Direct and inverse problems for a class of degenerate parabolic equations arising in climate science

Piermarco Cannarsa

Dipartimento di Matematica
Università di Roma "Tor Vergata"

International Conference
on **Partial Differential Equations and Applications**
in honor of the 70th birthday of Pierangelo Marcati

Gran Sasso Science Institute, L'Aquila (IT) 19-23 June 2023

Figure: UMI National Congress, Pavia 2019

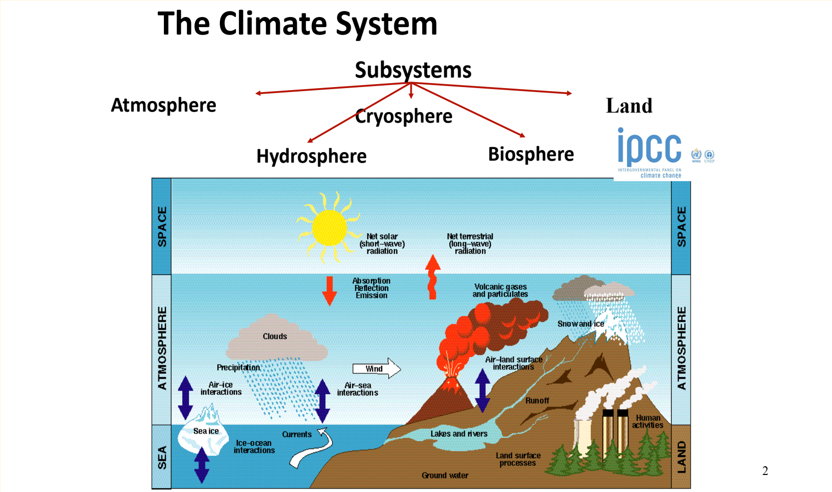
Figure: Piero's new appointment

Contents

“Climate is what we expect, weather is what we get”

When we talk about climate, we mean weather averaged over space and time so that local variations and diurnal and random fluctuations have been eliminated

Mathematics &Climate by H. Kaper and H. Engler



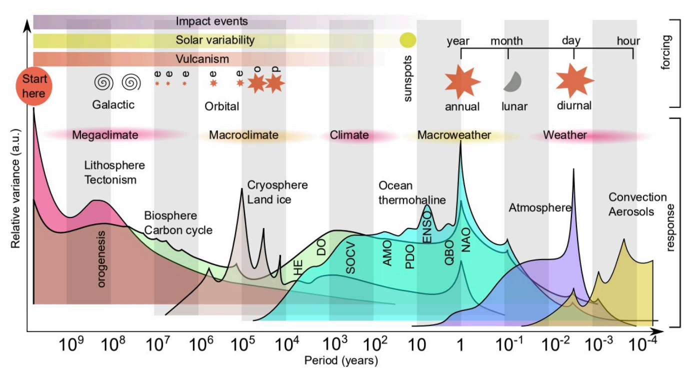
Climate models

Purposes

- Better understanding of past (and future) climate
- Better understanding the sensitivity to some relevant solar and terrestrial parameters

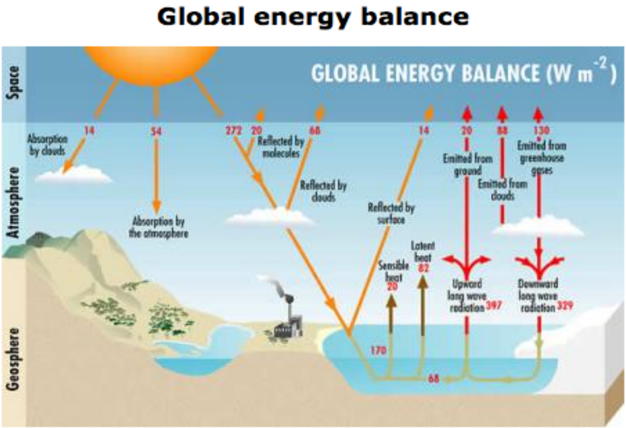
Main feature

- Multi-scale nature (in both time and space)



Energy balance models

General description

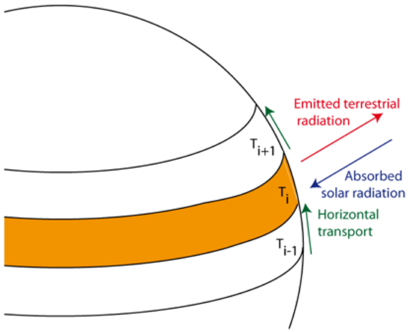


Source: Bureau of Meteorology, *The greenhouse effect and climate change*, Bureau of Meteorology, 2003, p. 7.

Energy balance models

introducing diffusion

Local Energy Budget



1

Hierarchy in the class of climate models :

- $0-D$: $u(t)$, mean Earth temperature
- $1-D$: $u(t,\varphi)$: mean temperature on the latitude circles around the Earth
- $2-D$: $u(t,m)$: temperature on Earth surface ($m \in S^2$)
- $3-D$: General Circulation Model $u(t,m,h)$ (coupled with Glaceology, Celestial Mechanics, Geophysics. . .)

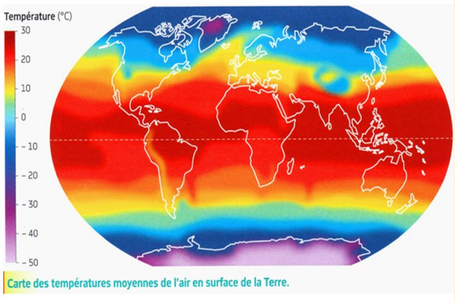


Figure: $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ parametrizes latitude

The Budyko (1969) and Sellers (1969) models

Temperature average on the Earth u satisfies

variation of $u =$ +absorbed energy −reflected energy +diffusion

hence a **reaction-diffusion equation** of the form

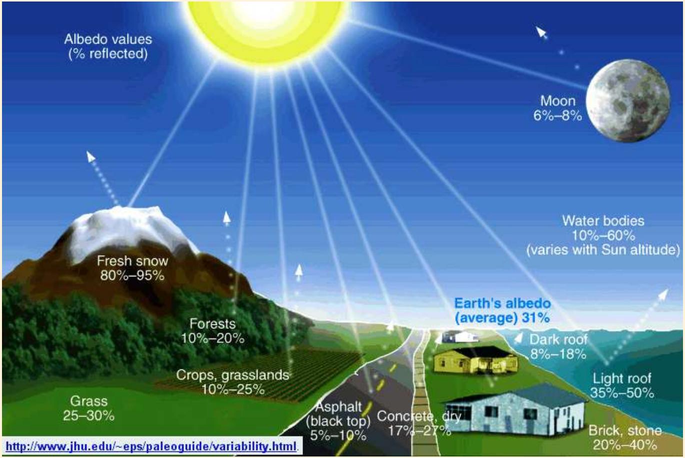
$$c(t,x)u_t - \text{ diffusion } = R_a - R_e$$

where

- $c(t,x)$: heat capacity,
- $R_a =$ absorbed solar radiation, $= QS(t,x)\beta(u)$:
  - Q : Solar constant,
  - $S(t,x)$: distribution of solar radiation,
  - $\beta(u)$: "planetary coalbedo" (= the fraction absorbed according the average temperature),
- $R_e =$: emitted radiation (depends on the amount of greenhouse gases, clouds and water vapor in the atmosphere, increases with u)

EBMs : absorbed solar radiation

Albedo



EBMs : absorbed solar radiation

Co-albedo and insolation function

$$R_a(t,x) = \beta\big(u(t,x)\big)\, Q(t,x) :$$

- Q : high-frequency solar radiation
- $\beta(u) = 1 - \alpha(u)$: co-albedo (fraction of absorbed energy) will be assumed to be piecewise linear

$$\beta(u) = \begin{cases} \beta_- & \text{for all } u \leq T_- \\ \beta_- + (\beta_+ - \beta_-)\frac{u - T_-}{T_+ - T_-} & \text{for all } u \in [T_-, T_+] \\ \beta_+ & \text{for all } u \geq T_+ \end{cases}$$

  for suitable constants $T_+ > T_- > 0$ and $\beta_+ > \beta_- > 0$
- α : albedo (fraction of reflected energy) ;
  - α nonincreasing, from α_+ to α_- (ice reflects more than non-iced surfaces),
  - Sellers : $\alpha(u)$ smooth / Budyko : $\alpha(u)$ discontinuous,
  - Bhattacharya-Ghil-Vulis (1982) : $\alpha(u,$ memory effect$)$; (memory effect : interesting to take into account the long response times of the ice sheets to temperature changes).

EBMs : diffusion and emitted radiation

- **diffusion** = div $(k(...)\nabla u)$:
  - $k = k_0$ positive constant, and averaging along the parallels, $x = \sin(\text{latitude})$: 1D model, degenerate parabolic equation (and possibly quasilinear) :

$$k_0((1-x^2)u_x)_x, \quad x \in (-1,1);$$

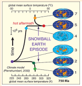
  - Sellers (1969), Ghil (1976) : 1D, $k(u)$,
  - Stone (1972) : $k(x,\nabla u) = k_1(x)|\nabla u|$ (manifold, rotating atmosphere),
  - Diaz (1993) : $k(x,\nabla u) = k_1(x)|\nabla u|^{p-2}$ (manifold) ;
- **emitted radiation** :

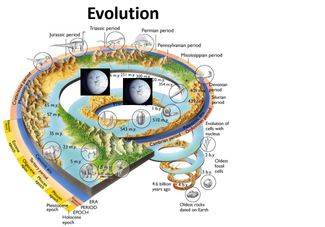
$$\textit{Sellers} : R_e = \sigma u^4 \quad / \quad \textit{Budyko} : R_e = a + bu,$$

(where σ : Stefan-Boltzmann constant).

EBMs : multistability of climate

Snow-ball and warm state

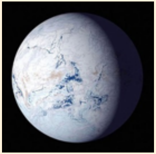


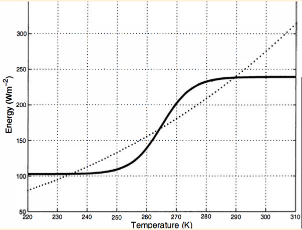


Even the simple ODE

$$u'(t) = \beta(u(t)) - \sigma u^4$$

has been used to prove the multistability of climate





EBMs : an overview

Main features

- $1-D$: degenerate diffusion coefficient
- $2-D$: on a manifold
- with nonlinear source terms, and possibly quasilinear
- possibly with discontinuous coefficients (Budyko)
- possibly with non local terms (memory)

What has been studied :

- existence and stability of multiple steady states (Ghil (1976))
- existence of solutions, uniqueness/non uniqueness (in prescribed classes) (Diaz (1993), Hetzer (1996, 2011))
- dynamics, long-time behavior (Hetzer (1991))
- free boundary value problem : snow lines (Diaz (1993))
- coeffs : uniqueness, inverse problems (Ghil et al (2014))

II. Sellers model with memory : well-posedness

1D Sellers climate model with memory

$$\begin{cases} u_t - (\rho_0(1-x^2)u_x)_x = r(t)q(x)\beta(u) - \varepsilon(u)|u|^3u + \mathbf{f}(\mathbf{H}), \\ \rho_0(1-x^2)u_x = 0, \quad x = \pm 1, \\ u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], \end{cases}$$

where

- 1-D parametrization $x = \sin(\varphi) \in (-1,1)$ with $\varphi =$ the latitude
- absorbed energy
- emitted energy
- **memory term** : to account for long response times of ice sheets to temperature changes (Ghil et al 1982, 2014)

$$\mathbf{H}(\mathbf{t},\mathbf{x},\mathbf{u}) = \int_{-\tau}^{0} k(s,x)u(t+s,x)\,ds.$$

An inverse problem for the Sellers model

- **Goal :** study an inverse problem that consists in recovering the insolation function $q(x)$ in the Sellers model with memory using partial measurements of the solution,
- **Difficulties :** degeneracy + nonlinearity + nonlocal,
- **Results :**
  - well-posedness,
  - uniqueness result under pointwise measurements,
  - Lipschitz stability under localized measurements.

Sellers model : precise assumptions

- $\rho(x) = \rho_0(1 - x^2)$, $\rho_0 > 0$, $x \in (-1, 1)$,
- $\beta \in \mathcal{C}^2(\mathbb{R})$, $\beta, \beta', \beta'' \in L^\infty(\mathbb{R})$, $\beta(\cdot) \geq \beta_1 > 0$,
- $q \in L^\infty(I)$,
- $r \in \mathcal{C}^1(\mathbb{R})$, $r, r' \in L^\infty(\mathbb{R})$, $r(\cdot) \geq r_1 > 0$,
- $\varepsilon \in \mathcal{C}^2$, $\varepsilon, \varepsilon', \varepsilon'' \in L^\infty(\mathbb{R})$, $\varepsilon(\cdot) \geq \varepsilon_1 > 0$,
- memory term :
  - kernel $k \in C^1([-\tau, 0] \times [-1, 1], \mathbb{R})$,
  - nonlinearity $f \in \mathcal{C}^2(\mathbb{R})$, $f, f', f'' \in L^\infty(\mathbb{R})$.

1D Sellers model : functional setting

- energy space : $I = (-1, 1)$

$$V = \left\{ w \in L^2(I) : w \in AC_{loc}(I), \sqrt{\rho} w_x \in L^2(I) \right\} \subset L^p(I), \forall p \geq 1$$

- operator $A : D(A) \subset L^2(I) \to L^2(I)$ in the following way :

$$\begin{cases} D(A) := \{u \in V : \rho u_x \in H^1(I)\} \\ Au := (\rho_0 (1 - x^2) u_x)_x, \quad u \in D(A) : \end{cases}$$

$(A, D(A))$ is a self-adjoint operator and it is the infinitesimal generator of an analytic and compact semigroup $\{e^{tA}\}_{t \geq 0}$ in $L^2(I)$ that satisfies

$$|||e^{tA}|||_{\mathcal{L}(L^2(I))} \leq 1.$$

(Campiti-Metafune-Pallara (1998))

1D Sellers model : functional setting

- energy space : $I = (-1, 1)$

$$V = \left\{ w \in L^2(I) : w \in AC_{loc}(I), \sqrt{\rho} w_x \in L^2(I) \right\} \subset L^p(I)\,, \forall p \geq 1$$

- operator $A : D(A) \subset L^2(I) \to L^2(I)$ in the following way :

$$\begin{cases} D(A) := \{ u \in V : \rho u_x \in H^1(I) \} \\ Au := (\rho_0 (1 - x^2) u_x)_x, \quad u \in D(A) : \end{cases}$$

$(A, D(A))$ is a self-adjoint operator and it is the infinitesimal generator of an analytic and compact semigroup $\{e^{tA}\}_{t \geq 0}$ in $L^2(I)$ that satisfies

$$|||e^{tA}|||_{\mathcal{L}(L^2(I))} \leq 1.$$

(Campiti-Metafune-Pallara (1998))

1D Sellers model : functional setting

- energy space : $I = (-1, 1)$

$$V = \left\{ w \in L^2(I) : w \in AC_{loc}(I), \sqrt{\rho} w_x \in L^2(I) \right\} \subset L^p(I), \forall p \geq 1$$

- operator $A : D(A) \subset L^2(I) \to L^2(I)$ in the following way :

$$\begin{cases} D(A) := \{u \in V : \rho u_x \in H^1(I)\} \\ Au := (\rho_0(1 - x^2) u_x)_x, \quad u \in D(A) : \end{cases}$$

$(A, D(A))$ is a self-adjoint operator and it is the infinitesimal generator of an analytic and compact semigroup $\{e^{tA}\}_{t \geq 0}$ in $L^2(I)$ that satisfies

$$|||e^{tA}|||_{\mathcal{L}(L^2(I))} \leq 1.$$

(Campiti-Metafune-Pallara (1998))

1D Sellers model : definition of mild solution

$$\begin{cases} \dot{u}(t) = Au(t) + G(t,u) + \mathbf{F}(\mathbf{u}^{(\mathbf{t})}) & t \in [0,T] \\ u(s) = u_0(s) & s \in [-\tau, 0], \end{cases} \qquad (1)$$

with

$$G(t,u) = \text{ local source terms}, \quad \mathbf{F}(\mathbf{u}^{(\mathbf{t})}) = \text{memory term}$$

Definition

Given $u_0 \in C([-\tau,0];V)$, a function

$$u \in H^1(0,T;L^2(I)) \cap L^2(0,T;D(A)) \cap C([-\tau,T];V)$$

is called a mild solution of (1) on $[0,T]$ if $u(s) = u_0(s)$ for all $s \in [-\tau,0]$, and if for all $t \in [0,T]$, we have

$$u(t) = e^{tA}u_0(0) + \int_0^t e^{(t-s)A}\left(G(s,u) + \mathbf{F}(\mathbf{u}^{(\mathbf{s})})\right) ds$$

Memory Sellers model : well-posedness result

$$\begin{cases} \dot{u}(t) = Au(t) + G(t,u) + \mathbf{F}(\mathbf{u}^{(\mathbf{t})}) & t \in [0, T] \\ u(s) = u_0(s) & s \in [-\tau, 0], \end{cases} \qquad (1)$$

Theorem (C-Malfitana-Martinez (2018))

Let u_0 be such that

$$u_0 \in C([-\tau, 0]; V) \quad \textit{and} \quad u_0(0) \in D(A) \cap L^\infty(I).$$

Then, for all $T > 0$, (1) has a unique mild solution on $[0, T]$.

Proof :

- local existence (fixed point, contraction)
- uniqueness (Gronwall's lemma),
- global existence

Memory Sellers model : local existence

Functional setting :

- the space of functions

$$\mathbb{X}_R := \left\{ v \in C([-\tau, t^*]; V) \mid \begin{cases} \|v(t)\|_V \leq R \ \forall t \in [-\tau, t^*], \\ v(t) = u_0(t) \ \forall t \in [-\tau, 0], \end{cases} \right\},$$

- and the associated application

$$\Gamma : \mathbb{X}_R \subset C([-\tau, t^*]; V) \to C([-\tau, t^*]; V)$$

defined by $\Gamma(u)(t) := u_0(t)$ for $t \in [-\tau, 0]$, and

$$\Gamma(u)(t) := e^{t\tilde{A}} u_0(0) + \int_0^t e^{(t-s)\tilde{A}} \left[\tilde{G}(s, u(s)) + F(u^{(s)}) \right] ds$$

for $t \in [0, t^*]$ (and $\tilde{A} = A - Id$ strictly dissipative) :

Γ well-defined, $\Gamma(\mathbb{X}_R) \subset \mathbb{X}_R$ and Γ contraction of $t^* > 0$ small.

Essential tool (Pazy) :

$$|||(-\tilde{A})^{1/2} e^{t\tilde{A}}|||_{\mathcal{L}(L^2(I))} \leq \frac{c}{\sqrt{t}}.$$

Memory Sellers model : uniqueness

u, $\tilde{u}$ solutions on $[0, T_0]$: the difference $w = u - \tilde{u}$ solves

$$\begin{cases} w_t - (\rho w_x)_x = G(t, u) - G(t, \tilde{u}) + f(H) - f(\tilde{H}), \\ \rho w_x = 0, \quad t \in (0, T_0), x = \pm 1, \\ w(s, x) = 0, \quad s \in [-\tau, 0], x \in (-1, 1), \end{cases}$$

and then

$$W(T') := \int_0^{T'} \|w(T)\|^2_{L^2(-1,1)} \, dT$$

is nondecreasing and satisfies (integration by parts, estimates)

$$W(T') \leq C \int_0^{T'} W(T) \, dT.$$

Gronwall $\Longrightarrow$ $W = 0$ $\Longrightarrow$ $w = 0$ $\Longrightarrow$ $u = \tilde{u}$.

Memory Sellers model : global existence

Maximal existence time :

$$T^*(u_0) := \sup\{T \geq 0 \text{ s.t. (1) has a mild solution on } [0,T]\},$$

Then

$$u_0 \in C([-\tau,0];V), u_0(0) \in D(A) \cap L^\infty(I) \quad \Longrightarrow \quad T^*(u_0) = +\infty :$$

based on the following boundedness property :

Theorem

Consider $u_0 \in C([-\tau,0];V)$ *and* $u_0(0) \in D(A) \cap L^\infty(I)$, $T > 0$ *and* u *a mild solution of* (1) *defined on* $[0,T]$*. Let us denote*

$$M_1 := \left(\frac{||q||_{L^\infty(I)} ||r||_{L^\infty(\mathbb{R})} ||\beta||_{L^\infty(\mathbb{R})} + ||f||_{L^\infty(\mathbb{R})}}{\varepsilon_1} \right)^{\frac{1}{4}}$$

and $M := \max\{||u_0(0)||_{L^\infty(I)}, M_1\}$*. Then* u *satisfies*

$$||u||_{L^\infty((0,T)\times I)} \leq M.$$

III. Sellers model : reconstruction of insolation function

1D Memory Sellers model : uniqueness/stability of the insolation function ?

$$\begin{cases} u_t - (\rho(x)u_x)_x = r(t)q(x)\beta(u) - \varepsilon(u)|u|^3 u + f(H) \\ \rho(x)u_x = 0, \quad x \in \partial I, \\ u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], x \in I \end{cases} \qquad (S)$$

$$\begin{cases} \tilde{u}_t - (\rho(x)\tilde{u}_x)_x = r(t)\tilde{q}(x)\beta(\tilde{u}) - \varepsilon(\tilde{u})|\tilde{u}|^3 \tilde{u} + f(\tilde{H}) \\ \rho(x)\tilde{u}_x = 0, \quad x \in \partial I \\ \tilde{u}(s,x) = \tilde{u}_0(s,x), \quad s \in [-\tau, 0], x \in I \end{cases} \qquad (\tilde{S})$$

$u = \tilde{u}$ on a "small" set $\implies$ $q = \tilde{q}$?

$u - \tilde{u}$ "small on a small set" $\implies$ $q - \tilde{q}$ small ?

Motivation for inverse problems (Roques et al 2014)

- Goal of the Energy Balance Models (with Memory) : toy models to understand the evolution of climate
- With suitable tuning of the parameters : EBMs simulations give reasonable results for the observed present climate (North-Mengel-Short (1983))
- Once fitted, EBM(M) can be used to estimate the temporal response patterns to various scenarios (climate change).
- BUT in practice, parameters cannot be measured directly (intertwined effects of several physical processes). So, one takes measurements of the solution and uses such measurements to reconstruct parameters (Yamamoto-Zou (2001), Roques et al (2014)).

Inverse source problem for the linear heat equation

Various approaches and results

- **"Simple" models** (constant coefficients/depending only on x or only on t...) : elegant and sharp techniques :
  - Fourier series (moment method, biorthogonal families),
  - Laplace transform,
  - Volterra integral equations...

  $\rightsquigarrow$ sharp results (explicit formula of the solution...)
  Cannon 1968, Lorenzi-Sinestrari (1988), Lorenzi (1989...), Bukhgeim (1993), Gentili (1991), Grasselli (1992), Yamamoto (1993), Janno-Wolfersdorf (1996), Choulli-Yamamoto (2006)...
- **nonlinear models** (or coefficients in x and t) : local/global Carleman estimates $\rightsquigarrow$ uniqueness, Holder/Lipschitz stability : Bukhgeim/Klibanov 1981, Klibanov (1992), Isakov (1990, 1998...), Imanuvilov/Yamamoto (1998)
- **Use of analyticity properties** $\rightsquigarrow$ uniqueness under measurements at one point (in $1-D$) (Roques-Cristofol (2010), Roques-Checkroun-Cristofol-Soubeyrand-Ghil (2014))

Inverse source problem for evolution equations

Literature on the subject

Founding papers using GCE :

Puel/Yamamoto 1996 + 1997 (linear wave equation)

Imanuvilov/Yamamoto 1998 (linear heat equation)

More on Lipschitz stability for parabolic equations :

Yamamoto/Zou 2001 (simultaneous reconstruction of 2 quantities)
Cristofol/Gaitan/Ramoul 2006 (systems)
Benabdallah/Dermenjian/Le Rousseau 2007 + Benabdallah/Gaitan/Le Rousseau 2009 (discontinuous diffusion coefficient)
C/Tort/Yamamoto 2010 (degenerate diffusion coefficient)
Ignat/Pazoto/Rosier 2012 (networks)

Lipschitz stability for other types of equations :

Hyperbolic equations : Imanuvilov/Yamamoto 2001, Komornik/Yamamoto 2002, Bellassoued/Yamamoto 2006, Liu/Triggiani 2011

Schrodinger equation : Baudouin/Puel 2002, Mercado/Osses/Rosier 2008, Liu/Triggiani 20111 ...

Uniqueness under pointwise measurements

Assumptions

(H1) admissible initial conditions

$$\mathcal{U}^{(pt)} = C^{1,2}([-\tau, 0] \times [-1, 1])$$

(H2) admissible coefficients

$$\mathcal{Q}^{(pt)} := \{q \text{ is Lipschitz-continuous and piecewise analytic on } I\}$$

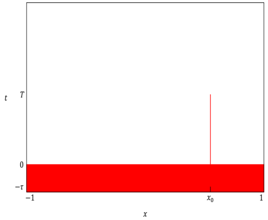
(H3) admissible memory kernels

$$\exists \delta \in (0, \tau) \quad \text{such that} \quad k(s, \cdot) \equiv 0 \quad \forall s \in [-\delta, 0]$$

Uniqueness under pointwise measurements

Main result

Theorem (C-Malfitana-Martinez)



Consider

- *two insolation functions* $q, \tilde{q} \in \mathcal{Q}^{(pt)}$
- *an initial condition* $u_0 = \tilde{u}_0 \in \mathcal{U}^{(pt)}$

Assume that

- *the memory kernel is admissible*
- r *and* β *are positive*

If u *and* $\tilde{u}$ *are the solutions of* (S) *and* $(\tilde{S})$*, respectively, and there exists* $x_0 \in I$ *and* $T > 0$ *such that*

$$\forall t \in (0, T), \quad u(t, x_0) = \tilde{u}(t, x_0), \text{ and } u_x(t, x_0) = \tilde{u}_x(t, x_0),$$

then $q \equiv \tilde{q}$ *on* $(-1, 1)$

This extends a result by Roques-Checkroun-Cristofol-Soubeyrand-Ghil (2014)

1D Memory Sellers model : Lipschitz stability

Set-up

$$\begin{cases} u_t - (\rho(x)u_x)_x = r(t)q(x)\beta(u) - \varepsilon(u)|u|^3u + f(H) \\ \rho(x)u_x = 0, \quad x \in \partial I, \\ u(s,x) = u_0(s,x), \quad s \in [-\tau, 0], x \in I \end{cases} \qquad (S)$$

$$\begin{cases} \tilde{u}_t - (\rho(x)\tilde{u}_x)_x = r(t)\tilde{q}(x)\beta(\tilde{u}) - \varepsilon(\tilde{u})|\tilde{u}|^3\tilde{u} + f(\tilde{H}) \\ \rho(x)\tilde{u}_x = 0, \quad x \in \partial I, \\ \tilde{u}(s,x) = \tilde{u}_0(s,x), \quad s \in [-\tau, 0], x \in I \end{cases} \qquad (\tilde{S})$$

Goal : to prove that

$$\|q - \tilde{q}\| \quad \leq \quad C \ \|u - \tilde{u}\|$$

1D Memory Sellers model : Lipschitz stability

Assumptions

- admissible initial conditions : given $M > 0$,

$$\mathcal{U}_M^{(loc)} = \Big\{ u_0 \in C([-\tau, 0]; V \cap L^\infty(-1,1)) \ : \ u_0(0) \in D(A), \ Au_0(0) \in L^\infty(I), \\ \sup_{t \in [-\tau, 0]} \big(\|u_0(t)\|_V + \|u_0(t)\|_{L^\infty} \big) + \|Au_0(0)\|_{L^\infty(I)} \le M \Big\}$$

- admissible coefficients : given $M' > 0$,

$$\mathcal{Q}_{M'}^{(loc)} := \{ q \in L^\infty(I) : \|q\|_{L^\infty(I)} \le M' \}$$

- admissible memory kernels : the same support condition

$$\exists \delta \in (0, \tau) \quad \text{such that} \quad k(s, \cdot) \equiv 0 \quad \forall s \in [-\delta, 0]$$

1D Memory Sellers model : Lipschitz stability

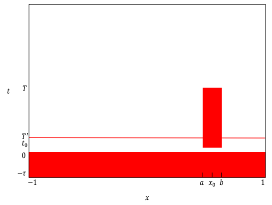
Main result

Theorem (C–Malfitana–Martinez (2018))

Let $0 \leq t_0 < T' < T$, *let* $M, M' > 0$, *and suppose*

$$0 < T' < \delta$$

Then there exists $C(t_0, T', T, M, M') > 0$ *such that, for all* $u_0, \tilde{u}_0 \in \mathcal{U}_M^{(loc)}$ *and all* $q, \tilde{q} \in \mathcal{Q}_{M'}^{(loc)}$, *the solutions* u *and* $\tilde{u}$ *of* (S) *and* $(\tilde{S})$, *respectively, satisfy*



$$\|q - \tilde{q}\|^2_{L^2(I)} \leq C\Big(\|u(T') - \tilde{u}(T')\|^2_{D(A)} + \|u_t - \tilde{u}_t\|^2_{L^2((t_0,T)\times(a,b))} + \|u_0 - \tilde{u}_0\|^2_{C([-\tau,0];V)}\Big)$$

Extension of Tort–Vancostenoble (2012)

1D Memory Sellers model : Lipschitz stability

Main tools for proof

- to reduce the problem to some (non standard) inverse source problem for a linear equation
- to follow the method by Imanuvilov-Yamamoto (1998) for inverse source problems
- to use adapted Global Carleman Estimates for degenerate parabolic equations C – Martinez-Vancostenoble (2008)
- appeal to maximum principles to deal with nonlinear terms

1D Memory Sellers model : Lipschitz stability

Steps 1 and 2 of the proof

Step 1 to identify the problem satisfied by the difference $w = u - \tilde{u}$ as

$$\begin{cases} w_t - (\rho w_x)_x = K^* + K + \tilde{K} + K^h, \quad t > 0, x \in (-1,1), \\ \rho w_x = 0, \quad x = \pm 1, \\ w(s,x) = u_0(s,x) - \tilde{u}_0(s,x), \quad s \in [-\tau, 0], x \in (-1,1), \end{cases}$$

where

$$K^*(t,x) = r(t)(q(x) - \tilde{q}(x))\beta(u), \quad K(t,x) = r(t)\tilde{q}(x)(\beta(u) - \beta(\tilde{u})),$$

$$\tilde{K} = -\varepsilon(u)|u|^3 u + \varepsilon(\tilde{u})|\tilde{u}|^3 \tilde{u}, \quad K^h = f(H) - f(\tilde{H})$$

Step 2 K^* satisfies (Imanuvilov-Yamamoto (1998))

$$\exists C_0 > 0 \text{ s.t.} \forall t, x, \quad \left| \frac{\partial K^*}{\partial t}(t,x) \right| \leq C_0 |K^*(T', x)|$$

(this follows from the fact that $u_t \in L^\infty((0,T) \times I)$)

1D Memory Sellers model : Lipschitz stability

Carleman estimate and conclusion

Step 3 Carleman estimate for $z := w_t$: there exists a smooth weight function $\theta : (t_0, T) \to (0,\infty)$, with $\theta(t) \to \infty$ as $t \downarrow t_0$ and $T \uparrow T$, such that

$$\int_{t_0}^{T}\int_{-1}^{1}\Big(R^3\theta^3(1-x^2)z^2 + R\theta(1-x^2)z_x^2 + \frac{1}{R\theta}z_t^2\Big)e^{-2R\sigma}$$
$$\le C\int_{t_0}^{T}\int_{-1}^{1}(K_t^*)^2e^{-2R\sigma} + C\int_{t_0}^{T}\int_{a}^{b}R^3\theta^3z^2e^{-2R\sigma}\Big)$$
$$+ C\|w(T')\|^2_{L^2(I)} + C\|u_0-\tilde u_0\|^2_{C([-\tau,0],V)}$$

Step 4 to estimate $\|q-\tilde q\|$ we use an upper bound for

$$K^*(T') = r(T')(q(x)-\tilde q(x))\beta(u)$$
$$= z(T') - (\rho w_x)_x(T') - K(T') - \tilde K(T') - K^h(T')$$

that is

$$\int_{-1}^{1}|K^*(T')|^2e^{-2R\sigma(T')}dx \le C\|w(T')\|^2_{D(A)} +$$
$$+C\int_{-1}^{1}\Big(\underbrace{|z(T')|^2}_{(CE)} + \underbrace{|K(T')|^2 + |\tilde K(T')|^2}_{C\|w(T')\|^2_{L^2(I)}} + \underbrace{|K^h(T')|^2}_{C\|u_0-\tilde u_0\|^2_{C([-\tau,0],V)}}\Big)e^{-2R\sigma(T')}dx$$

IV. Ongoing research : EBMs with vertical component

EBMs with vertical component

joint work with V. Lucarini, P. Martinez, C. Urbani, J. Vancostenoble

T_a : temperature of an atmosphere layer

T_s : surface temperature of the Earth

$$\begin{cases} \gamma_a \left[\frac{\partial T_a}{\partial t} - k_a \frac{\partial}{\partial x} \left((1-x^2) \frac{\partial T_a}{\partial x} \right) \right] = -\lambda (T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_a \\ \gamma_s \left[\frac{\partial T_s}{\partial t} - k_s \frac{\partial}{\partial x} \left((1-x^2) \frac{\partial T_s}{\partial x} \right) \right] = -\lambda (T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_s \\ (1-x^2) \frac{\partial T_a}{\partial x}_{|x=\pm 1} = 0 = (1-x^2) \frac{\partial T_s}{\partial x}_{|x=\pm 1} \quad T_a(0,x) = T_a^{(0)}(x), \ T_s(0,x) = T_s^{(0)}(x) \end{cases}$$

Exchange of energy between the layers :

- linear term : non-radiative vertical exchanges of energy due to the action of the geophysical fluids
- nonlinear term : emission of infrared radiation by one level being captured by the other layer

Relevant constants :

- $\lambda \geq 0$: coupling parameter for vertical exchanges
- $\varepsilon_a \in [0,1]$: absorptivity (depends on greenhouse gases CO_2, CH_4)

Well-posedness of the ODE problem

$$(2LEBM) \quad \begin{cases} \gamma_a \frac{\partial T_a}{\partial t} = -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_a, \\ \gamma_s \frac{\partial T_s}{\partial t} = -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_s, \\ T_a(0) = T_a^{(0)}, \quad T_s(0) = T_s^{(0)} \end{cases}$$

Assumptions :

- $\lambda \geq 0, \quad q > 0, \quad \sigma_B > 0, \quad \varepsilon_a \in (0,2)$
- $\beta_a, \beta_s : \mathbb{R} \to \mathbb{R}$ globally Lipschitz, $\beta_a \geq 0$ and $\beta_s > 0$, and

$$\mathcal{R}_a = q\beta_a(T_a), \quad \mathcal{R}_s = q\beta_s(T_s).$$

- $T_a^{(0)} \geq 0, \quad T_s^{(0)} \geq 0$

Proposition

$(2LEMB)$ has unique solution, defined and bounded for any $t \in [0, +\infty)$. Moreover

$$\forall\, t \in (0, +\infty), \quad T_a(t) > 0 \quad \textit{and} \quad T_s(t) > 0.$$

Asymptotic behaviour of solutions

Definition

A C^1 system of ODE on $\mathbb{R}^n$ $\quad \frac{d}{dt}x_i = F_i(x_1, \dots, x_n) = F_i(x), \ \ i = 1, \dots, n \quad$ is *competitive* if

$$\frac{\partial}{\partial x_j} F_i(x) \leq 0 \qquad i \neq j$$

and **cooperative** if the reverse inequalities hold.

For $\varepsilon_a \in (0, 2)$ and $\lambda \geq 0$

$$T_s \mapsto \frac{1}{\gamma_a}\left[\lambda T_s + \varepsilon_a \sigma_B |T_s|^3 T_s - \lambda T_a - 2\varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_a(T_a)\right]$$

$$T_a \mapsto \frac{1}{\gamma_s}\left[\lambda T_a - \sigma_B |T_s|^3 T_s - \lambda T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + \mathcal{R}_s(T_s)\right]$$

are nondecreasing on $[0, +\infty)$

$\Longrightarrow$ our system is cooperative

$\Longrightarrow$ [Smith] any initial condition $(T_a^{(0)}, T_s^{(0)})$ converges to an equilibrium.

Case $\lambda = 0$ and $\mathcal{R}_a = 0$

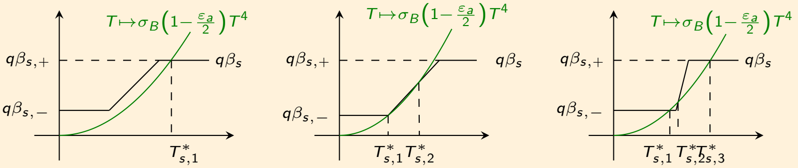
Equilibrium points are solutions of

$$\begin{cases} \varepsilon_a \sigma_B T_s^4 - 2\varepsilon_a \sigma_B T_a^4 = 0, \\ -\sigma_B T_s^4 + \varepsilon_a \sigma_B T_a^4 + \mathcal{R}_s(T_s) = 0 \end{cases}$$

that is equivalent to solve

$$\sigma_B(1 - \frac{\varepsilon_a}{2}) T_s^4 = q\beta_s(T_s)$$

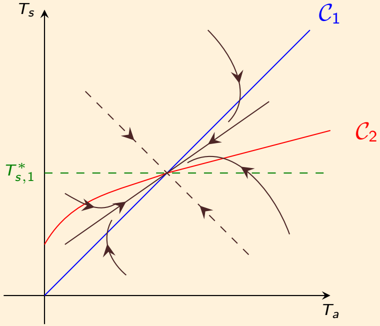
$\Longrightarrow$ one, two or three possible equilibria



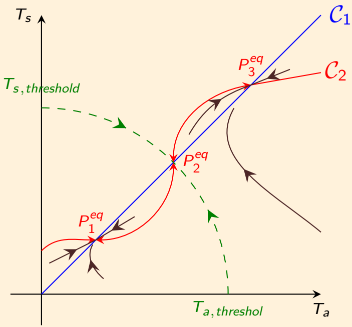
depending on parameters σ_B, ε_a, q.

Nature of the equilibrium points ($\lambda = 0$, $\mathcal{R}_a = 0$)

One equilibrium : $(T^*_{a,1}, T^*_{s,1})$ **asymptotically stable**



Three equilibria : $(T^*_{a,1}, T^*_{s,1})$, $(T^*_{a,3}, T^*_{s,3})$ **asymptotically exponentially stable**, $(T^*_{a,2}, T^*_{s,2})$ **unstable**



Case $\lambda > 0$ and $\mathcal{R}_a = 0$

Equilibrium points are solutions of

$$\begin{cases} -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a = 0, \\ -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + q\beta(T_s) = 0. \end{cases}$$

Lemma

- *There exists at least one warm and one cold equilibrium.*
- *There exists at most five equilibria.*
- *Fixed $\lambda > 0$ and $\varepsilon_a \in (0, \varepsilon_{a,0})$ (with $\varepsilon_{a,0} > 1.99$), there exist at most three equilibrium points.*
- *Any warm (resp. a cold) equilibrium is asymptotically exponentially stable.*

Case $\lambda > 0$ and $\mathcal{R}_a = 0$

Equilibrium points are solutions of

$$\begin{cases} -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a = 0, \\ -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + q\beta(T_s) = 0. \end{cases}$$

Lemma

- *There exists at least one warm and one cold equilibrium.*
- *There exists at most five equilibria.*
- *Fixed $\lambda > 0$ and $\varepsilon_a \in (0, \varepsilon_{a,0})$ (with $\varepsilon_{a,0} > 1.99$), there exist at most three equilibrium points.*
- *Any warm (resp. a cold) equilibrium is asymptotically exponentially stable.*

Case $\lambda > 0$ and $\mathcal{R}_a = 0$

Equilibrium points are solutions of

$$\begin{cases} -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a = 0, \\ -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + q\beta(T_s) = 0. \end{cases}$$

Lemma

- *There exists at least one warm and one cold equilibrium.*
- *There exists at most five equilibria.*
- *Fixed $\lambda > 0$ and $\varepsilon_a \in (0, \varepsilon_{a,0})$ (with $\varepsilon_{a,0} > 1.99$), there exist at most three equilibrium points.*
- *Any warm (resp. a cold) equilibrium is asymptotically exponentially stable.*

Case $\lambda > 0$ and $\mathcal{R}_a = 0$

Equilibrium points are solutions of

$$\begin{cases} -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a = 0, \\ -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + q\beta(T_s) = 0. \end{cases}$$

Lemma

- *There exists at least one warm and one cold equilibrium.*
- *There exists at most five equilibria.*
- *Fixed $\lambda > 0$ and $\varepsilon_a \in (0, \varepsilon_{a,0})$ (with $\varepsilon_{a,0} > 1.99$), there exist at most three equilibrium points.*
- *Any warm (resp. a cold) equilibrium is asymptotically exponentially stable.*

Case $\lambda > 0$ and $\mathcal{R}_a = 0$

Equilibrium points are solutions of

$$\begin{cases} -\lambda(T_a - T_s) + \varepsilon_a \sigma_B |T_s|^3 T_s - 2\varepsilon_a \sigma_B |T_a|^3 T_a = 0, \\ -\lambda(T_s - T_a) - \sigma_B |T_s|^3 T_s + \varepsilon_a \sigma_B |T_a|^3 T_a + q\beta(T_s) = 0. \end{cases}$$

Lemma

- *There exists at least one warm and one cold equilibrium.*
- *There exists at most five equilibria.*
- *Fixed $\lambda > 0$ and $\varepsilon_a \in (0, \varepsilon_{a,0})$ (with $\varepsilon_{a,0} > 1.99$), there exist at most three equilibrium points.*
- *Any warm (resp. a cold) equilibrium is asymptotically exponentially stable.*

Dependence of equilibria on ε_a

Proposition

Fixed $\lambda \geq 0$, let $(T_a^{eq,\varepsilon_a^*}, T_s^{eq,\varepsilon_a^*})$ be an asymptotically exponentially stable warm [resp. cold] equilibrium point with $\varepsilon_a = \varepsilon_a^*$.

Then, there exists a unique asymptotically exponentially stable warm [resp. cold] equilibrium $(T_a^{eq,\varepsilon_a}, T_s^{eq,\varepsilon_a})$ for ε_a close to ε_a^* and the following monotonicity property holds :

- $\varepsilon_a \mapsto T_s^{eq,\varepsilon_a}$ *is increasing : the surface temperature of the equilibrium increases as ε_a increases ;*
- $\varepsilon_a \mapsto T_a^{eq,\varepsilon_a}$ *is increasing if $\varepsilon_a^* \in (1,2)$: the atmosphere temperature of the equilibrium increases as ε_a increases.*

Dependence of equilibria on ε_a

Proposition

Fixed $\lambda \geq 0$, let $(T_a^{eq,\varepsilon_a^*}, T_s^{eq,\varepsilon_a^*})$ be an asymptotically exponentially stable warm [resp. cold] equilibrium point with $\varepsilon_a = \varepsilon_a^*$.

Then, there exists a unique asymptotically exponentially stable warm [resp. cold] equilibrium $(T_a^{eq,\varepsilon_a}, T_s^{eq,\varepsilon_a})$ for ε_a close to ε_a^* and the following monotonicity property holds :

- $\varepsilon_a \mapsto T_s^{eq,\varepsilon_a}$ *is increasing : the surface temperature of the equilibrium increases as ε_a increases ;*
- $\varepsilon_a \mapsto T_a^{eq,\varepsilon_a}$ *is increasing if $\varepsilon_a^* \in (1,2)$: the atmosphere temperature of the equilibrium increases as ε_a increases.*

Dependence of equilibria on ε_a and λ

Proposition

Fixed $\varepsilon_a \in (0,2)$, let $(T_a^{eq,\lambda^*}, T_s^{eq,\lambda^*})$ be an asymptotically exponentially stable warm [resp. cold] equilibrium with $\lambda = \lambda^* \geq 0$.

Then, there exists a unique asymptotically exponentially stable equilibrium point $(T_a^{eq,\lambda}, T_s^{eq,\lambda})$ for λ close to λ^* and the following monotonicity properties are satisfied :

- $\lambda \mapsto T_s^{eq,\lambda}$ *is decreasing : the surface temperature of the equilibrium decreases as λ increases ;*
- $\lambda \mapsto T_a^{eq,\lambda}$ *is*
  - *increasing if $\varepsilon_a \in (0,1)$,*
  - *decreasing if $\varepsilon_a \in (1,2)$.*

Hence, the atmosphere temperature of an equilibrium is monotone with respect to λ, and monotonicity depends on ε_a.

Dependence of equilibria on ε_a and λ

Proposition

Fixed $\varepsilon_a \in (0,2)$, let $(T_a^{eq,\lambda^*}, T_s^{eq,\lambda^*})$ be an asymptotically exponentially stable warm [resp. cold] equilibrium with $\lambda = \lambda^* \geq 0$.

Then, there exists a unique asymptotically exponentially stable equilibrium point $(T_a^{eq,\lambda}, T_s^{eq,\lambda})$ for λ close to λ^* and the following monotonicity properties are satisfied :

- $\lambda \mapsto T_s^{eq,\lambda}$ *is decreasing : the surface temperature of the equilibrium decreases as λ increases ;*
- $\lambda \mapsto T_a^{eq,\lambda}$ *is*
  - *increasing if $\varepsilon_a \in (0,1)$,*
  - *decreasing if $\varepsilon_a \in (1,2)$.*

  Hence, the atmosphere temperature of an equilibrium is monotone with respect to λ, and monotonicity depends on ε_a.

Future work directions

- Inverse problems for parameter reconstruction (ε_a, λ, q)
- Analysis of the PDE system
- Extension to a variable solar radiation $Q(t, x) = r(t)q(x)$, with r positive and periodic - allowing for seasonal cycle - and q the latitudinal-dependent insolation function
- Extension to a space-dependent absorptivity $\varepsilon_a(x)$

Happy Birthday Piero !