H^1 scattering mass-subcritical NLS

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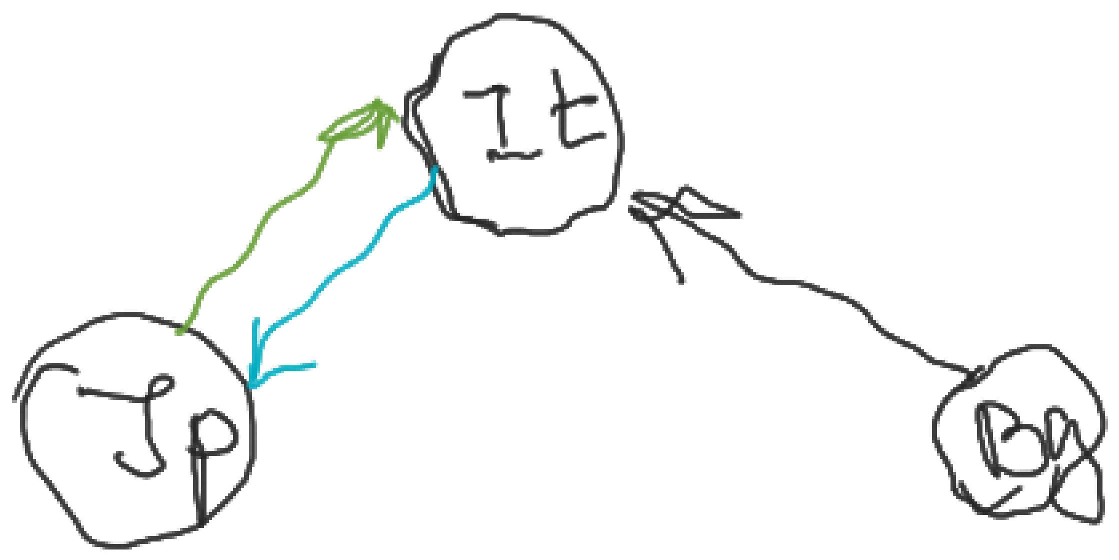
Workshop dedicated to 70th birthday of Piero Marcati



NLS

$$(i\partial_t + \Delta)u = \pm u|u|^p$$

$+$ defocusing

$-$ focusing

For $0 < p < \frac{4}{n-2}$, $n \geq 3$ global solution on $[0; T]$ with T depending on $\dot{H}^1$ data.

We consider Defocussing case.

$$E(t) := \frac{1}{2}\|\nabla u(t)\|^2_{L^2} + \frac{1}{p+2}\|u(t)\|^{p+2}_{L^{p+2}}$$

The case $p = 4/(n-2)$ + radial data

global sol. + scatt → J. Bourgain (99)

non-rad. 2008 Colliander, Keel, Staffilani, Takaoka, Tao.

Non-linear semigroup: Let $u(t,x) = U(t,t_0)\varphi$ be the solution operator associated with the Cauchy problem

$$\begin{cases} (i\partial_t + \Delta)u = u|u|^p, \\ u(t_0) = \varphi \end{cases}$$

We have the following properties of

$$\underbrace{\mathcal{U}(t_1, t_2) : H^1 \longrightarrow H^1}$$

(a) $\mathcal{U}(t_1, t_2)\,\mathcal{U}(t_2, t_3) = \mathcal{U}(t_1, t_3)$

(b) $\mathcal{U}(t_1 + T, t_2 + T) = \mathcal{U}(t_1, t_2) = \underline{\mathcal{U}(t_1 - t_2)}$

(c) $\operatorname{s-lim}_{t \to 0} \mathcal{U}(t, 0) = I$

(d) $\|\mathcal{U}(t) f\|^2_{H^1} \lesssim \|f\|^2_{H^1} + \|f\|^{p+2}_{H^1}$, $p < \frac{4}{n-2}$

Wave operators connect the free evolution

$\mathcal{U}_0(t)f = e^{it\Delta} f$ with non-linear

semi-group $\mathcal{U}(t)$:

Formally

$$W_- = \operatorname*{s-lim}_{t \to +\infty} \mathcal{U}(-t) e^{it\Delta}$$

Completeness of wave operators

$$f \in H^1 \Rightarrow \exists\, g \in H^1, \quad f = W g$$

$$\operatorname{Im} W = H^1(\mathbb{R}^n)$$

(a) $$\boxed{\| U(t) f - e^{it\Delta} g \|_{H^1} \xrightarrow{t \to +\infty} 0}$$

(b) $$\| f - U(-t) e^{it\Delta} g \|_{H^1} \xrightarrow[t \to +\infty]{} 0$$

Case $\frac{4}{n} < p < \frac{4}{n-2}$ Mass supercritical
Energy subcritical

Existence and completeness of wave operators $n \geq 3$

Ginibre-Velo 85

$n = 1, 2$ Nakanishi 99

Mass critical case $p = 4/n$, Dodson (12-16)

Mass-subcritical case $p < 4/n$,

new critical exponent $p = 2/n$

$0 < p \leq 2/n$ No scattering Barab 84

$\frac{2}{n} < p < \frac{4}{n}$ scattering in

$$\Sigma_1 = H^1(\mathbb{R}^n) \cap |x|^{-1} L^2(\mathbb{R}^n)$$

Yajima - Tsutsumi (84) $\boxed{p \in (2/n, 4/n), \text{ deloc.}}$

If $\varphi \in \Sigma_1$, then $\exists\ \varphi_+ \in L^2$ so that

$$\| U(t)\varphi - e^{it\Delta}\varphi_+ \|_{L^2} \longrightarrow 0$$

Our main result

T2 If $\frac{2}{n} < p < \frac{4}{n}$ and $\varphi \in \Sigma_1 \Rightarrow \exists\, \varphi_+ \in H^1$ so that

$$\| U(t)\varphi - e^{it\Delta}\varphi_+ \|_{H^1} \xrightarrow[t\to\infty]{} 0$$

Cazenave, Weissler 92
Tsutsumi 03

$$\boxed{n p_*^2 + (n-2)p_* - 2 = 0}$$

$p > p_*$

$\varphi \in \Sigma_2 \Rightarrow \varphi_+ \in \Sigma_1$ so that

$$\| U(t)\varphi - e^{it\Delta}\varphi_+ \|_{\Sigma^1} \xrightarrow{t\to+\infty} 0$$

Idea of proof

To show $\|\nabla U(t,\cdot)\varphi\|_{L^2} - \|\nabla e^{it\Delta}\varphi_+\|_{L^2} \xrightarrow{t\to\infty} 0$

we use pseudo-conformal transform.

$$\boxed{\begin{gathered} i\partial_t u + \Delta u = u|u|^p \\ u(0) = \varphi \end{gathered}}$$

$$\begin{gathered} i\partial_s w + \Delta w = t^{\alpha} w|w|^p \\ w(1) \in \Sigma_1 \end{gathered}$$

$$\boxed{u(t,x) = \frac{1}{t^{n/2}}\bar{w}\left(\frac{1}{t},\frac{x}{t}\right)e^{ix^2/4t}}$$

$$(1) \quad u(t,x) = \frac{1}{t^{n/2}} w\left(1/t, x/t\right) e^{ix^2/4t}$$

$$\frac{1}{t} = T, \quad \frac{x}{t} = X.$$

$$i\partial_t u + \Delta_x u = \dots \boxed{\frac{1}{t^{n/2+2}}} \nabla_x^2 w\left(1/t, x/t\right) e^{ix^2/4t}$$

$$u|u|^p = \boxed{\frac{1}{t^{\frac{n}{2}(p+1)}}} w|w|^{p-1} e^{ix^2/4t}$$

$$(2) \quad \left(i\partial_t + \Delta\right) w = t^{-\alpha} w|w|^p, \qquad \alpha = 2 - \frac{np}{2}$$

Information about w

$$(i\partial_t + \Delta) w = t^{-\alpha} w|w|^p$$

$$w(1) = \tilde{\varphi}$$

$$\frac{d}{dt}\left(\frac{t^\alpha \|\nabla w\|^2}{2} + \frac{\|w(t)\|_{L^{p+2}}^{p+2}}{p+2} \right) =$$

$$= \frac{\|\nabla w\|^2}{2} \partial_t t^\alpha > 0$$

$$\boxed{\begin{aligned} (i\partial_t + \Delta) u &= u|u|^p \\ u(1) &= \varphi \in \Sigma_1 \end{aligned}}$$

$\Downarrow$? find $\varphi_+ \in H^1$

$$\| U(t-1)\varphi - e^{i(t-1)\Delta}\varphi_+ \|_{H^1} \to 0$$

as $t \to +\infty$.

$\boxed{\alpha \leq 1}$

$$t \|\nabla w\|_{L^2} \xrightarrow{t \to 0} 0$$

$$w(t,x) = \frac{1}{t^{n/2}} u(1/t, x/t) e^{i x^2/4t}$$

$$\nabla w(t,x) = \frac{1}{t^{n/2}} e^{i x^2/4t} \left(\frac{\nabla u(1/t, x/t)}{t} - \frac{i x}{2} u \right)$$

$$t \|\nabla w(t)\|_{L^2} = \| \nabla u(1/t) - i t x u(1/t, \cdot) \|_{L^2}$$

$T = 1/t$

$$\left\| \nabla u(T) - \frac{ix}{2T} u(T) \right\|_{L^2} \xrightarrow{T \to \infty} 0$$

Finally, we use virial type argument to show

$$\left\| \frac{x}{T} \nabla u(T) \right\|_{L^2} \xrightarrow{T \to \infty} \| \nabla \varphi_+ \|_{L^2}$$

Lemma 1 $\forall \varepsilon > 0$, $\exists t_\varepsilon, R_\varepsilon$ so that

$$\sup_{t > t_\varepsilon} \left\| \frac{|x|}{t} \mathcal{U}(t)\varphi \right\|_{L^2(|x| > R_\varepsilon t_\varepsilon)} \leq \varepsilon .$$

Idea of proof $u(t,x) = \mathcal{U}(t)\varphi \xrightarrow{PCT} w(t,x)$

Our goal $\sup_{t < t_\varepsilon} \| |x| w(t,x) \|_{L^2(|x| > R_\varepsilon)} \leq \varepsilon .$

St.1 $\sup_{0 < t < 1} \| \psi_R(|x|) w \|^2_{L^2} \lesssim 1$ $\boxed{\psi_R(|x|) = |x|, \ |x| > R}$

$$\left| \frac{d}{dt} \| \psi_R w \|^2 \right| \lesssim \| \nabla w \|_{L^2} \, \| |x| w \|_{L^2(|x|>R)} \lesssim t^{-d/2}$$

$$\| \psi_R w(t) \|^2 \lesssim \| \psi_R w(t_\varepsilon) \|_{L^2}^2 + \int_t^{t_\varepsilon} \tau^{-d/2} \, d\tau$$

$$\varepsilon \Longrightarrow \exists \, t_\varepsilon, \ \int_0^{t_\varepsilon} \tau^{-d/2} \, d\tau < \varepsilon/2$$

$$R_\varepsilon \quad \| |x| w(t_\varepsilon) \|_{L^2(|x|>R_\varepsilon)} < \varepsilon/2$$

$\#$

Key Proposition.

$$\left\| \frac{|x|}{t} u(t,x) \right\|^2_{L^2(|x| < Rt)} \xrightarrow{t \to \infty} 4 \int_{|x| < Rt} |\nabla \varphi_+(x)|^2 \, dx$$

Idea of proof:

$$\left\| \frac{|x|}{t} \left(|u(t,x)| - \left| \frac{\hat{\varphi}_+(x/2t)}{(2t)^{n/2}} \right| \right) \right\| \leq \text{two terms}$$

$$\leq \left\| \frac{|x|}{t} \left(|u(t,x)| - |e^{it\Delta} \varphi_+| \right) \right\|_{L^2(|x|<Rt)} +$$

$$+ \left\| \frac{|x|}{t} \left(|e^{it\Delta} \varphi_+| - \left| \frac{\hat{\varphi}_+(x/2t)}{(2t)^{n/2}} \right| \right) \right\|_{L^2(|x|<Rt)}$$

First term tends to 0 due to Yajima-Tsutsumi result $\left(\frac{|x|}{t} < R\right)$

second is a well-known $\longrightarrow 0$ #

Happy Birthday,

Piero!