \Omega, \\ \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \Omega]\, \Omega. \end{cases}$$

comes from the microscopic conservation laws for each collision: of number of particles, $1 + 1 = 1 + 1$, of momentum, $\xi + \xi_* = \xi' + \xi'_*$, and of energy, $|\xi|^2 + |\xi_*|^2 = |\xi'|^2 + |\xi'_*|^2$. This microscopic conservation laws yield the macroscopic conservation laws

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \xi \\ \frac{1}{2}|\xi|^2 \end{pmatrix} Q(\mathrm{f},\mathrm{f})(\boldsymbol{x}, t, \xi) d\xi = \mathbf{0}, \quad \begin{pmatrix} \text{conservation of mass} \\ \text{conservation of momentum} \\ \text{conservation of energy} \end{pmatrix}$$

Collision invariants: $1,\ \xi,\ |\xi|^2$

Conservation Laws

Integrate the Boltzmann equation times $(1, \xi, |\xi|^2)^T$ to yield

$$
\begin{aligned}
&\partial_t \rho + \partial_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v}) = 0, \text{ conservation of mass},\\
&\partial_t (\rho \boldsymbol{v}) + \partial_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v} + \mathbb{P}) = 0, \text{ conservation of momentum},\\
&\partial_t (\rho E) + \partial_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v} E + \mathbb{P} \boldsymbol{v} + \boldsymbol{q}) = 0, \text{ conservation of energy}.
\end{aligned}
$$

- For $\boldsymbol{x} \in \mathbb{R}^3$, this is an under-determined system of 5 equations for 13 unknowns. The Boltzmann equation is essentially a system of infinite number of partial differential equations.
- In most part of the gas region, the Boltzmann equation can be accurately approximated by some finite dimensional fluid dynamics equations.
- The study of singular layers is needed for the boundary values and boundary condition for the fluid dynamics equations.

H-Theorem

Integrate the Boltzmann equation times $1 + \log \mathrm{f}$ to yield the important H-Theorem

$$H_t + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{H} = \frac{1}{4k} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S_+^2} \log \frac{\mathrm{f}\mathrm{f}_*}{\mathrm{f}'\mathrm{f}'_*} [\mathrm{f}'\mathrm{f}'_* - \mathrm{f}\mathrm{f}_*] B d\Omega(\alpha) d\boldsymbol{\xi}_* d\boldsymbol{\xi} \leq 0,$$

$$H = H(\boldsymbol{x}, t) = \int_{\mathbb{R}^3} \mathrm{f} \log \mathrm{f} d\boldsymbol{\xi}, \ \boldsymbol{H} = \boldsymbol{H}(\boldsymbol{x}, t) = \int_{\mathbb{R}^3} \boldsymbol{\xi} \mathrm{f} \log \mathrm{f} d\boldsymbol{\xi}.$$

The inequality becomes an equality if and only if the state is in thermal equilibrium:

$$\mathrm{f}(\boldsymbol{x}, t, \boldsymbol{\xi}) = \frac{\rho(\boldsymbol{x}, t)}{(2\pi R\theta(\boldsymbol{x}, t))^{3/2}} e^{-\frac{|\boldsymbol{\xi} - \boldsymbol{v}(\boldsymbol{x},t)|^2}{2R\theta(\boldsymbol{x},t)}} \equiv \mathrm{M}_{(\rho, \boldsymbol{v}, \theta)}, \ \text{Maxwellian.}$$

Here θ is the temperature defined by the ideal gas relation $p = R\rho\theta$.

Boltzmann Equation and Fluid Dynamics

- On the 5-dimensional thermal equilibrium manifold $\{\mathrm{M}_{(\rho,\boldsymbol{v},\theta)},\ \rho,\theta \in \mathbb{R}_+,\ \boldsymbol{v} \in \mathbb{R}^3\},\ Q(\mathrm{M},\mathrm{M}) = 0$,
- the conservation laws become the Euler equations in gas dynamics:

$$\begin{aligned}
&\partial_t \rho + \partial_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v}) = 0, \\
&\partial_t (\rho \boldsymbol{v}) + \partial_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v} + p\mathbb{I}) = 0, \\
&\partial_t (\rho E) + \partial_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v} E + p \boldsymbol{v}) = 0,
\end{aligned}$$

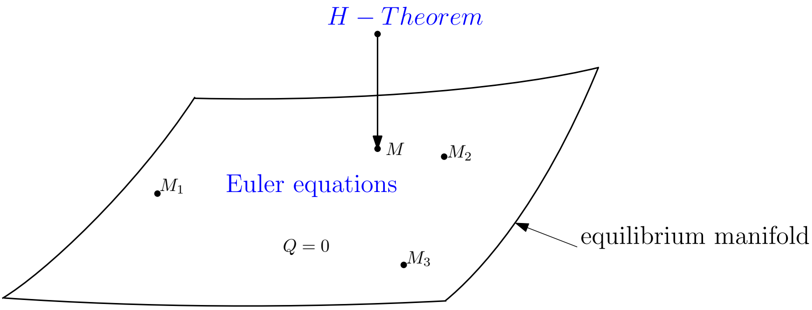
- and thermodynamics applies:

$$\rho s = -\int_{\mathbb{R}^3} \mathrm{M}_{(\rho,\boldsymbol{v},\theta)} \log \mathrm{M}_{(\rho,\boldsymbol{v},\theta)} d\xi,\ \text{entropy}.$$

Boltzmann Equation and Euler equations

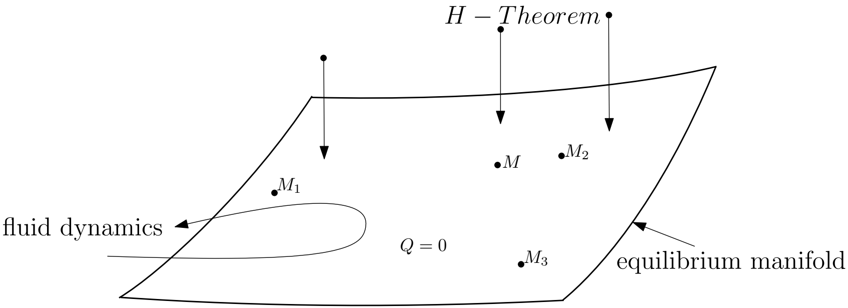
The H-Theorem $\Rightarrow$

- A space homogeneous Boltzmann solution $\partial_t \mathsf{f} = \frac{1}{k} \mathsf{Q}(\mathsf{f}, \mathsf{f})$ tends to the thermal equilibrium manifold.
- On the equilibrium manifold the Boltzmann equation is reduced to the Euler equations in gas dynamics.



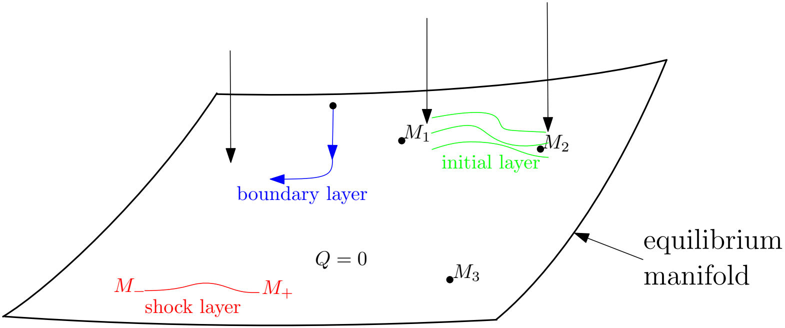
Boltzmann Equation and Fluid Dynamics

The solutions of the Boltzmann equation $\partial_t \mathrm{f} + \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{x}} \mathrm{f} = \frac{1}{k} \mathrm{Q}(\mathrm{f}, \mathrm{f})$ tends to a neighborhood of the equilibrium manifold, where the Fluid Dynamics phenomena, such as the the viscous, heat conducting, and thermal creep effects, occur.



Grad 1963: Singular Layers

- Boundary layer: due to boundary condition.
- Shock layer: due to fluid nonlinearity.
- Initial layer: Due to variation of initial values.



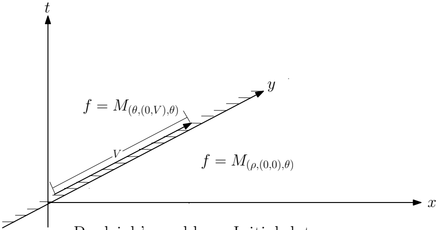
Coupling of Singular Layers

- For most of the physical region $(\boldsymbol{x}, t)$, the Boltzmann equation can be well approximated by fluid-like equations
- Singular layers of the Boltzmann equation are constructed to connect to the fluid-like region in order to obtain global picture in the physical space.
- Physically natural setups often involve coupling of singular layers, e.g., Coupling of shock and initial layers, Coupling of shock and boundary layers, Coupling of boundary and initial layers, etc.

Example 1. Rayleigh's Problem

Coupling of Intial layer and Boundary Layer.

$$
\begin{aligned}
&\mathrm{f}_t + \xi_1 \mathrm{f}_x = \frac{1}{k} Q(\mathrm{f}, \mathrm{f}),\\
&\mathrm{f}(x, y, 0) = \mathrm{M}_(\rho, (0,0), \theta),\ \ x > 0, -\infty < y < \infty,\\
&\mathrm{f}(0, y, t) = \mathrm{M}_(\rho, (0, V), \theta),\ -\infty < y < \infty,\ \xi_1 > 0.
\end{aligned}
$$

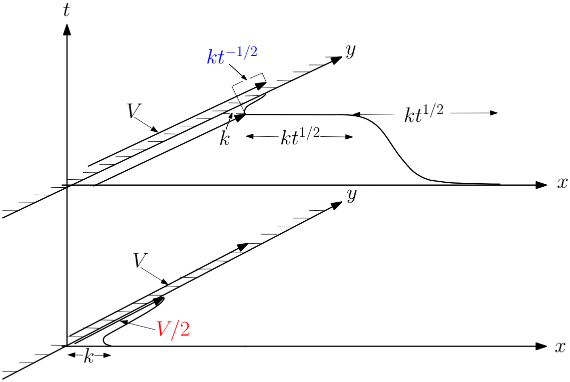


Rayleigh's problem: Initial data

Goal: Comparison with classical fluid dynqmics.

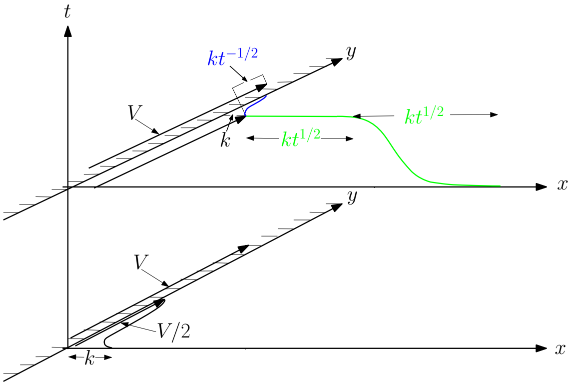
Coupling of Initial and Boundary Layers:

Coupling of initial and boundary layers: Rayleigh's Problem:



For small time, in the initial layer, gas flows at speed $V/2$ around the boundary, due to the diffuse reflection boundary condition.

Rayleigh's Problem:



After initial-boundary coupling, a decay at rate $t^{-1/2}$ for the boundary slip, followed by fluid diffusion wave at rate $t^{1/2}$.

Rayleigh's Problem:

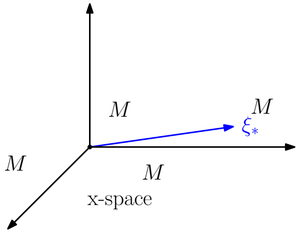
- Classical fluid dynamics:
  Lord.Rayleigh, Phil. Mag. Ser. 6, 21 (1911) 647.
- Modern fluid dynamics: viewed from kinetic theory, Laplace transform, asymptotic expansion, solution behavior:
  Sone, Y, Kinetic Theory Analysis of Linearized Rayleigh Problem. J. Phys. Soc. Jpn. 19, pp. 1463-1473 (1964)
- $\Rightarrow$ Hilbert expansion with boundary by Sone 1969, 1971.
- Laplace transform, existence of solutions:
  Cercignani, C.; Sernagiotto, F. The method of elementary solutions for time-dependent problems in linearized kinetic theory. Ann. Physics 30 (1964), 154-167.
- Open problem: To understand analytically the coupling of initial and boundary layers.

Green's function $\mathbb{G}(\boldsymbol{x}, t, \xi; \xi_*)$

For study of coupling of singular layers, strongly quantitative method, such as the Green's function approach, is needed.

$$\begin{cases} \mathbb{G}_t + \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{x}} \mathbb{G} = \mathrm{L}\mathbb{G}, \ x \in \mathbb{R}, \\ \mathbb{G}(x, 0, \xi; \xi_*) = \delta(\boldsymbol{x})\delta(\boldsymbol{\xi} - \boldsymbol{\xi}_*) \\ \mathrm{L}: \ \text{linearized collision operator around a Maxwelian M.} \end{cases} .$$

$\mathbb{G}(\boldsymbol{x}, t, \xi; \xi_*)$: The dispersion around the ambient Maxwellian M of particles starting at origin and with speed $\boldsymbol{\xi}_*$.



Linearized Boltzmann Equation

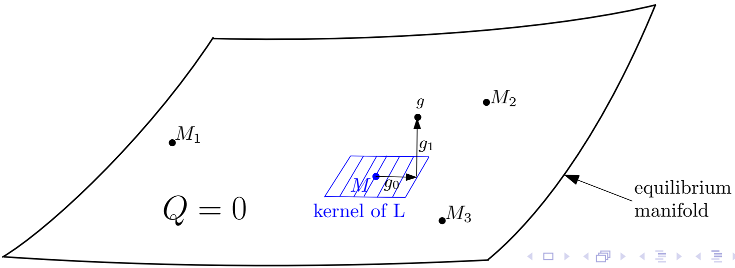
Perturbation around Maxwellian M: $f = M + \sqrt{M}g$.

$$g_t + \boldsymbol{\xi} \cdot \nabla_{\boldsymbol{x}} g = \frac{1}{k} Lg, \quad \text{Linear Boltzmann equation},$$

$$Lg = \frac{2\boldsymbol{Q}(\sqrt{M}g, M)}{\sqrt{M}}, \quad \text{Linear collision operator},$$

$\text{kernel}(L) = \text{span}\{1, \boldsymbol{\xi}, |\boldsymbol{\xi}|^2\}\sqrt{M}$, tangent to equilibrium manifold.

Macro-projection P_0: projection onto kernel of L.
Micro-projection $P_1 = I - P_0$, $g = P_0 g + P_1 g = g_0 + g_1$.



Euler Characteristics

Macro-projection of the linear Boltzmann equation:

$$(\mathrm{P}_0\mathrm{g})_t + \nabla_{\boldsymbol{x}} \cdot \mathrm{P}_0\boldsymbol{\xi}\mathrm{P}_0\mathrm{g} = 0, \quad \text{linear Euler equations}.$$

1-D Euler characteristics

$$\mathrm{P}_0\xi_1\mathrm{E}_j = \lambda_j\mathrm{E}_j,\ \lambda_1 = v - \boldsymbol{c},\ \lambda_2 = v,\ \lambda_3 = v + \boldsymbol{c}.$$

Sound speed for monatomic gases $\boldsymbol{c} = \sqrt{\frac{5\theta}{3}}$. $(\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3)$ orthogonal, the macro-projections:

$$\mathrm{P}_0\mathrm{g} = \sum_{j=1}^{3}(\mathrm{g}, \mathrm{E}_j)\mathrm{E}_j = \sum_{j=1}^{3}(\mathrm{g}_0, \mathrm{E}_j)\mathrm{E}_j = \sum_{j=1}^{3}\mathrm{g}_{0j}\mathrm{E}.$$

Collision operator

For hard spheres, Hilbert:

$$\mathrm{L}g = \big(-\nu(\xi) + \mathrm{K}\big)g = \big(-\nu(\xi) + \mathrm{K}_2 - \mathrm{K}_1\big)g,$$

$$\nu(\xi) = \frac{2\rho_0\sqrt{R\theta_0}}{\sqrt{2\pi}}\Big(e^{-\frac{|\boldsymbol{c}|^2}{2}} + \big(|\boldsymbol{c}| + \frac{1}{|\boldsymbol{c}|}\big)\int_0^{|\boldsymbol{c}|} e^{-\frac{y^2}{2}}dy\Big),$$

$$\mathrm{K}_j g(\xi) = \int_{\mathbb{R}^3} \mathrm{K}_j(\xi, \xi_*) g(\xi_*) d\xi_*, \; j = 1, 2,$$

$$\mathrm{K}_1(\xi, \xi_*) = \frac{\rho_0}{\sqrt{2\pi}R\theta_0}|\boldsymbol{c} - \boldsymbol{c}_*|e^{-\frac{|\boldsymbol{c}|^2}{4}}e^{-\frac{|\boldsymbol{c}_*|^2}{4}},$$

$$\mathrm{K}_2(\xi, \xi_*) = \frac{2\rho_0}{\sqrt{2\pi}R\theta_0}|\boldsymbol{c}_* - \boldsymbol{c}|^{-1}e^{-\frac{(|\boldsymbol{c}|^2 - |\boldsymbol{c}_*|^2)^2}{8|\boldsymbol{c} - \boldsymbol{c}_*|^2}}e^{-\frac{|\boldsymbol{c} - \boldsymbol{c}_*|^2}{8}},$$

$$\boldsymbol{c} \equiv (\xi - \boldsymbol{v}_0)/\sqrt{2R\theta_0}, \; \boldsymbol{c}_* \equiv (\xi_* - \boldsymbol{v}_0)/\sqrt{2R\theta_0}.$$

1-D Green's function

$$\begin{cases} \mathbb{G}_t + \xi_1 \partial_x \mathbb{G} = \mathrm{L}\mathbb{G}, \ x \in \mathbb{R}, \\ \mathbb{G}(x, 0, \boldsymbol{\xi}; \boldsymbol{\xi}_*) = \delta^1(x)\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_*). \end{cases}.$$

$$\mathbb{G}(x, t; \boldsymbol{\xi}, \boldsymbol{\xi}_*) = \sum_{j=1}^{3} \frac{1}{\sqrt{4\pi A_j(t+1)}} e^{-\frac{(x-\lambda_j t)^2}{4A_j(t+1)}} \mathrm{E}_j \otimes \langle \mathrm{E}_j | \text{ fluid part}$$

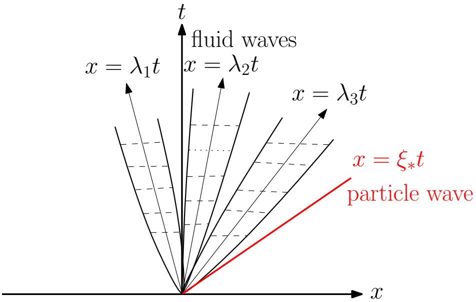
$$+ e^{-\frac{\nu(\boldsymbol{\xi}_*)}{k} t} \delta^1(x - \xi_{*1} t)\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_*) \text{ particle-like}$$

$$+ \text{Remainder},$$

$$A_i \equiv -k(\mathrm{P}_1 \xi_1 \mathrm{E}_i, \mathrm{L}^{-1} \mathrm{P}_1 \xi_1 \mathrm{E}_i), \text{ Navier-Stokes dissipation coefficients.}$$

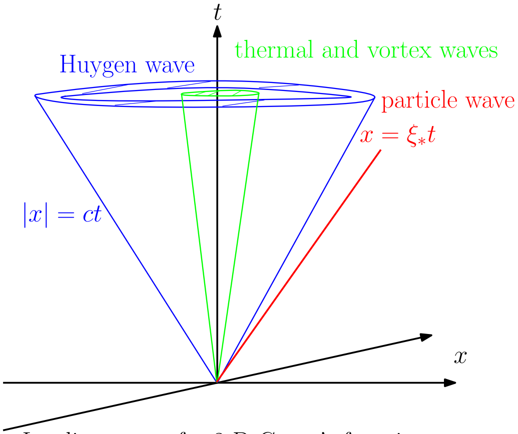
- fluid part: away from boundary and initial time.
- particl-like: initial and boundary layers.
- Remainder: smoother and decaying faster. Liu-Yu 2004, Liu-Yu 2006.

1-D Green's function



Leading waves for 1-D Green's function

3-D Green's function

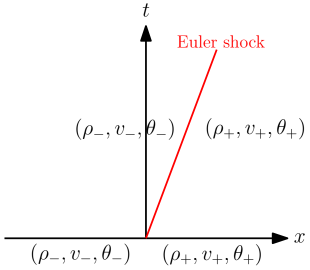


Leading waves for 3-D Green's function

Example 2. Riemann Problem

Coupling of Shock and Initial Layers.

$$f_t + \xi_1 f_x = \frac{1}{k} Q(f,f),\ f(x,0) = \begin{cases} M_{(\rho_-,v_-,\theta_-)}, & x < 0, \\ M_{(\rho_+,v_+,\theta_+)}, & x > 0. \end{cases}$$

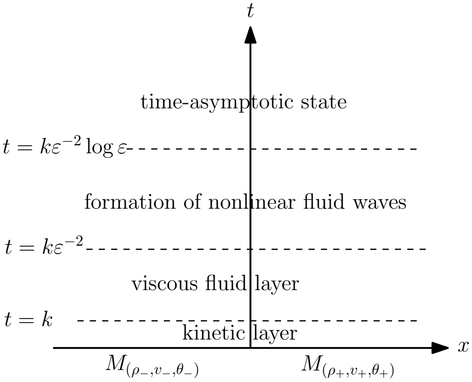


For the Boltzmann solution, the coupling of initial and shock layers induces several time scales. Yu 2014

Coupling of Shock and Initial Layers.

$$f_t + \xi_1 f_x = \frac{1}{k} Q(f, f), \; f(x, 0) = \begin{cases} M_{(\rho_-, v_-, \theta_-)}, & x < 0, \\ M_{(\rho_+, v_+, \theta_+)}, & x > 0. \end{cases}$$

The coupling induces four layers.



Initial Layer, I

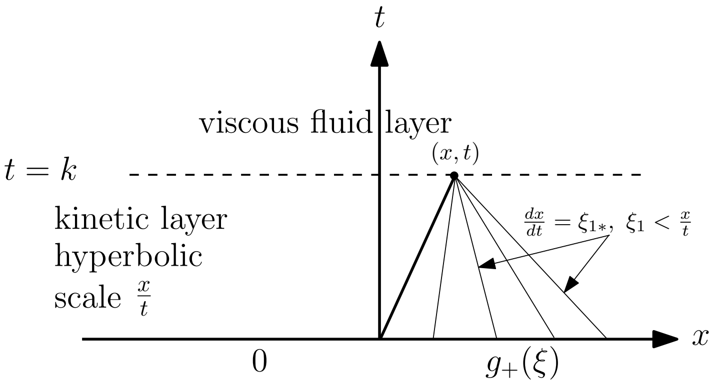
For small time and weak shocks, linearization, around $\mathrm{M}_- = \mathrm{M}_{(\rho_-, v_-, \theta_-)}$, is accurate:

$$
\begin{aligned}
&\mathrm{g}_t + \xi_1 \mathrm{g}_x = \frac{1}{k}\mathrm{L}(\mathrm{g}),\\
&\mathrm{g}(x,0,\boldsymbol{\xi}) = \begin{cases} 0, \ x < 0, \\ \mathrm{g}_+ = \nabla_{\rho,v,\theta}\mathrm{M}_- \cdot (\rho_+ - \rho_-, v_+ - v_-, \theta_+ - \theta_-), \ x > 0. \end{cases}
\end{aligned}
$$

The phenomena is hyperbolic and dominated by particle-like propagator $e^{-\frac{\nu(\boldsymbol{\xi}_*)}{k}t}\delta^1(x - \xi_{*1}t)\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_*)$ of the Green's function:

$$
\mathrm{f}(x,t,\boldsymbol{\xi}) \tilde{=} \begin{cases} \mathrm{M}_{(\rho_-, v_-, \theta_-)}, \ \xi_1 > \frac{x}{t}, \\ \mathrm{M}_{(\rho_-, v_-, \theta_-)} + e^{-\frac{\nu(\boldsymbol{\xi}_*)}{k}t}\mathrm{g}_+(\boldsymbol{\xi}), \ \xi_1 < \frac{x}{t}. \end{cases}
$$

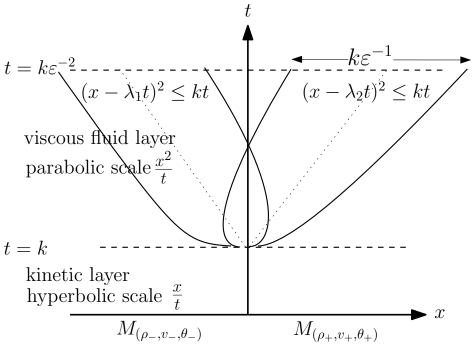
Initial Layer, I



Macroscopic variables such as the density is

$$\rho(x,t) \tilde{=} \rho_- + \int_{-\infty}^{x/t} \Big(\int_{\mathbb{R}^2} g_+(\xi) d\xi_2 d\xi_3 \Big) d\xi_1 .$$

Initial Layer, II

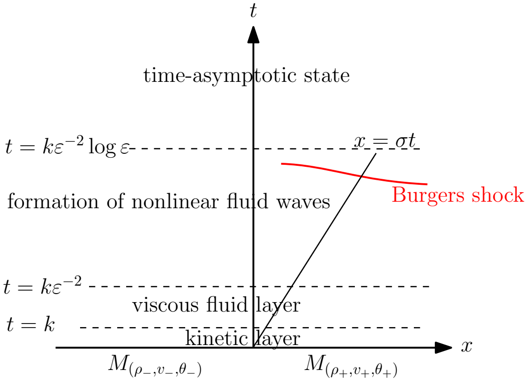


The width of Boltzmann shock is $k\varepsilon^{-1}$ for shock of strength ε. It takes time

$$(k\varepsilon^{-1})^2 = kt \;\Rightarrow\; t = k\varepsilon^{-2}$$

to reach this width.

Initial Layer, III

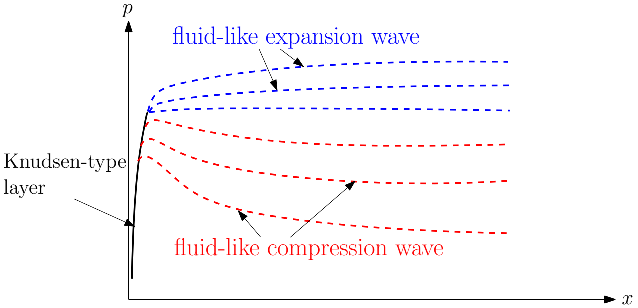


- Hoff - Liu, 1989: Approximate the Boltzmann shock by Burgers shock and use Hopf-Cole transformation to show that the Burgers shock formation time is $k\varepsilon^{-2}\log\varepsilon$.
- Yu, 2013: After time $k\varepsilon^{-2}\log\varepsilon$, study time-asymptotic stability of Boltzmann shock using the fluid part of the Green's function.

Example 3. Stationary Layers

Coupling of Knudsen-type layer and fluid-like waves.

- Knudsen boundary layer: Width proportional to the mean free path k.
- Fluid-like waves: Width k/ε, ε strength of the fluid wave.
- Coupling occurs when one of the Euler characteristics is near zero. Wave patterns for transonic condensation:



Knudsen Boundary Layers

Near the smooth boundary, the boundary layer is locally 1-dimensional near the boundary. Thus consider stationary Boltzmann equation with the gas region $x > 0$:

$$\xi_1 \mathrm{f}_x = \frac{1}{k} Q(\mathrm{f}, \mathrm{f}), \; x > 0.$$

For this transport equation, the most direct problem is to prescribe the boundary values b_+ for characteristics pointing to the gas region, $\xi_1 > 0$,

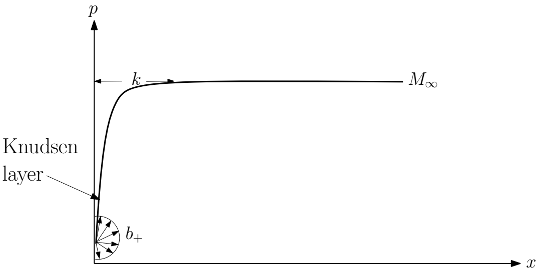
$$\mathrm{f}(0, \boldsymbol{\xi}) = \mathrm{b}_+(\boldsymbol{\xi}), \; \xi_1 > 0.$$

The Knudsen boundary layer tends to a Maxwellian so that it can connect the boundary values to the fluid-like flows:

$$\mathrm{f}(x, \boldsymbol{\xi}) \to \mathrm{M}_\infty(\boldsymbol{\xi}) \text{ as } x \to \infty.$$

Knudsen Layers

The boundary value problem is solvable when the boundary values $b_+(\xi)$ satisfies certain n_+ conditions, n_+ the number of positive Euler characteristics for the Maxwellian $M_\infty(\xi)$. The width of the Knudsen layer is of the order of the mean free path k.



e.g. Coron, Golse, Sulem 1988.

Knudsen Boundary Layers

- Knudsen layer at $x = 0$: $\begin{cases} \xi_1 g_x = \frac{1}{k} L(g), \ x > 0, \\ g(0, \boldsymbol{\xi}) = b_+(\boldsymbol{\xi}), \ \xi_1 > 0, \\ g(x, \boldsymbol{\xi}) \to 0 \text{ as } x \to \infty. \end{cases}$
- flux conservation: $(\xi_1 g(x), E_j) = 0, \ \Rightarrow$ macro part g_0 in terms of micro part g_1, effective when $\lambda_j, \ j = 1, 2, 3$, are away from zero:

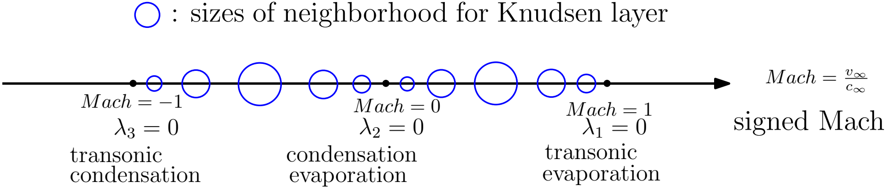
$$g_{0j}(x) = \frac{(\xi_1 g_1(x), E_j)}{\lambda_j}, \ j = 1, 2, 3.$$

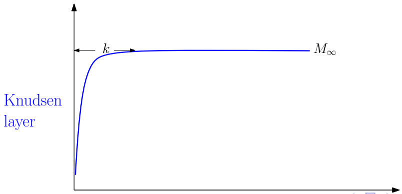
- Linear version of H-Theorem $(g, Lg) \leq -\nu_0((1 + |\boldsymbol{\xi}|)g_1, g_1)$.
- Weighted energy estimate using above yields the linear spectral gap for Knudsen layer:

$$(g, g)(x) = e^{-\alpha x}, \text{ as } x \to \infty, \ \alpha \tilde{=} \min\{|\lambda_j|, \ j = 1, 2, 3\}.$$

Degeneracy of Nonlinear Knudsen Layers

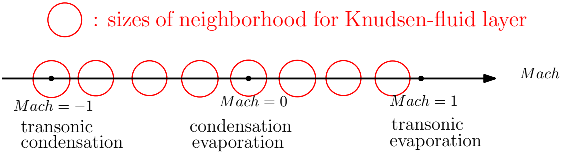
Linear spectral gap $\alpha \simeq \min\{|\lambda_j|,\ j = 1, 2, 3\}$, dynamical system approach yields exponentially decaying Knudsen layers for full Boltzmann equation $\xi_1 f = Q(f, f)$, but with the validity neighborhood of the size of α.

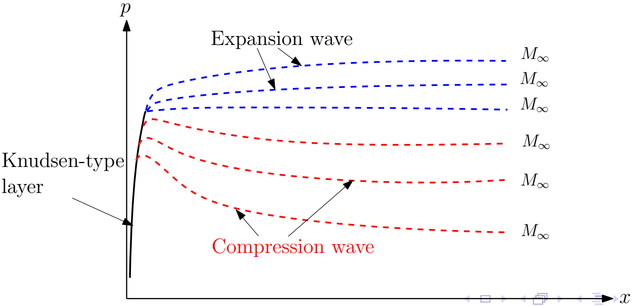




Knudsen-like Layer and Fluid-type wave

For a neighborhood of definite size around $\lambda_j = 0,\ j = 1, 2, 3$, there are Knudsen-fluid layers. For transonic condensation, the fluid-like waves are compression and expansion waves.





Green's Function Approach

To study this resonance and bifurcation phenomena, one considers all the stationary Boltzmann flows. Quantitative estimates are necessary for the strongly nonlinear phenomena, The Green's function approach is used through time-asymptotic analysis:

Multiply the linear Boltzmann equation $\xi_1 \mathrm{f}_x = \frac{1}{k}\mathrm{L}(\mathrm{f})$ (with given data b at $x = 0$) by the Green's function $\mathbb{G}$ and integrate over $x, t > 0$ to obtain the Green's identity

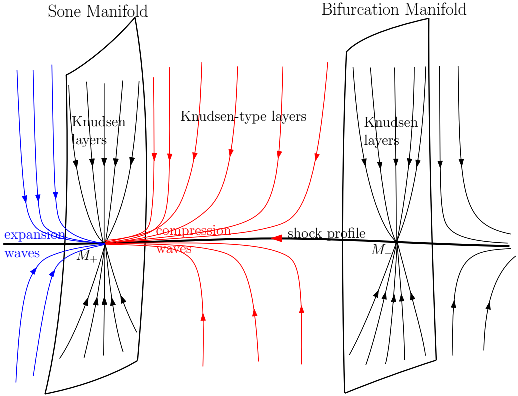
$$\mathrm{f}(x) = \int_0^t \mathbb{G}(x, t-\tau)[\xi_1 \mathrm{b}]d\tau + \int_0^\infty \mathbb{G}(x-y, t)\mathrm{f}(y)dy$$

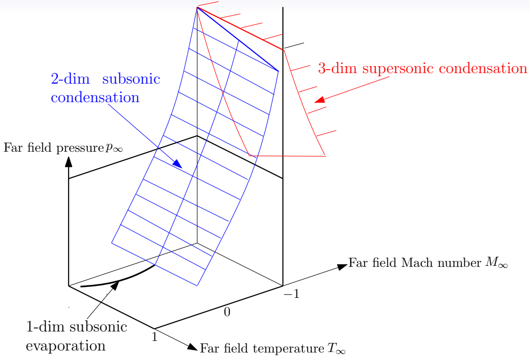
Use time-asymptotic approach and the explicit form of the Green's function to analyze

$$\int_0^\infty \mathbb{G}(x, t-\tau)[\xi_1 \mathrm{b}]d\tau$$ boundary integral operator.

Global diagram of trajectories, Liu-Yu, 2013

Trajectories for transonic condensation.





Strongly nonlinear bifurcation diagram

- Kyoto School 1980-2000, asymptotic analysis and computations.
- , Liu-Yu 2013, Green's function approach, Center manifold reduction. Fluid nonlinearity.

Happy 70th Birthday, Piero.