Uniqueness and error estimates for Hyperbolic Conservation Laws

Alberto Bressan

Department of Mathematics, Penn State University
http://www.math.psu.edu/bressan/

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celebrating the 70-th birthday of Piero Marcati

Why should we care about uniqueness ?

$$u_t + f(u)_x = 0, \qquad u(0,x) = \bar{u}(x)$$

- Glimm approximations
- front tracking approximations
- vanishing viscosity approximations

all converge to the same trajectory of a Lipschitz semigroup $u(t) = S_t \bar{u}$.

- It is entirely clear which one is the right solution.
- Proving a uniqueness result is only a matter of coming up with a good definition, which characterizes the semigroup trajectories.

Why is uniqueness important ?

$$u_t + f(u)_x = 0, \qquad u(0,x) = \bar{u}(x)$$

Approximate solutions can also be constructed by

(i) relaxation approximations
(ii) semidiscrete schemes
(iii) periodic mollifications
(iv) diffusion approximations (possibly degenerate parabolic)
(v) Backward Euler approximations
(vi) fully discrete numerical schemes (Lax-Friedrichs, Godunov, ...)

Question. Let $(u_n)_{n\geq 1}$ be a sequence of approximate solutions taking values in the domain of the semigroup: $u_n(t,\cdot) \in \mathcal{D}$.
Does it converge to the semigroup trajectory: $u_n(t) \to S_t\bar{u}$?
What is the convergence rate ?

An error estimate

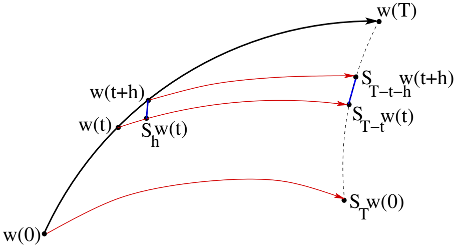
Theorem. *Let* $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$ *be a Lipschitz semigroup satisfying*

$$\|S_t u - S_s v\| \leq L \cdot \|u - v\| + L' \cdot |t - s|$$

Then, for every Lipschitz continuous map $w : [0, T] \mapsto \mathcal{D}$ *one has*

$$\left\| w(T) - S_T w(0) \right\| \leq L \cdot \int_0^T \left\{ \liminf_{h \to 0+} \frac{\left\| w(t+h) - S_h w(t) \right\|}{h} \right\} dt$$

$$= L \cdot \int_0^T [\text{instantaneous error rate at time } t]\, dt$$



Characterization of semigroup trajectories

Question: if $u = u(t,x)$ is a weak solution of

$$u_t + f(u)_x \;=\; 0 \qquad\qquad u(0,x) = \bar{u}(x)$$

what additional properties guarantee that $u(t,\cdot) = S_t\bar{u}$?

NOTE: throughout the following, w.l.o.g. we assume that all wave speeds satisfy

$$|\lambda| < 1$$

This can always be achieved by a time rescaling: $t' = ct$.

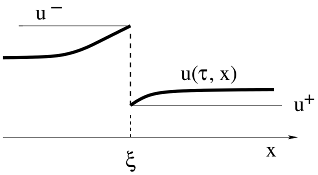
1. Comparison with solutions to a Riemann problem

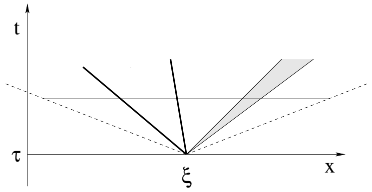
Fix (τ, ξ). Define $U^\sharp = U^\sharp_{(\tau,\xi)}$ as the solution of the Riemann problem

$$w_t + f(w)_x = 0, \qquad\qquad w(\tau, x) = \begin{cases} u^+ \doteq u(\tau, \xi+) & \text{if} \quad x > \xi \\ u^- \doteq u(\tau, \xi-) & \text{if} \quad x < \xi \end{cases}$$

Then we expect

$$\lim_{h \to 0+} \frac{1}{h} \int_{\xi-h}^{\xi+h} \left| u(\tau+h,\ x) - U^\sharp_{(\tau,\xi)}(\tau+h,\ x) \right| dx \ = \ 0 \qquad (E1)$$

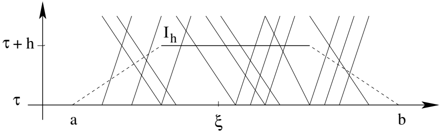




Fix (τ, ξ). Define $U^\flat = U^\flat_{(\tau,\xi)}$ as the solution of the linear Cauchy problem

$$w_t + \widetilde{A} w_x = 0 \qquad\qquad w(\tau, x) = u(\tau, x)$$

with "frozen" coefficients: $\widetilde{A} \doteq A\big(u(\tau,\xi)\big)$



Then, choosing $\xi \in]a, b[$, for any $h > 0$ we expect

$$\frac{1}{h}\int_{a+h}^{b-h} \Big| u(\tau+h,\ x) - U^\flat(\tau+h,\ x) \Big|\, dx \;=\; \mathcal{O}(1)\cdot \Big(\text{Tot.Var.}\,\{u(\tau,\cdot);\ \ ]a,\ b[\,\}\Big)^2 \quad (E2)$$

$$\approx\ \frac{1}{h}\int_\tau^{\tau+h} [\text{total amount of waves}] \times [\text{error in the speed}]\, dt$$

Theorem (A.B., *Arch. Rational Mech. Anal.*, 1995)

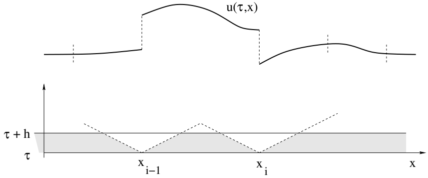
Let $u : [0, T] \mapsto \mathcal{D}$ be Lipschitz continuous w.r.t. the $\mathbf{L}^1$ distance.
Then u is a weak solution to the system of conservation laws

$$u_t + f(u)_x = 0$$

obtained as limit of front tracking approximations if and only if the estimates (E1)-(E2) are satisfied for a.e. $\tau \in [0, T]$.

$$\big\| u(T) - S_T u(0) \big\| \leq L \cdot \int_0^T \left\{ \liminf_{h \to 0+} \frac{\big\| u(\tau + h) - S_h u(\tau) \big\|}{h} \right\} d\tau$$

$$(\text{E1}) + (\text{E2}) \quad \Longrightarrow \quad \lim_{h \to 0+} \frac{\big\| u(\tau + h) - S_h u(\tau) \big\|_{\mathbf{L}^1}}{h} = 0$$



Hence $u(t) = S_t u(0)$

Uniqueness of weak solutions

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \bar{u}(x)$$

- introduce a suitable set of admissibility + regularity assumptions
- show that these assumptions imply

$$\liminf_{h\to 0+} \frac{\|u(\tau+h) - S_h u(\tau)\|_{\mathbf{L}^1}}{h} = 0 \qquad\qquad \text{for a.e. } \tau$$

$$\Longrightarrow \quad u(t) = S_t \bar{u} \quad \text{for all} \quad t \geq 0$$

Points of approximate jump

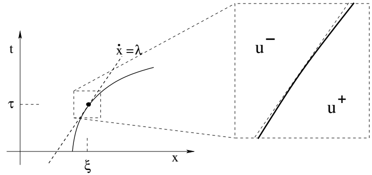
We say that $u = u(t,x)$ has an **approximate jump** at the point $(\tau,\xi) \in \mathbb{R}^2$ if there exists vectors $u^+ \neq u^-$ and a speed λ such that, setting

$$U(t,x) \doteq \begin{cases} u^- & \text{if} \quad x < \xi + \lambda(t-\tau) \\ u^+ & \text{if} \quad x > \xi + \lambda(t-\tau) \end{cases}$$

one has:
$$\lim_{r\to 0+} \frac{1}{r^2} \int_{-r}^{r} \int_{-r}^{r} \Big| u(\tau+t,\, \xi+x) - U(t,x) \Big| \, dxdt = 0 \qquad (1)$$

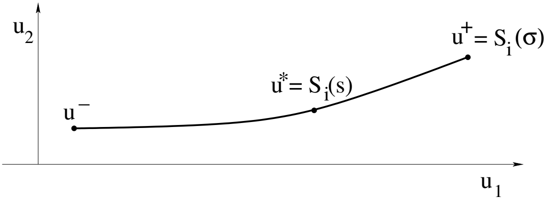
Moreover, we say that u is **approximately continuous** at the point (τ,ξ) if (1) holds with $u^+ = u^-$ (and λ arbitrary).

NOTE: the above definitions depend only on the $\mathbf{L}^1$ equivalence class of u



The Liu admissibility condition for hyperbolic systems

(T. P. Liu, *J. Math. Anal. Appl.* 1976)



Given a left state u^-, let $s \mapsto S_i(s)$ be the curve of right states that can be connected to u^- by a shock of the i-th family.

Call $\lambda_i(s)$ the Rankine-Hugoniot speed of these shocks

A shock of the i-th family, connecting the states u^- and $u^+ = S_i(\sigma)$ is **Liu-admissible** if

$$\lambda_i(s) \ \geq \ \lambda_i(\sigma) \qquad \text{for all} \quad s \in [0, \sigma]$$

The **Liu admissibility condition** selects shocks which can be obtained as limits of vanishing viscosity approximations

S. Bianchini, On the Riemann problem for non-conservative hyperbolic systems, *Arch. Rational Mech. Anal.* **166** (2003), 1-26.

A set of assumptions

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \bar{u}(x)$$

(A1) (Conservation Equations)

$u : [0, T] \mapsto \mathcal{D}$ is continuous w.r.t. the $\mathbf{L}^1$ distance.

The initial condition $u(0,x) = \bar{u}(x)$ holds.

Moreover, u is a weak solution:

$$\iint \left\{ u\varphi_t + f(u)\varphi_x \right\} dxdt = 0 \qquad\qquad \text{for all } \varphi \in \mathcal{C}^1_c\big(]0, T[\times\mathbb{R}\big)$$

$\Longrightarrow$ the map $t \mapsto u(t,\cdot)$ is **Lipschitz continuous** from $[0, T]$ into $\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)$.

(A2) (Admissibility Conditions) u satisfies the Liu admissibility conditions at each point (τ, ξ) of approximate jump.

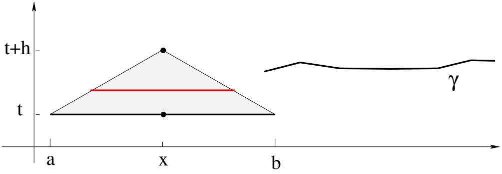
(A3) (Tame Variation) For some constant C the following holds. For every open interval $]a, b[$ and every $t, h > 0$ one has

$$\text{Tot.Var.}\Big\{u(t+h,\cdot)\,;\ \ ]a+h\,,\ b-h[\,\Big\} \ \le\ C\cdot\text{Tot.Var.}\Big\{u(t,\cdot)\,;\ \ ]a,b[\,\Big\}$$

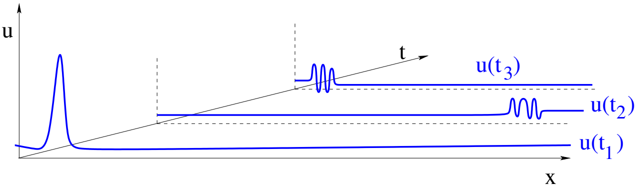
(A4) (Tame Oscillation) For some constant C the following holds. For every point $x \in \mathbb{R}$ and every $t, h > 0$ one has

$$\Big|u(t+h,x)-u(t,x)\Big| \ \le\ C\cdot\text{Tot.Var.}\Big\{u(t,\cdot)\,;\ \ [x-h,\ x+h]\Big\}$$

(A5) (Bounded Variation along space-like curves) There exists $\delta > 0$ such that, for every space-like curve $\big\{t = \tau(x)\big\}$ with $|d\tau/dx| \le \delta$ a.e., the function $x \mapsto u\big(\tau(x), x\big)$ has locally bounded variation.



need to rule out:



Uniqueness results

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \bar{u}(x)$$

Theorem. *Let the system be strictly hyperbolic, with each characteristic field either linearly degenerate or genuinely nonlinear.*

- *Every weak solution* $u = u(t,x)$ *obtained as limit of front tracking approximations satisfies all conditions (A1)–(A5).*
- *If* $u : [0,T] \mapsto \mathcal{D}$ *satisfies (A1),(A2) and any one of the three regularity conditions (A3), (A4), (A5) then* $u(t) = S_t\bar{u}$ *for* $t \geq 0$.

- **Tame Variation:** (A.B. & P. LeFloch, *Arch. Rational Mech. Anal.* 1997)
- **Tame Oscillation:** (A.B. & P. Goatin, *J. Differential Equations*, 1999)
- **Bounded Variation along space-like curves:** (A.B. & M. Lewicka, *Discr. Cont. Dyn. Syst.* 2000)

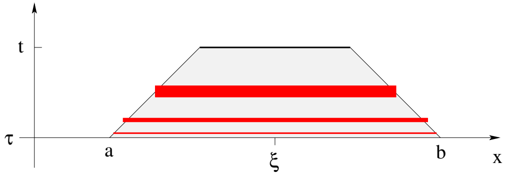
Do we need any regularity assumption ???

For a class of 2×2 systems, without any of the regularity assumptions (A3)–(A5), uniqueness is proved in

G.Chen, S.Krupa, and A.Vasseur, Uniqueness and weak-BV stability for 2x2 conservation laws, *Arch. Rational Mech. Anal.* **246** (2022), 299–332.

(Indeed, the assumption (A5) on bounded variation along space-like curves is always satisfied)

A direct approach, for general $n \times n$ systems



Assume:

- Tot.Var.$\Big\{u(\tau,\cdot)\,;\ ]a,b[\Big\} \ \leq \varepsilon$
- the time τ is a **Lebesgue point** of the function

$$V(t) \ = \ \text{Tot.Var.}\Big\{u(t,\cdot)\,;\ \big]a+(t-\tau)\,,\ b-(t-\tau)\big[\Big\}$$

Then we again have

$$\limsup_{h\to 0+} \frac{1}{h}\int_{a+h}^{b-h} \Big|u(\tau+h,x) - U^\flat(\tau+h,x)\Big|dx \ = \ \mathcal{O}(1)\cdot\varepsilon^2$$

Systems with a strictly convex entropy

$$u_t + f(u)_x \;=\; 0$$

Assume:

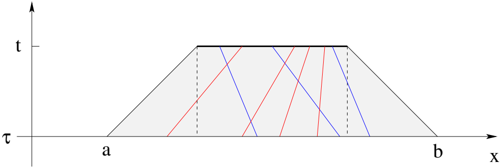
- strictly hyperbolic $n \times n$ system
- each characteristic field is either linearly degenerate or genuinely nonlinear
- there exists a strictly convex $\mathcal{C}^2$ entropy η with entropy flux q

$$\eta(v) \;\geq\; \eta(u) + \nabla\eta(u)\cdot(v-u) + c_0|v-u|^2$$

for all $u, v \in \Omega \subset \mathbb{R}^n$

Lemma. For some constant $C > 0$ the following holds.
Let $u = u(t,x)$ be any entropy weak solution. Then

$$\int_{a+(t-\tau)}^{b-(t-\tau)} \big|u(t,x) - u(\tau,x)\big|\,dx \;\leq\; C(t-\tau)\cdot \text{Tot.Var.}\big\{u(\tau,\cdot)\,;\;\,]a,b[\big\}.$$



NOTE: if the **Tame Variation** condition holds, then

$$\int_{a+(t-\tau)}^{b-(t-\tau)} \big|u(t,x) - u(\tau,x)\big|\,dx$$

$$\leq\; C' \int_\tau^t \text{Tot.Var.}\Big\{u(s,\cdot)\,;\;\big]a+(t-\tau)\,,\; b-(t-\tau)\big[\Big\}\,ds$$

$$\leq\; C(t-\tau)\cdot \text{Tot.Var.}\big\{u(\tau,\cdot)\,;\;\,]a,b[\big\}$$

Proving uniqueness without regularity assumptions

$$u_t + f(u)_x = 0 \tag{1}$$

Theorem (A.B., G.Guerra, 2023)

Let (1) be a strictly hyperbolic $n \times n$ system, where each characteristic field is either genuinely nonlinear or linearly degenerate, and which admits a strictly convex entropy $\eta(\cdot)$.

Then every entropy-weak solution $u : [0, T] \mapsto \mathcal{D}$, taking values within the domain of the semigroup, coincides with a semigroup trajectory.

Proof. 1. By the structure theorem for BV functions, there is a null set of times $\mathcal{N} \subset [0, T]$ such that every $(\tau, \xi) \in [0, T] \times \mathbb{R}$, with $\tau \notin \mathcal{N}$, is either a point of approximate continuity, or a point of approximate jump of the function u.

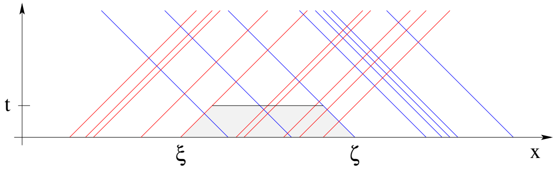
2. For every couple of rational points $\xi, \zeta \in \mathbb{Q}$, the scalar function

$$W^{\xi,\zeta}(t) \doteq \begin{cases} \text{Tot.Var.}\{u(t)\,;\ ]\xi + t,\ \zeta - t[\,\} & \text{if} \quad \xi + t < \zeta - t\,, \\ 0 & \text{otherwise,} \end{cases}$$

is bounded and measurable (indeed, it is lower semicontinuous)

$\Longrightarrow$ a.e. $t \in [0, T]$ is a Lebesgue point.

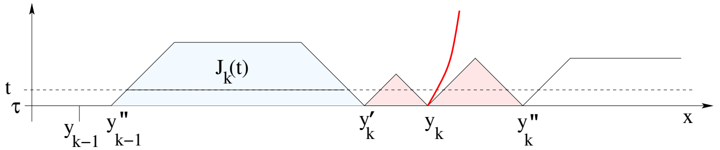
Denote by $\mathcal{N}' \subset [0, T]$ the null set of all times t which are NOT Lebesgue for at least one of the countably many functions $W^{\xi,\zeta}$.



The theorem is proved by showing that

For every $\tau \in [0, T] \setminus (\mathcal{N} \cup \mathcal{N}')$ *and* $\varepsilon > 0$, *one has*

$$\limsup_{h \to 0+} \frac{1}{h} \Big\| u(\tau + h) - S_h u(\tau) \Big\|_{\mathbf{L}^1} \leq \varepsilon$$



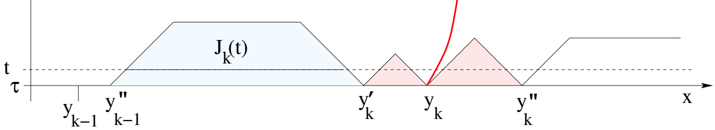
- Choose points $y_0 < y_1 < \cdots < y_N$, with $N \leq C\varepsilon^{-1}$, so that

$$\text{Tot.Var.}\big\{ u(\tau, \cdot)\,;\ ]y_{k-1}, y_k[\,\big\} \leq \varepsilon \qquad \text{for all } k$$

- Choose additional points $y'_k < y_k < y''_k$ such that

$$y'_k + \tau \in \mathbb{Q}, \qquad y''_k - \tau \in \mathbb{Q}$$

$$\text{Tot.Var.}\big\{ u(\tau, \cdot)\,;\ ]y'_k, y_k[\big\} < \varepsilon^2, \qquad \text{Tot.Var.}\big\{ u(\tau, \cdot)\,;\ ]y_k, y''_k[\big\} < \varepsilon^2$$



To estimate $\left\| u(t) - S_{t-\tau} u(\tau) \right\|_{\mathbf{L}^1}$, three types of integrals need to be considered:

(I) For each $y \in \{y_0, y_0'', y_1', y_1, y_1'', \ldots, y_N', y_N\}$, the integral of $\left| u(t,\cdot) - U^\sharp(t,\cdot) \right|$ over the interval

$$J_y(t) \doteq \big[y - (t-\tau),\ y + (t-\tau) \big] \qquad \text{is } o(t-\tau)$$

(II) The integral of $\left| u(t,x) - U^\flat(t,x) \right|$ over the interval

$$J_k(t) = \big[y''_{k-1} + (t-\tau),\ y'_k - (t-\tau) \big] \qquad \text{is } \mathcal{O}(1) \cdot \varepsilon^2 (t-\tau) + o(t-\tau)$$

(III) The integral of $\left| u(t,x) - u(\tau,x) \right|$ over the intervals

$$\begin{cases} J'_k(t) = \big[y'_k + (t-\tau),\ y_k - (t-\tau) \big] \\ J''_k(t) = \big[y_k + (t-\tau),\ y''_k - (t-\tau) \big] \end{cases} \qquad \text{is } \mathcal{O}(1) \cdot \varepsilon^2 (t-\tau)$$

Since the same estimates hold for semigroup trajectories, and $N \leq C\varepsilon^{-1}$ we conclude

$$\limsup_{h \to 0+} \frac{1}{h} \Big\| u(\tau + h) - S_h u(\tau) \Big\|_{\mathbf{L}^1} \ = \ \mathcal{O}(1) \cdot \varepsilon \qquad \text{for a.e. } \tau \in [0, T]$$

$$\Longrightarrow \qquad u(t, \cdot) \ = \ S_t \bar{u} \qquad \text{for all } t \in [0, T]$$

$$u_t + f(u)_x = 0 \qquad\qquad u(0,x) = \bar{u}(x)$$

A general class of ε-approximate solutions

- Fix a time step $\varepsilon = \Delta t$

Two properties are assumed:

(AL) Approximate Lipschitz continuity:

$$\|u(\tau,\cdot) - u(\tau',\cdot)\|_{\mathbf{L}^1} \leq M\,|\tau - \tau'| \qquad\qquad \tau,\tau' \in \varepsilon\mathbb{N}\,.$$

($\mathbf{P}_\varepsilon$) Approximate conservation law and approximate entropy inequality:

For every strip $[\tau, \tau'] \times \mathbb{R}$ with $\tau, \tau' \in \varepsilon\mathbb{N}$, and every test function $\varphi \in \mathcal{C}^1_c(\mathbb{R}^2)$, there holds

$$\left| \int u(\tau, x)\varphi(\tau, x)\, dx - \int u(\tau', x)\varphi(\tau, x)\, dx + \int_\tau^{\tau'} \int \{u\varphi_t + f(u)\varphi_x\}\, dx\, dt \right|$$
$$\leq \ \varepsilon \|\varphi\|_{W^{1,\infty}} \cdot (\tau' - \tau)$$

Moreover, given a uniformly convex entropy η with flux q, assuming $\varphi \geq 0$, one has the entropy inequality

$$\int \eta(u(\tau, x))\varphi(\tau, x)\, dx - \int \eta(u(\tau', x))\varphi(\tau', x)\, dx + \int_\tau^{\tau'} \int \{\eta(u)\varphi_t + q(u)\varphi_x\}\, dxdt$$
$$\geq \ -\varepsilon \|\varphi\|_{W^{1,\infty}} \cdot (\tau' - \tau)$$

A.B., M.T.Chiri, W.Shen, A posteriori error estimates for numerical solutions to hyperbolic conservation laws. *Arch. Rational Mech. Anal.* **241** (2021), 357–402.

Uniform convergence rate

$$u_t + f(u)_x = 0 \qquad u(0,x) = \bar{u}(x) \qquad (CP)$$

Corollary (A.B., G.Guerra, 2023).

Consider an $n \times n$ strictly hyperbolic system, endowed with a strictly convex entropy which selects the Liu admissible shocks, and which generates a Lipschitz semigroup $S : \mathcal{D} \times \mathbb{R}_+ \mapsto \mathcal{D}$

Then, given $T > 0$ and an interval $[-R, R]$, there exists a function $\varepsilon \mapsto \varrho(\varepsilon)$ with the following properties.

(i) ϱ is continuous, nondecreasing, with $\varrho(0) = 0$.

(ii) If $t \mapsto u_\varepsilon(t) \in \mathcal{D}$ is an ε-approximate solution to (CP) supported on $[-R, R]$, then

$$\left\| u_\varepsilon(t) - S_t \bar{u} \right\|_{\mathbf{L}^1} \leq \varrho(\varepsilon) \qquad \text{for all } t \in [0, T]$$

compactness + uniqueness $\Longrightarrow$ uniform convergence rate

Proof. If the conclusion fails, there exists a sequence of ε_n-approximate solutions $(u_n)_{n\geq 1}$, with $\varepsilon_n \downarrow 0$ but

$$\sup_{t\in[0,T]} \big\| u_n(t) - S_t \bar{u} \big\|_{\mathbf{L}^1} \;\geq\; \delta_0 \;>\; 0 \qquad\qquad \text{for all } n \geq 1.$$

By compactness, taking a subsequence we have the $\mathbf{L}^1$-convergence $u_n(t) \to u(t)$ for all $t \in [0, T]$.

But then the limit function u would be an entropy solution distinct from $S_t\bar{u}$. This contradicts the uniqueness theorem. □

Rate of convergence

$$u_t + f(u)_x = 0, \qquad u(0,x) = \bar{u}(x)$$

In the setting of the Corollary, we have

$$\left\| u_\varepsilon(t) - S_t \bar{u} \right\|_{\mathbf{L}^1} \leq \varrho(\varepsilon) \qquad \text{for all } t \in [0, T]$$

The function $\varrho(\varepsilon)$ is a **universal rate of convergence** of ε-approximate solutions, taking values in the domain of the semigroup

Problem 1.

Give upper and lower estimates on the function $\varrho(\cdot)$.

Guess: $\sqrt{\varepsilon}\, |\ln \varepsilon| \leq \varrho(\varepsilon) \leq$???

Uniqueness for general $n \times n$ hyperbolic systems

(no genuine nonlinearity, no entropy)

Theorem (A.B., C. De Lellis, 2023)

Consider a strictly hyperbolic $n \times n$ system of conservation laws

$$u_t + f(u)_x = 0$$

Let $S : \mathcal{D} \times \mathbb{R}_+ \mapsto \mathcal{D}$ be the semigroup of vanishing viscosity solutions.

Then every weak solution $u : [0, T] \mapsto \mathcal{D}$,

- whose shocks satisfy the Liu admissibility conditions,

coincides with a semigroup trajectory.

Theorem (S.Bianchini, A.B., *Annals of Math.* 2005)

Every trajectory of the semigroup generated by vanishing viscosity approximations is a weak solution satisfying the Liu admissibility conditions.

Proof. 1. Observe that $u = u(t, x)$ is a BV functions of two variables.

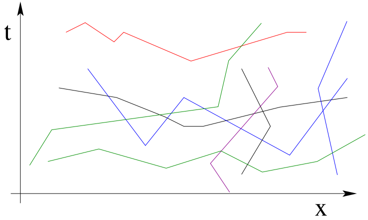
Let $\mathcal{N} \subset [0, T]$ be a null set of times such that, for every $\tau \notin \mathcal{N}$ and $\xi \in \mathbb{R}$, the point (τ, ξ) is either a point of approximate continuity, or a point of approximate jump of u

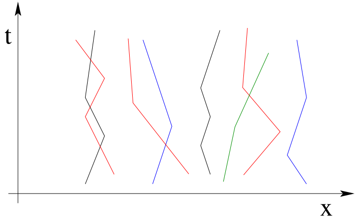
2. The set of points of approximate jump is countably rectifiable.
Indeed, there exists countably many Lipschitz functions $(\phi_k)_{k\geq 1}$ such that

- all points of approximate jump are contained in the set

$$\Big\{(t,x)\,; \quad x = \phi_k(t) \quad \text{for some } k\Big\}$$

- $\big|\phi_k(t) - \phi_k(s)\big| \leq |t-s|$ for all $s, t \in [0,1]$, $k \geq 1$





3. Consider the countable set of **good** Lipschitz functions $\{\psi_j\,;\ j \geq 1\}$ containing all functions ϕ_k together with the functions $\xi + t$, $\xi - t$, for all $\xi \in \mathbb{Q}$.

For every couple of integers $i, j \geq 1$, the scalar function

$$W_{ij}(t) \ \doteq \ \begin{cases} \text{Tot.Var.}\big\{u(t)\,;\ ]\psi_i(t)\,,\ \psi_j(t)[\,\big\} & \text{if} \quad \psi_i(t) < \psi_j(t) \\ 0 & \text{otherwise} \end{cases}$$

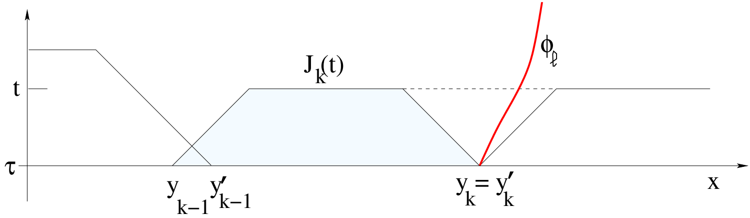
is bounded and measurable (indeed, it is lower semicontinuous)

$\Longrightarrow$ a.e. $t \in [0, T]$ is a Lebesgue point.

Denote by $\mathcal{N}' \subset [0, T]$ the null set of all times t which are NOT Lebesgue for at least one of the countably many functions W_{ij}.

Assuming $\tau \notin \mathcal{N} \cup \mathcal{N}'$, we claim

$$\limsup_{h\to 0+} \frac{1}{h} \Big\| u(\tau + h) - S_h u(\tau) \Big\|_{\mathbf{L}^1} \ = \ 0$$



Given $\varepsilon > 0$, by induction on k, we construct points

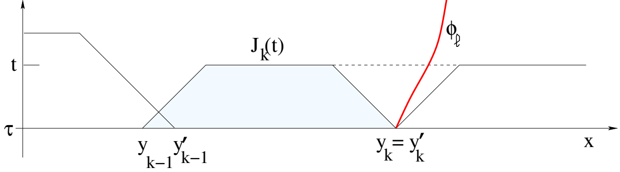
$$y_0 \le y_0' \; < \; y_1 \le y_1' \; < \; y_2 \le y_2' \; < \; \cdots \; < \; y_N \le y_N'$$

with the following properties.

(i) Either $y_k = y_k' = \phi_\ell(\tau)$ for some j, or else $y_k < y_k'$ and

$$y_k' + \tau \in \mathbb{Q}, \qquad\qquad y_k - \tau \in \mathbb{Q}$$

(ii) $\text{Tot.Var.}\{u(\tau,\cdot)\,;\;]y_{k-1}, y_k'[\,\} < 2\varepsilon$



Claim: $$\limsup_{h\to 0+} \frac{1}{h}\Big\| u(\tau+h) - S_h u(\tau)\Big\|_{\mathbf{L}^1} \;=\; \mathcal{O}(1)\cdot\varepsilon$$

Two types of integrals need to be estimated:

(I) For each k such that $y_k = y'_k$ the integral of $\big|u(t,\cdot) - U^\sharp(t,\cdot)\big|$ over the interval

$$\big[y_k - (t-\tau)\,,\; y_k + (t-\tau)\big] \qquad \text{is}\;\; o(t-\tau)$$

(II) The integral of $\big|u(t,x) - U^\flat(t,x)\big|$ over the intervals

$$J_k(t) \;=\; \big[y_{k-1}+(t-\tau)\,,\; y'_k-(t-\tau)\big] \qquad \text{is}\;\; \mathcal{O}(1)\cdot\varepsilon^2(t-\tau) + o(t-\tau)$$

This proves the claim, because $N \leq 2\text{Tot.Var.}\{u(\tau)\}/\varepsilon$.

Universal convergence rate for general $n \times n$ systems ?

To prove the existence of a **universal rate of convergence** for approximate solutions to general $n \times n$ systems (without any entropy), we would need to quantify

"by how much the Liu conditions are not satisfied"

This leads to

Question 2.

Does it make sense to say that, in an approximate solution, the Liu condition is satisfied "ε-approximately" ?

Let $u = u(t, x)$ be a (possibly smooth) approximate solution to

$$u_t + f(u)_x = 0, \qquad u(0, x) = \bar{u}(x)$$

ε-approximate entropy condition:

Given a convex entropy η with flux q, for every interval $[\tau, \tau']$ and every test function $\varphi \in \mathcal{C}^1_c(\mathbb{R}^2)$ with $\varphi \geq 0$, one has

$$\int \eta(u(\tau,x))\varphi(\tau,x)\,dx - \int \eta(u(\tau',x))\varphi(\tau',x)\,dx + \int_\tau^{\tau'}\int \big\{\eta(u)\varphi_t + q(u)\varphi_x\big\}\,dxdt$$

$$\geq \; -\varepsilon\|\varphi\|_{W^{1,\infty}} \cdot (\tau' - \tau)$$

ε-approximate Liu condition:

? ? ?

NOTE: as $\varepsilon \to 0$, in the limit this should reduce to the Liu condition.

Happy birthday Piero !