

HYPERBOLIC SYSTEMS OF BALANCE LAWS

WITH STIFF SOURCE

CAUCHY PROBLEM FOR SYSTEMS OF CONSERVATION LAWS

$$\begin{cases} \partial_t U + \partial_x F(U) = 0 & -\infty < x < \infty, t > 0 \\ U(x, 0) = U_0(x) & -\infty < x < \infty \end{cases}$$

$DF(U)$: eigenvalues $\lambda_1(U) < \dots < \lambda_n(U)$

eigenvectors $R_1(U), \dots, R_n(U)$

$TV U_0(\cdot) = \delta \ll 1 \Rightarrow \exists$ BV solution $U(x, t)$

$$TV U(\cdot, t) \leq c\delta, \quad 0 \leq t < \infty$$

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CAUCHY PROBLEM FOR SYSTEMS OF BALANCE LAWS

$$\begin{cases} \partial_t U + \partial_x F(U) + G(U) = 0 & -\infty < x < \infty, t > 0 \\ U(x, 0) = U_0(x) & -\infty < x < \infty \end{cases}$$

$TV U_0(\cdot) = \delta \ll 1 \Rightarrow \exists$ BV solution $U(x, t)$ on $[0, T)$

$$TV U(\cdot, t) \leq c \delta e^{pt}, \quad 0 \leq t < T$$

solution may be extended for as long as

$$TV U(\cdot, t) \ll 1$$

STRONG DISSIPATION

$$\begin{cases} \partial_t U + \partial_x F(U) + G(U) = 0 & -\infty < x < \infty, t > 0 \\ U(x, 0) = U_0(x) & -\infty < x < \infty \end{cases}$$

$$R(U) = [R_1(U) \dots R_n(U)]$$

$$A = R(0)^{-1} \triangleright G(0) R(0)$$

- A strictly diagonal dominant
- $TV U_0(\cdot) = \delta \ll 1$

$\Rightarrow \exists$ BV solution $U(x, t)$ on $[0, \infty)$

$$TV U(\cdot, t) \leq c \delta e^{-\nu t}, \quad 0 \leq t < \infty$$

PARTIAL DISSIPATION

$$\begin{cases} \partial_t U + \partial_x F(U) + G(U) = 0 & -\infty < x < \infty, t > 0 \\ U(x, 0) = U_0(x) & -\infty < x < \infty \end{cases}$$

- There exists convex entropy $\eta(U)$
- Dissipative source: $D\eta(U) G(U) \geq |G(U)|^2$
- $A_{ii} > 0, i=1, \dots, n$
- $TV U_0(\cdot) = \delta \ll 1$
- $\int_{-\infty}^{\infty} (1+x^2) |U_0(x)|^2 dx = \varepsilon^2 \ll 1$

$\Rightarrow \exists$ BV solution $U(x, t)$ on $[0, \infty)$

$$TV U(\cdot, t) \leq c_1 \delta e^{-\nu t} + c_2 \varepsilon$$

$$TV U(\cdot, t) \rightarrow 0, \text{ as } t \rightarrow \infty$$

STIFF SOURCE - ZERO RELAXATION LIMIT

$$\begin{cases} \partial_t U + \partial_x F(U) + \frac{1}{\mu} G(U) = 0, & -\infty < x < \infty, t > 0 \\ U(x, 0) = U_0(x) & -\infty < x < \infty \end{cases}$$

TASKS :

- For fixed U_0 , existence of solution $U_\mu(x, t)$ for all $\mu > 0$.
- $TV U_\mu(\cdot, t)$ bounded small for $t \geq 0$ uniformly in $\mu > 0$
- Limits of $\{U_\mu\}$ as $\mu \rightarrow 0$ - uniqueness??

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MODEL SYSTEM

$$\begin{cases} \partial_t U - \partial_x V = 0 \\ \partial_t V - \partial_x S(U) + \frac{1}{\mu} V = 0 \end{cases}$$

$$S(U) = D\varphi(U), \quad \varphi(U) \text{ convex}$$

$$\text{entropy: } \quad \varphi(U) + \frac{1}{2} |V|^2$$

$$n=1:$$

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v + \partial_x p(u) + \frac{1}{\mu} v = 0 \end{cases}$$

SPECIAL SYSTEM

$$\begin{cases} \partial_t u - \partial_x v = 0 \\ \partial_t v + \partial_x \left(\frac{1}{u} \right) + \frac{1}{\mu} v = 0 \end{cases}$$

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x)$$

$$\bullet \quad TV u_0(\cdot) + TV v_0(\cdot) = \delta$$

$\Rightarrow \exists$ BV solution $(u_\mu(x, t), v_\mu(x, t))$ on $[0, \infty)$

$$TV u_\mu(\cdot, t) + TV v_\mu(\cdot, t) \leq c\delta, \quad 0 \leq t < \infty$$

$$\int_{-\infty}^{\infty} |u_\mu(x, \sigma) - u_\mu(x, \tau)| dx \leq c_1 \delta (\sigma - \tau), \quad 0 \leq \tau < \sigma < \infty$$

$$\int_{-\infty}^{\infty} |v_\mu(x, \sigma) - v_\mu(x, \tau)| dx \leq \left[c_1 \delta + c_2 \frac{1}{\mu} e^{-\frac{\lambda t}{\mu}} \right] (\sigma - \tau)$$

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MAIN RESULT

$$\begin{cases} \partial_t U - \partial_x V = 0 \\ \partial_t V - \partial_x S(U) + \frac{1}{\mu} V = 0 \end{cases} \quad -\infty < x < \infty, t > 0$$

$$U(x, 0) = U_0(x), \quad V(x, 0) = V_0(x), \quad -\infty < x < \infty$$

$$\int_{-\infty}^{\infty} \left\{ |U_0(x)|^2 + |V_0(x)|^2 + |U_0'(x)|^2 + |V_0'(x)|^2 \right\} dx = \varepsilon^2 < 1$$

\Rightarrow There are $\varepsilon_0, c_0, c_1, \lambda$ such that, for any $\mu > 0$

and $\varepsilon < \varepsilon_0$ \exists BV solution (U_μ, V_μ) with

$$TV U_\mu(\cdot, t) + TV V_\mu(\cdot, t) \leq c_0 \varepsilon$$

$$\int_{-\infty}^{\infty} |U_\mu(x, \sigma) - U_\mu(x, \tau)| dx \leq c_0 \varepsilon (\sigma - \tau)$$

$$\int_{-\infty}^{\infty} |V_\mu(x, \sigma) - V_\mu(x, \tau)| dx \leq c_1 \left[\varepsilon + \frac{1}{\mu} e^{-\lambda t/\mu} \|V_0\|_{L^1} \right] (\sigma - \tau)$$

REDISTRIBUTION OF DAMPING FAILS

$$\begin{cases} \partial_t U - \partial_x V = 0 \\ \partial_t V - \partial_x S(U) + \frac{1}{\mu} V = 0 \end{cases} \quad -\infty < x < \infty, t > 0$$

$$U(x, 0) = U_0(x), \quad V(x, 0) = V_0(x), \quad -\infty < x < \infty$$

$$\Phi(x, t) = \frac{1}{\mu} \int_{-\infty}^x U(y, t) dy$$

$$\Phi_0(x) = \frac{1}{\mu} \int_{-\infty}^x U_0(y) dy$$

$$Y(x, t) = V(x, t) + \frac{1}{2} \Phi(x, t)$$

$$\begin{cases} \partial_t U - \partial_x Y + \frac{1}{2\mu} U = 0 \\ \partial_t Y - \partial_x S(U) + \frac{1}{2\mu} Y = \frac{1}{4\mu} \Phi \end{cases}$$

$$TV \Phi(\cdot, t) = \frac{1}{\mu} \int_{-\infty}^{\infty} |U(y, t)| dy$$

PARTITION OF INITIAL DATA

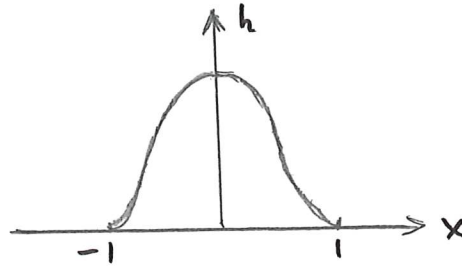
$$U_0(x) = P_0(x) + W_0(x), \quad V_0(x) = Q_0(x) + \Sigma_0(x), \quad \Phi_0(x) = \Psi_0(x) + \Omega_0(x)$$

$$P_0(x) = \frac{1}{\mu} \int_{-\infty}^{\infty} h\left(\frac{x-\xi}{\mu}\right) U_0(\xi) d\xi$$

$$Q_0(x) = \frac{1}{\mu} \int_{-\infty}^{\infty} h\left(\frac{x-\xi}{\mu}\right) V_0(\xi) d\xi$$

$$\Psi_0(x) = \frac{1}{\mu} \int_{-\infty}^{\infty} h\left(\frac{x-\xi}{\mu}\right) \Phi_0(\xi) d\xi = \frac{1}{\mu} \int_{-\infty}^x P_0(y) dy$$

$$h(-x) = h(x)$$



$$\int_{-1}^1 h(x) dx = 1$$

$$\int_{-\infty}^{\infty} (1+x^2) \{ |P_0(x)|^2 + |P_0'(x)|^2 + \mu^2 |P_0''(x)|^2 + \mu^4 |P_0'''(x)|^2 \} dx \leq c \varepsilon^2$$

$$\int_{-\infty}^{\infty} (1+x^2) \{ |Q_0(x)|^2 + |Q_0'(x)|^2 + \mu^2 |Q_0''(x)|^2 + \mu^4 |Q_0'''(x)|^2 \} dx \leq c \varepsilon^2$$

$$\int_{-\infty}^{\infty} (1+x^2) \{ |W_0(x)|^2 + |\Sigma_0(x)|^2 + |\Omega_0(x)|^2 \} dx \leq c \mu^2 \varepsilon^2$$

PARTITION OF STATE VECTOR

$$U(x,t) = P(x,t) + W(x,t), \quad V(x,t) = R(x,t) + Z(x,t), \quad \Phi(x,t) = \Psi(x,t) + \Omega(x,t)$$

$$\begin{cases} \partial_t P = \mu \partial_x^2 S(P) + e^{-\frac{t}{\tau}} \partial_x Q_0 \\ P(x,0) = P_0(x) \end{cases}$$

$$R(x,t) = \mu \partial_x S(P(x,t)) + e^{-\frac{t}{\tau}} Q_0(x)$$

$$\Psi(x,t) = \frac{1}{\mu} \int_{-\infty}^x P(y,t) dy$$

$$\begin{cases} \partial_t W - \partial_x Z = 0 \\ \partial_t Z - \partial_x [S(P+W) - S(P)] + \frac{1}{\mu} Z = -\mu \partial_t \partial_x S(P) \end{cases}$$

$$W(x,0) = W_0(x), \quad Z(x,0) = \underline{Z}_0(x) - \mu \partial_x S(P_0(x))$$

ENERGY ESTIMATES FOR THE POROUS MEDIA EQUATION

$$\begin{cases} \partial_t P = \mu \partial_x^2 S(P) + e^{-\frac{t}{\mu}} \partial_x Q_0 \\ P(x, 0) = P_0(x) \end{cases}$$

$$|\partial_x P(x, t)| \leq \frac{c\varepsilon}{\sqrt{\mu}}, \quad |\partial_t P(x, t)| \leq \frac{c\varepsilon}{\sqrt{\mu}}$$

$$t |\partial_t P(x, t)| \leq c\varepsilon$$

$$\text{TV} P(\cdot, t) \leq c\varepsilon$$

$$\text{TV} \partial_x S(P(\cdot, t)) \leq \frac{c\varepsilon}{\mu}$$

$$\text{TV} \partial_x^2 S(P(\cdot, t)) \leq \frac{c\varepsilon}{\mu^2}$$

$$\int_{-\infty}^{\infty} |\partial_t P(x, t)| dx \leq c\varepsilon$$

$$\int_0^{\infty} \int_{-\infty}^{\infty} (1+x^2) |\partial_t \partial_x S(P(x, t))|^2 dx dt \leq \frac{c\varepsilon^2}{\mu}$$

ENERGY ESTIMATES FOR THE REDUCED BALANCE LAW

$$\begin{cases} \partial_t W - \partial_x Z = 0 \\ \partial_t Z - \partial_x [S(P+W) - S(P)] + \frac{1}{\mu} Z = -\mu \partial_t \partial_x S(P) \end{cases}$$

$$W(x,0) = W_0(x), \quad Z(x,0) = \bar{X}_0(x) - \mu \partial_x S(P_0(x))$$

$$\eta(W, Z) = \varphi(P+W) - \varphi(P) - S(P) \cdot W + \frac{1}{2} |Z|^2$$

$$H(W, Z, \Omega) = \eta(W, Z) + \nu |\Omega|^2 + 2\nu \Omega \cdot Z$$

$$\partial_t [(1+x^2) \eta(W, Z)] + \dots$$

$$\partial_t [(1+x^2) H(W, Z, \Omega)] + \dots$$

$$\partial_t [t(1+x^2) \eta(W, Z)] + \dots$$

$$\int_{-\infty}^{\infty} (1+x^2) \{ |W(x,t)|^2 + |Z(x,t)|^2 \} dx \leq c \mu^2 \varepsilon^2$$

$$\int_{-\infty}^{\infty} |W(x,t)| dx \leq c \mu \varepsilon$$

BOUNDS ON THE VARIATION

$$\begin{cases} \partial_t W - \partial_x Z = 0 \\ \partial_t Z - \partial_x [S(P+W) - S(P)] + \frac{1}{\mu} Z = -\mu \partial_t \partial_x S(P) \end{cases}$$

$$\bar{X} = Z + \mu \partial_x S(P) - \frac{1}{2} \int_0^t e^{-\frac{t-\tau}{2\mu}} \partial_x S(P) d\tau + \frac{1}{2} \Omega - \frac{1}{2} e^{-\frac{t}{2\mu}} \Omega_0$$

$$\begin{cases} \partial_t W - \partial_x \bar{X} + \frac{1}{2\mu} W = \frac{1}{\mu} \Theta \\ \partial_t \bar{X} - \partial_x [S(P+W) - S(P)] + \frac{1}{2\mu} \bar{X} = \frac{1}{4\mu} \Omega \end{cases}$$

$$\Theta = -\mu^2 \partial_x^2 S(P) + \frac{1}{2} \mu \int_0^t e^{-\frac{t-\tau}{2\mu}} \partial_x^2 S(P) d\tau + \frac{1}{2} e^{-\frac{t}{2\mu}} W_0$$

$$TV \Omega(\cdot, t) = \frac{1}{\mu} \int_{-\infty}^{\infty} |W(x, t)| dx \leq c \varepsilon$$

$$TV \Theta(\cdot, t) \leq c \varepsilon$$