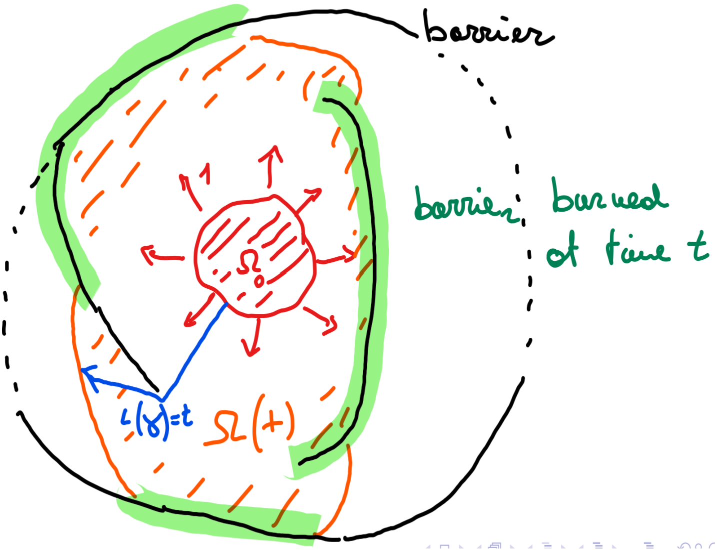
Spiral strategies for blocking fire

Martina Zizza, S.B.

SISSA, Trieste

June 21, 2023

Fire blocking problem



Fire blocking problem

- The *barrier* $\Gamma \subset \mathbb{R}^2$ is a Borel set with finite length

Fire blocking problem

- The *barrier* $\Gamma \subset \mathbb{R}^2$ is a Borel set with finite length
- The *fire* is propagating in every direction with speed 1 starting from an open set $\Omega_0 \subset \mathbb{R}^2$, but it cannot cross the barrier Γ

Fire blocking problem

- The *barrier* $\Gamma \subset \mathbb{R}^2$ is a Borel set with finite length
- The *fire* is propagating in every direction with speed 1 starting from an open set $\Omega_0 \subset \mathbb{R}^2$, but it cannot cross the barrier Γ
- The *burned region* is the relatively open set $\Omega(t) \subset \mathbb{R}^2\Gamma$ which is reached by the fire at time t: $\forall x \in \Omega(t)$ there is a curve with length $< t$ which connects Ω_0 to x avoiding Γ

Fire blocking problem

- The *barrier* $\Gamma \subset \mathbb{R}^2$ is a Borel set with finite length
- The *fire* is propagating in every direction with speed 1 starting from an open set $\Omega_0 \subset \mathbb{R}^2$, but it cannot cross the barrier Γ
- The *burned region* is the relatively open set $\Omega(t) \subset \mathbb{R}^2\Gamma$ which is reached by the fire at time t: $\forall x \in \Omega(t)$ there is a curve with length $< t$ which connects Ω_0 to x avoiding Γ
- The barrier is *admissible* if the part $\Gamma(t) = \Gamma \cap \mathrm{clos}(\Omega(t))$ reached by the fire has length $\leq \sigma t$,

$$\mathcal{H}^1(\Gamma(t)) = \mathcal{H}^1(\Gamma \cap \mathrm{clos}(\Omega(t))) \leq \sigma t$$

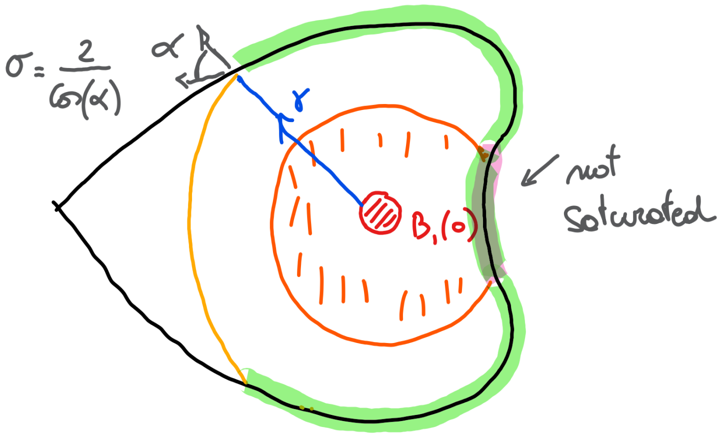
Fire blocking problem

- The *barrier* $\Gamma \subset \mathbb{R}^2$ is a Borel set with finite length
- The *fire* is propagating in every direction with speed 1 starting from an open set $\Omega_0 \subset \mathbb{R}^2$, but it cannot cross the barrier Γ
- The *burned region* is the relatively open set $\Omega(t) \subset \mathbb{R}^2\Gamma$ which is reached by the fire at time t: $\forall x \in \Omega(t)$ there is a curve with length $< t$ which connects Ω_0 to x avoiding Γ
- The barrier is *admissible* if the part $\Gamma(t) = \Gamma \cap \mathrm{clos}(\Omega(t))$ reached by the fire has length $\leq \sigma t$,

$$\mathcal{H}^1(\Gamma(t)) = \mathcal{H}^1(\Gamma \cap \mathrm{clos}(\Omega(t))) \leq \sigma t$$

The *fire blocking problem* is whether there exists an admissible barrier (strategy) encircling the fire in finite time.

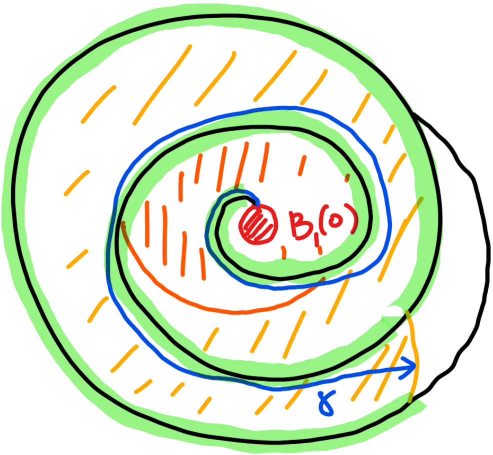
Simple closed curve



$\sigma > 2$ OK

[Bressan (..., Chiri), optimal strategies Bressan-De Lellis (Robyr)]

Spiral-like barrier



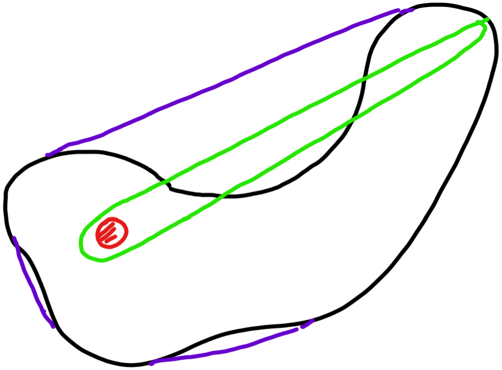
$\sigma > 2.61...$ OK

[Bressan (...), Klein-Langetepe-Levcopoulos]

If $\sigma \leq 2$ then the fire cannot be blocked

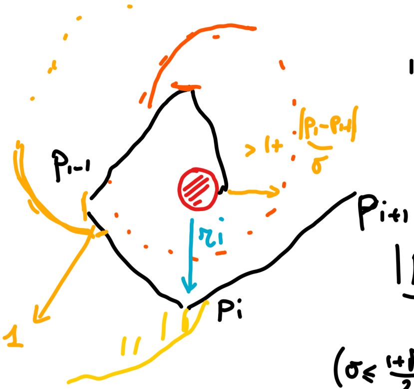
[Bressan:] if $\sigma \leq 1$ then the fire cannot be blocked, if $\sigma \leq 2$ it cannot be blocked with a simple closed curve

- minimal length convex
- if purple ok, $\sigma \leq 2$ ⇒ green better
- contradiction



If $\sigma \leq 2.61\ldots$ then the fire cannot be blocked by a spiral

[Klein-Langetepe-Levcopoulos:] if $\sigma \leq \frac{1+\sqrt{5}}{2}$ then a spiral cannot block the fire



$$1 + \frac{|p_i - p_{i-1}|}{\sigma} + \frac{|p_{i+1} - p_i|}{\sigma} \leq |p_{i+1}|$$

$$\leq |p_{i+1} - p_i|$$

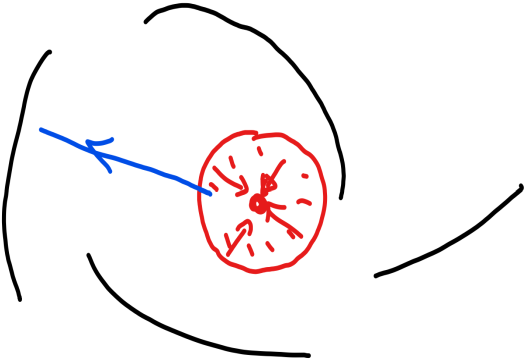
$$\Downarrow$$

$$\frac{|p_{i+1} - p_i|}{\sigma} \geq \frac{1}{\sigma - 1} + \frac{|p_i - p_{i-1}|}{\sigma(\sigma-1)}$$

$$\left(\sigma \leq \frac{1+\sqrt{5}}{2}\right) \geq 1 + |p_i - p_{i-1}|$$

Some simplifications

- The fire can be assumed to start in $(0,0)$, i.e. u is the distance from the origin avoiding Γ
- The barrier is outside the unit ball $B_1(0)$



Some notation

Let

$$\mathcal{A}(t[x]) = \underbrace{t[x]}_{\text{time taken by the fire}} - \frac{1}{\sigma} \underbrace{L(t[x])}_{\text{length of burned barrier}}$$

be the *admissibility functional*: an admissible barrier satisfies $\mathcal{A} \geq 0$ by definition.

Some notation

Let

$$\mathcal{A}(t[x]) = \underbrace{t[x]}_{\text{time taken by the fire}} - \frac{1}{\sigma} \underbrace{L(t[x])}_{\text{length of burned barrier}}$$

be the *admissibility functional*: an admissible barrier satisfies $\mathcal{A} \geq 0$ by definition.

Saturated set $S = \{x \in \Gamma : \mathcal{A}(t[x]) = 0\}$

Unsaturated set $U = \{x \in \Gamma : \mathcal{A}(t[x]) > 0\}$

Some notation

Let

$$\mathcal{A}(t[x]) = \underbrace{t[x]}_{\text{time taken by the fire}} - \frac{1}{\sigma} \underbrace{L(t[x])}_{\text{length of burned barrier}}$$

be the *admissibility functional*: an admissible barrier satisfies $\mathcal{A} \geq 0$ by definition.

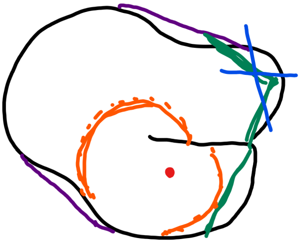
Saturated set $S = \{x \in \Gamma : \mathcal{A}(t[x]) = 0\}$

Unsaturated set $U = \{x \in \Gamma : \mathcal{A}(t[x]) > 0\}$

In the following an *optimal barrier* is the barrier with blocks the fire with minimal length.

A segment as internal barrier

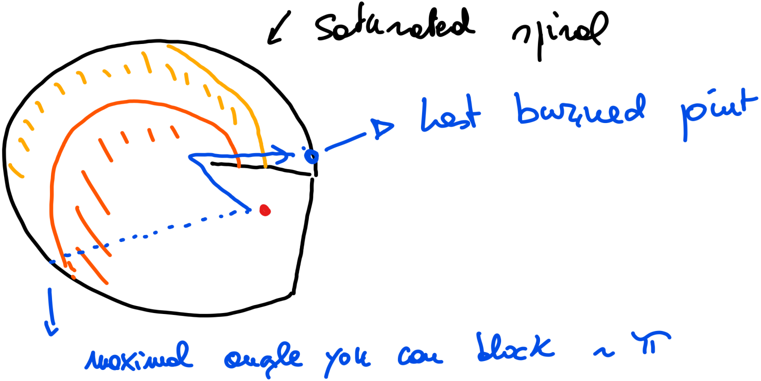
Reduction to a simple case



- Outer part is convex
- The part of barrier where two points are burning at the same time is unsaturated ($\sigma < 2$) and it is made of segments
- Other simplifications

A segment as internal barrier

Computation for the specific situation

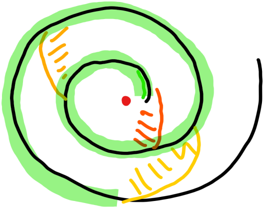


Definition of spiral

Definition

A *spiral barrier* Γ is a simple curve $\gamma(s)$, $s \in [0, L)$, such that

- $\gamma(0) = (1, 0)$, $\dot{\gamma}(0) = e^{i\theta}$, $\theta \in [0, \pi/2]$,
- $\dot{\gamma} \wedge \ddot{\gamma}(s) \geq 0$,
- $\{u = t\} \cap \Gamma$ is a connected arc.

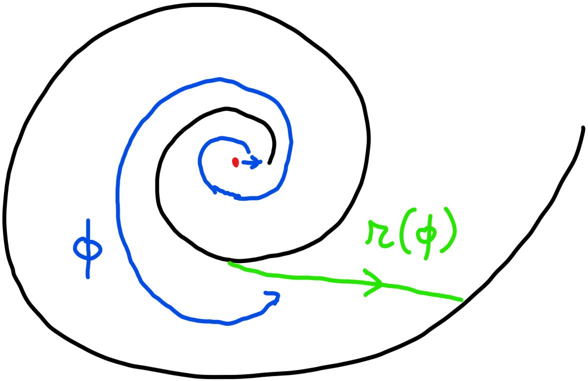


For optimal barriers some of the above assumptions are redundant.

Definition of spiral

The barrier can be described by means of the coordinates:

- the rotation angle ϕ of the optimal ray,
- the length $r(\phi)$ of the free part of the optimal ray.



Definition of spiral

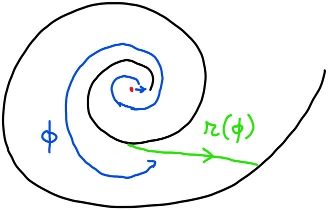
The barrier can be described by means of the coordinates:

- the rotation angle ϕ of the optimal ray,
- the length $r(\phi)$ of the free part of the optimal ray.

The equations is the delayed DE

$$\frac{dr(\phi)}{d\phi} = \cot(\beta(\phi))r(\phi) - R(\phi'),$$

with $R(\phi')$ radius of curvature, and $\beta \in [0, \pi/2]$ is a control parameter.



Definition of spiral

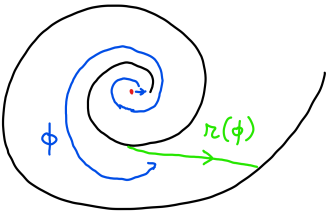
The barrier can be described by means of the coordinates:

- the rotation angle ϕ of the optimal ray,
- the length $r(\phi)$ of the free part of the optimal ray.

The equations is the delayed DE

$$\frac{dr(\phi)}{d\phi} = \cot(\beta(\phi))r(\phi) - R(\phi'),$$

with $R(\phi')$ radius of curvature, and $\beta \in [0, \pi/2]$ is a control parameter.



The *admissibility functional* is

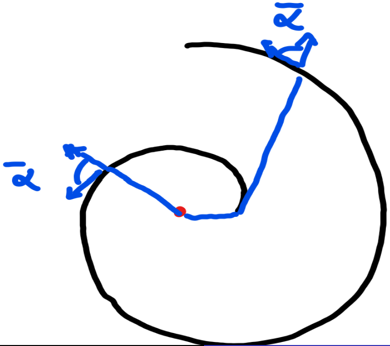
$$\mathcal{A}(\phi) = 1 + L(\phi') + r(\phi) - \cos(\alpha)L(\phi) \geq 0, \quad \cos(\alpha) = \frac{1}{\sigma}.$$

A different optimality condition

Definition

We say that the admissible Γ is optimal at the angle $\bar{\phi}$ if $r(\bar{\phi})$ is minimal.

In particular, if $r(\bar{\phi}) < 0$ then the barrier block the fire before $\bar{\phi}$.
An example of solution is for $\beta = \bar{\alpha}$,



A different optimality condition

Definition

We say that the admissible Γ is optimal at the rotation $\bar{\phi}$ if $r(\bar{\phi})$ is minimal.

In particular, if $r(\bar{\phi}) < 0$ then the barrier block the fire before $\bar{\phi}$.

An example of solution is for $\beta = \alpha$,

$$\frac{dr(\phi)}{d\phi} = \cot(\alpha) r(\phi) - \frac{r(\phi - 2\pi - \alpha)}{\sin(\alpha)},$$

$$r(\phi) = \begin{cases} e^{\cot(\alpha)\phi} & \phi \in [0, 2\pi), \\ e^{\cot(\alpha)2\pi} - 1 & \phi = 2\pi. \end{cases}$$

A different optimality condition

Definition

We say that the admissible Γ is optimal at the rotation $\bar{\phi}$ if $r(\bar{\phi})$ is minimal.

In particular, if $r(\bar{\phi}) < 0$ then the barrier block the fire before $\bar{\phi}$.

An example of solution is for $\beta = \alpha$,

$$\frac{dr(\phi)}{d\phi} = \cot(\alpha) r(\phi) - \frac{r(\phi - 2\pi - \alpha)}{\sin(\alpha)},$$

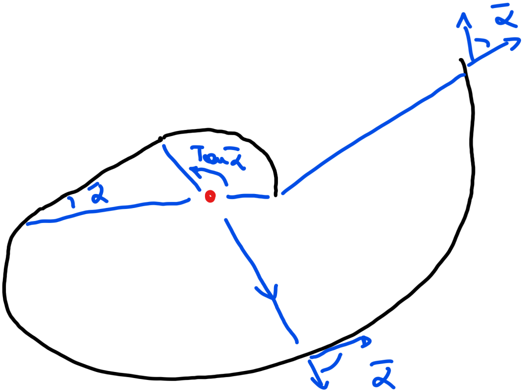
$$r(\phi) = \begin{cases} e^{\cot(\alpha)\phi} & \phi \in [0, 2\pi), \\ e^{\cot(\alpha)2\pi} - 1 & \phi = 2\pi. \end{cases}$$

One can prove that this $r(\phi)$ blows up if there is a real solution to

$$\lambda = \cot(\alpha) - \frac{e^{-\lambda(2\pi + \alpha)}}{\sin(\alpha)},$$

and this happens only if $\alpha \leq \bar{\alpha} = 1.17\ldots$, $\sigma = 1/\cos(\bar{\alpha}) = 2.61\ldots$

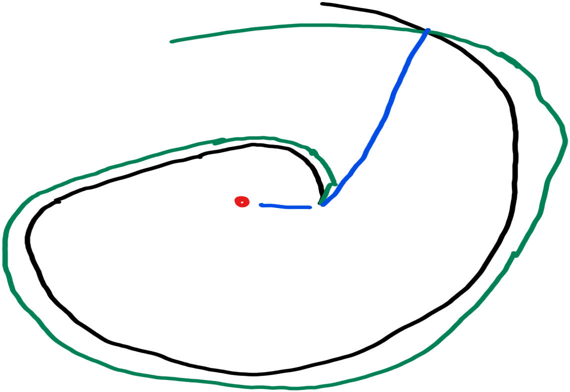
An example of (locally) optimal barrier



Lemma

The above barrier is optimal for angles $[0, 2\pi]$.

An example of (locally) optimal barrier

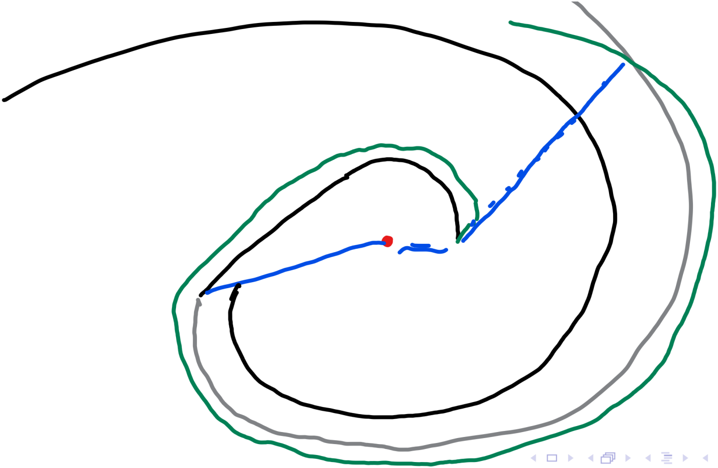


Corollary

It is not optimal for $\phi > 2\pi$, and the optimal barrier depends on $\bar{\phi}$.

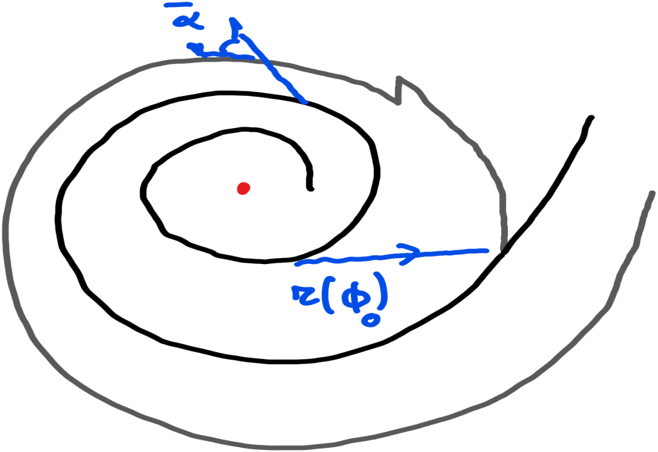
The (almost) fastest closing spiral

Problem: given a spiral barrier $r(\phi)$, replace the part $\phi \geq \phi_0$ with the (almost) fastest closing spiral.



The (almost) fastest closing spiral

Problem: given a spiral barrier $r(\phi)$, replace the part $\phi \geq \phi_0$ with the (almost) fastest closing spiral.



The (almost) fastest closing spiral

Problem: given a spiral barrier $r(\phi)$, replace the part $\phi \geq \phi_0$ with the (almost) fastest closing spiral.

Conclude the proof by showing that:

- the guess is almost optimal, in the sense that by adding a small perturbation (subtract a small quantity) one has a family of curves which is increasing
- the initial perturbed spiral is exponentially growing
- reduce to a specific situation, which is the worst one and only for $0 \leq \phi \leq 2\pi + \pi/2 - \bar{\alpha}$
- compute this case and prove that it cannot close

Happy birthday, Piero!