Poiseuille Time Periodic Flows in Space-Periodic Pipes.

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In collaboration with
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International Conference on Partial Differential Equations and Applications
In honor of the the 70th birthday of
Pierangelo Marcati
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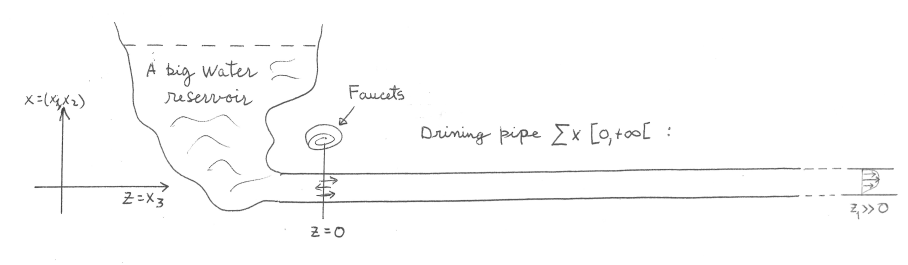
The Problem.

A supply of water, variable according to a daily time table, is provided.



The T-time periodic total flux $g(t)$ is well known, by construction.
But $g(x,0,t)$ is chaotic and unknown (and without practical interest).

At long distances the flow tends to forget the pointwise distribution of the velocity at the entrance and merely remembers the total flux, tending to become pointwisely $T-$periodic.

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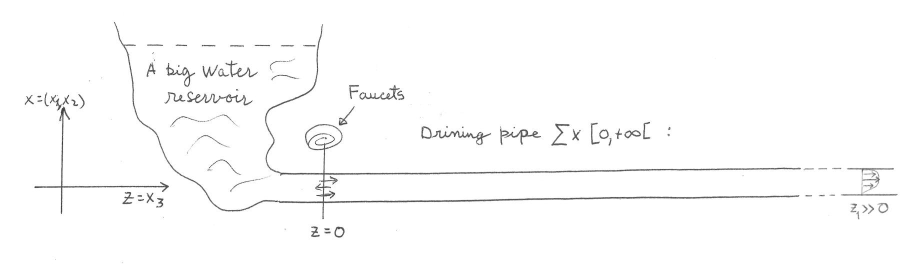

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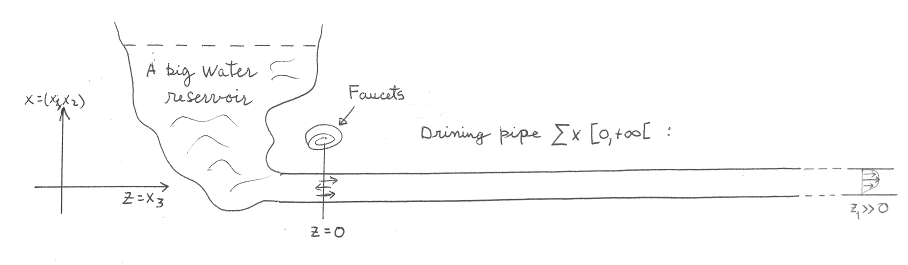

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So we face the problem of the existence of fully developed flows (Poiseuille flows) in infinite tubes $z \in (-\infty, +\infty)$, in correspondence to given T-periodic real functions $g(t)$.
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In a 2008 contribution G.P. Galdi wrote that the problem of the flow of a viscous liquid in an unbounded piping system, under a given time-periodic flow-rate, has been "discovered" only in 2005, thanks to H.B.V. reference [3], Arch. Ration. Mech. Anal., 2005 (denoted in the sequel simply by ARMA).

According to Galdi and Robertson, [14], there are two ways of determining a Poiseuille periodic flow, namely, by prescribing either the axial pressure gradient Γ or the flow rate $g(t) = \int_{\Sigma_z} v_z(x, z, t)\, dx$.
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If, conversely, we prescribe $g(t)$, then the problem becomes complicated, a fact that was emphasized and detailed in the 2005 ARMA's paper of H.B.V. who showed that the problem of determining $\boldsymbol{v}$ and Γ can be reduced to solving a *non-standard parabolic equation involving a non-local term* of the solution.

Moreover it is shown that this equation has one and only one solution, and so the problem is completely solved.

Furthermore, very interesting applications of the above result are given, in particular the resolution of the so-called "Leray's problem".

Actually, our proofs have not to do with the typical proofs of existence of time periodic solutions by appealing for fixed points of a map from a variable initial data to the value of the corresponding solution at a time T.

It is worth noting that Galdi and Robertson simplified the proof developed in the ARMA's paper by introducing a significant relationship between flow rate and axial pressure gradient, which depends only on the cross-section.

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A New Contribution

Let's pass to the extension of the results to pipes of varying cross section.
As emphasized by Galdi and Robertson, generalized Poiseuille flows are also important in the study of motions in "bent" pipes or in pipes of a varying cross-section, which appears in many problems of real life.

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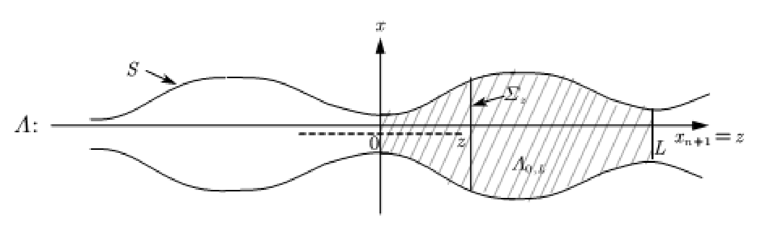
Notation and Main Results.

$\Lambda =$ $(n+1)$-dimensional infinite pipe, L-periodic in the $z = x_{n+1}$ direction.



$$x = (x_1, x_2, \cdots, x_n), \quad z = x_{n+1}.$$

Σ_z $=$ Orthogonal cross section at the level z.

$$\Lambda_{0,L} = \{(x,z) \in \Lambda : \ z \in (0,L)\}, \tag{0.1}$$

is the reference's pipe element (cell) normalized by $|\Lambda_{0,L}| = 1$. $S_{0,L}$ $=$ Lateral boundary of $\Lambda_{0,L}$.

Norms, scalar products, and other quantities of $z-$periodic functions in Λ are defined in $\Lambda_{0,L}$.

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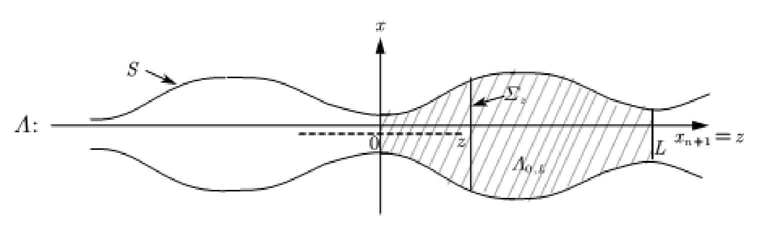

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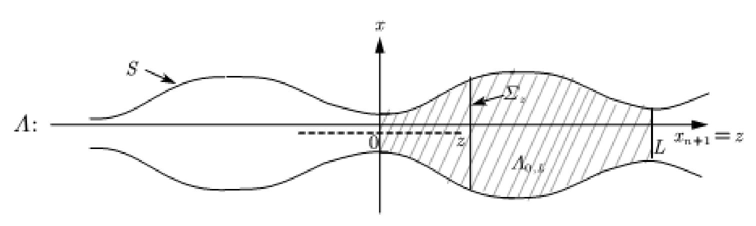

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We look for solutions $\boldsymbol{v}(x,z,t)$ with time-periodic total flux

$$\int_{\Sigma_z} v_z(x,z,t)\,dx = g(t)\,,$$

which should be simultaneously T-periodic with respect to time and L-periodic with respect to z. For convenience $T = 2\pi$.

Notation:

Lower symbols $\#$ means $T-$time periodicity.
Lower symbols $*$ means $L-$space-periodicity.

$$(\phi,\psi) =: \int_{\Lambda_{0,L}} \phi(\underline{x})\cdot\psi(\underline{x})\,d\underline{x}\,; \quad \|\phi\|^2 = (\phi,\phi)\,; \quad ((\phi,\psi)) = (\nabla\phi,\, \nabla\phi)\,.$$

$$\mathbb{H}(\Lambda) = \{\boldsymbol{u} \in L^2_*(\Lambda) : \nabla\cdot\boldsymbol{u} = 0\,, \quad (\boldsymbol{u}\cdot\mathbf{n})|_S = 0\}\,,$$

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Theorem

Let a $T-$periodic function $g \in H^1_\#(\mathbb{R}_t)$ be given. There is a unique solution $\boldsymbol{v} \in L^2_\#(\mathbb{R}_t; H^1_{0,*}(\Lambda))$ of the double periodic evolution Stokes problem

$$\begin{cases} \frac{\partial \boldsymbol{v}}{\partial t} - \nu \Delta \boldsymbol{v} + \nabla p = 0 & \text{in } \Lambda\,, \\ \nabla \cdot \boldsymbol{v} = 0 & \text{in } \Lambda\,, \\ \boldsymbol{v} = 0 & \text{on } S\,, \\ \int_{\Sigma_z} v_z \, d\Sigma_z = g(t)\,, \end{cases} \tag{0.2}$$

for which $\boldsymbol{v}' \in L^2_\#(\mathbb{R}_t; L^2_*(\Lambda))$.
Furthermore $\boldsymbol{v} \in L^2_\#(\mathbb{R}_t; H^2_*(\Lambda)) \cap C_\#(\mathbb{R}_t; H^1_{0,*}(\Lambda))$.
The result holds under suitable $T-$periodic external forces.

In the case of pipes with a fixed section Σ the solution of the evolution Stokes problem still solves the evolution Navier-Stokes problem. This situation is not true if the section depends on z. However the proof still lies in the study of the Stokes problem. The extension to the Navier-Stokes equations is obtained by a classical fixed point argument.

Theorem

There is a positive constant $c(\nu)$ such that the result in Theorem 1 holds for the Navier-Stokes equations if

$$\|g\|_{H^1_\#(\mathbb{R}_t)} < \frac{1}{4c^2(\nu)}. \qquad (0.3)$$

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The Stokes problem.

The L-space-periodic stationary Stokes problem

$$\begin{cases} -\Delta \mathbf{v} + \nabla p = \mathbf{f} & \text{in } \Lambda\,, \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \Lambda\,, \\ \mathbf{v} = 0 & \text{on } \partial\Lambda\,, \end{cases} \tag{0.4}$$

is written in a suitable abstract formulation

$$\mathcal{A}_H \boldsymbol{v} = \mathbf{f}\,, \tag{0.5}$$

where the Stokes operator

$$\mathcal{A}_H : \mathbb{V}_2(\Lambda) \to \mathbb{H}$$

is an isomorphism.

Proposition

Given $\mathbf{f} \in \mathbb{H}$ *there is a unique solution* $\boldsymbol{v} \in \mathbb{V}_2(\Lambda)$ *of the L-space-periodic stationary Stokes problem* (0.4) *in its formulation* (0.5).

Lemma

Pressure's Structure Result: If the double periodic evolution Stokes problem (0.2) *is solvable, then necessarily*

$$p(x,z,t) = -\psi(t)z + p_0(t) + \tilde{p}(x,z,t)\,, \tag{1.1}$$

where $p_0(t)$ *and* $\psi(t)$ *are arbitrary functions, and* $\tilde{p}(x,z,t)$ *is a z-periodic function. Decomposition* (1.1) *is unique up to* $p_0(t)$.

By appealing to (1.1) we replace the first equation (0.2) by

$$\frac{\partial \mathbf{v}}{\partial t} - \nu \Delta \mathbf{v} + \nabla \tilde{p} = \psi(t)\mathbf{e}_z \quad \text{in } \Lambda\,, \tag{1.2}$$

where $\mathbf{e}_z = \nabla z$ is the unit vector in the z-direction.
Next we eliminate the unknown $\psi(t)$ from equation (1.2):

$$\begin{cases} \frac{d\mathbf{v}}{dt} + \nu \mathcal{A}_H \mathbf{v} - \nu\,(\mathcal{A}_H \mathbf{v}\,, \mathbf{e})\mathbf{e} = \frac{L}{\|\mathbb{P}\mathbf{e}_z\|} g'(t)\mathbf{e}\,, \\ \int_{\Sigma_z} v_z \, d\Sigma_z = g(t)\,, \end{cases} \tag{1.3}$$

where $\mathbf{e} = \frac{\mathbb{P}\mathbf{e}_z}{\|\mathbb{P}\mathbf{e}_z\|}$. The elliptic part vanishes for $\mathcal{A}_H \mathbf{v} = \mathbf{e}$, and coerciveness fails (a force parallel to z is the strongest one against space-periodic flows).

Lemma

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$$p(x,z,t) = -\psi(t)z + p_0(t) + \tilde{p}(x,z,t)\,, \tag{2.1}$$

where $p_0(t)$ and $\psi(t)$ are arbitrary functions, and $\tilde{p}(x,z,t)$ is a z-periodic function. Decomposition (1.1) *is unique up to $p_0(t)$.*

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$$\frac{\partial \boldsymbol{v}}{\partial t} - \nu \Delta \boldsymbol{v} + \nabla \tilde{p} = \psi(t)\mathbf{e}_z \quad \text{in } \Lambda\,, \tag{2.2}$$

where $\mathbf{e}_z = \nabla z$ is the unit vector in the z-direction.

Next we eliminate the unknown $\psi(t)$ from equation (1.2):

$$\begin{cases} \frac{d\boldsymbol{v}}{dt} + \nu \mathcal{A}_H \boldsymbol{v} - \nu\,(\mathcal{A}_H \boldsymbol{v}\,,\mathbf{e})\mathbf{e} = \frac{L}{\|\mathbb{P}\mathbf{e}_z\|} g'(t)\mathbf{e}\,, \\ \int_{\Sigma_z} v_z\, d\Sigma_z = g(t)\,, \end{cases} \tag{2.3}$$

where $\mathbf{e} = \frac{\mathbb{P}\mathbf{e}_z}{\|\mathbb{P}\mathbf{e}_z\|}$. The elliptic part vanishes for $\mathcal{A}_H \boldsymbol{v} = \mathbf{e}$, and coerciveness fails (a force parallel to z is the strongest one against space-periodic flows).

Lemma

Pressure's Structure Result: If the double periodic evolution Stokes problem (0.2) *is solvable, then necessarily*

$$p(x,z,t) = -\psi(t)z + p_0(t) + \tilde{p}(x,z,t)\,, \qquad (3.1)$$

where $p_0(t)$ *and* $\psi(t)$ *are arbitrary functions, and* $\tilde{p}(x,z,t)$ *is a z-periodic function. Decomposition* (1.1) *is unique up to* $p_0(t)$.

By appealing to (1.1) we replace the first equation (0.2) by

$$\frac{\partial \mathbf{v}}{\partial t} - \nu\Delta\mathbf{v} + \nabla\tilde{p} = \psi(t)\mathbf{e}_z \quad \text{in } \Lambda\,, \qquad (3.2)$$

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The following ”negative” result will be crucial.

Proposition

One has

$$\mathbf{e} \notin \mathbb{V}(\Lambda). \tag{3.4}$$

Define $\mathbf{w} \in D(\mathcal{A}_H)$ as the unique solution of the equation

$$\mathcal{A}_H \mathbf{w} = \mathbf{e}\,. \tag{3.5}$$

We look for solutions $\boldsymbol{v} \in L^2_\#(\mathbb{R}_t; D(\mathcal{A}_H))$ of the Stokes evolution problem (1.3) in Fourier series form

$$\boldsymbol{v}(t) = \mathbf{a}_0 + \sum_{k=1}^{\infty} \mathbf{a}_k \cos kt + \sum_{k=1}^{\infty} \mathbf{b}_k \sin kt\,, \tag{3.6}$$

where the unknowns $\mathbf{a}_k$ and $\mathbf{b}_k$ will be looked for in $D(\mathcal{A}_H) = \mathbb{V}_2(\Lambda)$. The data $g \in L^2_\#(\mathbb{R}_t)$ is written in the Fourier series form

$$g(t) = p_0 + \sum_{k=1}^{\infty} p_k \cos kt + \sum_{k=1}^{\infty} q_k \sin kt\,, \tag{3.7}$$

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Substitution of equations (3.6) and (3.7) in equation (1.3), and orthogonality, leads to the $k-$infinite sequence of systems, $k \geq 1$,

$$\begin{cases} k\,\mathbf{b}_k + \nu \mathcal{A}_H \mathbf{a}_k - \nu(\mathcal{A}_H \mathbf{a}_k, \mathbf{e})\mathbf{e} = k\, \frac{L}{\|\mathbb{P}\mathbf{e}_z\|} q_k \mathbf{e}\,, \\ -k\,\mathbf{a}_k + \nu \mathcal{A}_H \mathbf{b}_k - \nu(\mathcal{A}_H \mathbf{b}_k, \mathbf{e})\mathbf{e} = -k\, \frac{L}{\|\mathbb{P}\mathbf{e}_z\|} p_k \mathbf{e}\,, \end{cases} \tag{3.8}$$

and to $\mathcal{A}_H \mathbf{a}_0 - (\mathcal{A}_H \mathbf{a}_0\,, \mathbf{e})\mathbf{e} = 0\,.$

We prove that this last equation implies $\mathbf{a}_0 = \tilde{c}\,\mathbf{w}$, for some constant $\tilde{c}$.

The systems (3.8) have all the same form

$$\begin{cases} k\,\boldsymbol{v} + \nu \mathcal{A}_H \boldsymbol{u} - \nu(\mathcal{A}_H \boldsymbol{u}, \mathbf{e})\mathbf{e} = k\, \frac{L}{\|\mathbb{P}\mathbf{e}_z\|} q\, \mathbf{e}\,, \\ -k\,\boldsymbol{u} + \nu \mathcal{A}_H \boldsymbol{v} - \nu(\mathcal{A}_H \boldsymbol{v}, \mathbf{e})\mathbf{e} = -k\, \frac{L}{\|\mathbb{P}\mathbf{e}_z\|} p\, \mathbf{e}\,, \end{cases} \tag{3.9}$$

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We prove the following result.

Theorem

Consider the system (3.9), *where* $k \geq 1$, p *and* q *are fixed. This problem has one and only one solution* $(\boldsymbol{u}, \boldsymbol{v}) \in D(\mathcal{A}_H) \times D(\mathcal{A}_H)$. *Moreover,*

$$\|\mathcal{A}_H \boldsymbol{u}\|^2 + \|\mathcal{A}_H \boldsymbol{v}\|^2 \leq \tilde{C} \left(1 + \left(\frac{L}{\nu \|\mathbb{P} \mathbf{e}_z\|} \right)^2 k^2 \right) (p^2 + q^2) \,. \tag{3.10}$$

The quite long and tricky proof is based on the approximation of the problem with a sequence of problems in increasing finite dimensional spaces V_m , with appeal to a special basis. We assume the above result. Some remarks on the proof would be shown below.

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Recall that we look for solutions of (0.2) under the form

$$\mathbf{v}(t) = \mathbf{a}_0 + \sum_{k=1}^{\infty} \mathbf{a}_k \cos kt + \sum_{k=1}^{\infty} \mathbf{b}_k \sin kt\,.$$

Hence

$$\mathcal{A}_H \mathbf{v}(t) = \tilde{c}\,\mathbf{e} + \sum_{k=1}^{\infty} (\mathcal{A}_H \mathbf{a}_k) \cos k\, t + \sum_{k=1}^{\infty} (\mathcal{A}_H \mathbf{b}_k) \sin k\, t\,. \tag{3.11}$$

$\mathcal{A}_H \mathbf{a}_0 = \tilde{c}\,\mathbf{e}$ follows from $\mathbf{a}_0 = \tilde{c}\,\mathbf{w}$. Below we prove that $\tilde{c}$ is uniquely determined.

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The Theorem 4 applies to each of the k-systems (3.8) and shows that the coefficients $\mathbf{a}_k$ and $\mathbf{b}_k$ in (3.8) are uniquely determined, and that for each $k \in \mathbb{N}$ one has the fundamental regularity estimates

$$\|\mathcal{A}_H \mathbf{a}_k\|^2 + \|\mathcal{A}_H \mathbf{b}_k\|^2 \leq \tilde{C} \left(1 + \left(\frac{kL}{\nu \|\mathbb{P}\mathbf{e}_z\|} \right)^2 \right) (p_k^2 + q_k^2) \, . \tag{3.12}$$

By appealing to (3.11), and orthogonality, it follows that

$$\|\mathbf{v}\|^2_{L^2_\#(\mathbb{R}_t;\mathcal{A}_H)} = \int_0^T (\mathcal{A}_H \mathbf{v}(t)\,, \mathcal{A}_H \mathbf{v}(t))dt = \tilde{c}^2\, T + \frac{T}{2} \sum_{k=1}^{\infty} \left(\|\mathcal{A}_H \mathbf{a}_k\|^2 + \|\mathcal{A}_H \mathbf{b}_k\|^2 \right) \,, \tag{3.13}$$

where $T = 2\pi$. Next, by equation (3.12), one gets the estimate

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So we have proved the very main estimate

$$\|\mathbf{v}\|^2_{L^2_\#(\mathbb{R}_t;D(\mathcal{A}_H))} \leq +C\, \|g\|^2_{L^2_\#(\mathbb{R}_t)} + \frac{C}{\nu^2} \|g'\|^2_{L^2_\#(\mathbb{R}_t)} \, . \tag{3.15}$$

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REMARKS ON THE PROOF OF MAIN THEOREM:

We find an increasing sequence of strictly positive, real eigenvalues λ_j of $\mathcal{A}_H$, and corresponding normalized eigenfunctions $\boldsymbol{w}_j \in \mathbb{H}(\Lambda)$, $j = 1, 2, \cdots$,

$$\mathcal{A}_H \boldsymbol{w}_j = \lambda_j \boldsymbol{w}_j \,. \tag{3.16}$$

We set $V_m = \text{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_m\}$ and look for $m-$approximate solutions

$$\boldsymbol{u}_m = \sum_{j=1}^{m} \alpha_j \boldsymbol{w}_j \,, \quad \boldsymbol{v}_m = \sum_{j=1}^{m} \beta_j \boldsymbol{w}_j \tag{3.17}$$

of the truncated equations (3.9), namely

$$\begin{cases} (k\,\boldsymbol{v}_m + \nu \mathcal{A}_H \boldsymbol{u}_m - \nu(\mathcal{A}_H \boldsymbol{u}_m, \mathbf{e})\mathbf{e}\,, \phi) = \frac{L}{\|\mathbb{P}\mathbf{e}_z\|} q_k(\mathbf{e}, \phi)\,, \\ (-k\,\boldsymbol{u}_m + \nu \mathcal{A}_H \boldsymbol{v}_m - \nu(\mathcal{A}_H \boldsymbol{v}_m, \mathbf{e})\mathbf{e}\,, \phi) = -\frac{L}{\|\mathbb{P}\mathbf{e}_z\|} p_k(\mathbf{e}\,, \phi)\,, \end{cases} \tag{3.18}$$

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Suitable calculations show that (3.18) is equivalent to the $2m$ dimensional system

$$\begin{cases} k\,\beta_l + \nu \sum_{j=1}^{m} [\delta_{jl} - (\mathbf{w}_j, \mathbf{e})(\mathbf{e}, \mathbf{w}_l)]\, \lambda_j \alpha_j = \ k\, \frac{L}{\|\mathbb{P}\mathbf{e}_z\|} q\,(\mathbf{e}, \mathbf{w}_l)\,, \\ -\,k\,\alpha_l + \nu \sum_{j=1}^{m} [\delta_{jl} - (\mathbf{w}_j, \mathbf{e})(\mathbf{e}, \mathbf{w}_l)]\, \lambda_j \beta_j = -\,k\, \frac{L}{\|\mathbb{P}\mathbf{e}_z\|} p\,(\mathbf{e}, \mathbf{w}_l)\,, \end{cases} \tag{3.19}$$

where l runs from 1 to m. We interpret (3.19) as a system on the unknown $2m$–dimensional column vector

$$X = (\lambda_1\alpha_1, \ldots, \lambda_m\alpha_m, \lambda_1\beta_1, \ldots, \lambda_m\beta_m) =: (X_1, X_2).$$

Set $\gamma_{jl} = \delta_{jl} - (\mathbf{w}_j, \mathbf{e})(\mathbf{e}, \mathbf{w}_l)$, $j, l = 1, \ldots, m$, and denote by M the corresponding $m \times m$ matrix. We prove that the $2m \times 2m$ matrix $\mathcal{M}$ of the system (3.19) is positive definite if and only if M is positive definite.

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Let $\bar{\mathbf{e}} = (e_1, ..., e_m, 0, 0, 0, ...)$ denote the orthogonal projection (in $\mathbb{H}$) of **e** onto V_m. One has

$$\sum_{j,l=1}^{m} \gamma_{jl}\, \xi_j\, \xi_l = |\xi|^2 - \sum_{j,l=1}^{m} (\xi_j e_j)(\xi_l e_l) \geq (1 - \|\bar{\mathbf{e}}\|^2)|\xi|^2 \,,$$

for each $\xi \in \mathbb{R}^m$.

Note that $\mathbf{e} \notin V_m$ since $\mathbf{e} \notin \mathbb{V}$. Since $\|\mathbf{e}\| = 1$, it follows that $\|\bar{\mathbf{e}}\| < 1$. Hence we have proved that the approximating $m-$problem (3.19), for each fixed k, admits one and only one solution (α, β) in $V_m \times V_m$.

REMARK: The strict positivity of M holds since $\mathbf{e} \notin \mathbb{V}$. However, if we try to pass to the limit as $m \to \infty$ we could not obtain a suitable estimate since $\|\bar{\mathbf{e}}\|$ converges to 1 as m goes to infinity.

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Multiplication of the first m equations (3.19) by $\lambda_l \alpha_l$, of the last m equations by $\lambda_l \beta_l$, summation for $l = 1, \cdots, m$, followed by a sequence of suitable arguments, and related calculations, lead to the (apparently bad estimate)

$$\|\mathcal{A}_H \boldsymbol{u}_m\|^2 + \|\mathcal{A}_H \boldsymbol{v}_m\|^2 \leq \left(\frac{Lk}{2\nu \|\mathbb{P}\mathbf{e}_z\|} \right)^2 (p^2 + q^2) + 2[(\mathcal{A}_H \boldsymbol{u}_m, \mathbf{e})^2 + (\mathcal{A}_H \boldsymbol{v}_m, \mathbf{e})^2] \,. \tag{3.20}$$

However, by exploiting the peculiarities of the vector $\mathbf{e}$, we prove that

$$C_1^4 [(\mathcal{A}_H \boldsymbol{u}_m, \mathbf{e})^2 + (\mathcal{A}_H \boldsymbol{v}_m, \mathbf{e})^2] \leq 16 \left[1 + \left(\frac{C_0}{C_1} \right)^2 \left(\frac{L}{\|\mathbb{P}\mathbf{e}_z\|} \right)^2 \left(\frac{k}{\nu} \right)^2 \right] (p^2 + q^2) \,. \tag{3.21}$$

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Thanks to this estimate and to (3.20) we show that

$$\|\mathcal{A}_H \boldsymbol{u}_m\|^2 + \|\mathcal{A}_H \boldsymbol{v}_m\|^2 \leq \tilde{C} \left(1 + \left(\frac{L}{\nu \|\mathbb{P}\mathbf{e}_z\|} \right)^2 k^2 \right) (p^2 + q^2) \ ,$$

which is just the main estimate (3.10) with $\boldsymbol{u}$ and $\boldsymbol{v}$ replaced by $\boldsymbol{u}_m$ and $\boldsymbol{v}_m$, and p and q by p_k and q_k, respectively.

From this estimate the weak convergence in $D(\mathcal{A}_H) \times D(\mathcal{A}_H)$ of the pair $(\boldsymbol{u}_m\,, \boldsymbol{v}_m)$ to a solution $(\boldsymbol{u}\,, \boldsymbol{v})$ of (3.9) follows.

To end, we prove that $\tilde{c}$ must be given by

$$\tilde{c} = \frac{L\, p_0}{\|\mathbb{P}\mathbf{e}_z\| (\boldsymbol{w}\,, \mathbf{e})} \,. \qquad (3.22)$$

The last step consists in proving the UNIQUENESS of the solution. In the absence of a suitable coercivity estimate, uniqueness is proved by a specific direct proof.

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The last step consists in proving the UNIQUENESS of the solution. In the absence of a suitable coercivity estimate, uniqueness is proved by a specific direct proof.

Concerning the extension to the Navier-Stokes equations, we have considered the Stokes evolution problem also under the effect of a suitable external force $\mathbf{f}$. The next two steps consist in replacing the external force $\mathbf{f}$ by $-\boldsymbol{w} \cdot \nabla \boldsymbol{w}$ and in proving the Theorem 2 by a contraction's map argument applied to the map $\boldsymbol{w} \to \boldsymbol{v}$.

Some main related references

In [14] G.P.Galdi and A.M.Robertson give a proof of the main result in the ARMA's paper by introducing a significant relationship between flow rate and axial pressure gradient, which depends only on the cross-section.

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Very interesting problems, but mathematically quite distinct, have been studied by M. Chipot, N. Klovienė, K. Pileckas and S. Zube in [9], by K. Kaulakytė, N. Klovienė, M. Skujus in [11], and by L.V. Kapitanski in [15].

A final remark.

A careful analysis of the structure of the proofs easily convince us that they can be extended to the case where the z-axis is replaced by an arbitrary, sufficiently regular, L-periodic parametric curve $\underline{x} = \underline{\gamma}(z)\,,\, \underline{\gamma} = (\gamma_1, ..., \gamma_{n+1})\,.$

Let's consider a very simple case to which the above theory applies but the result obtained in this way is weaker than that expected by appealing to the above curved-axis approach. Assume that the parametric curve $\underline{\gamma}(z)$ is a classical circular helix, see fig.2 (roughly, a spring. L is the pitch) and consider the pipe generated by the motion of a given $n-$dimensional flat, circular, surface Σ moving orthogonally to the given curve $\underline{\gamma}(z)$. Clearly the center of the "moving" circular Σ lies on $\underline{\gamma}(z)$.

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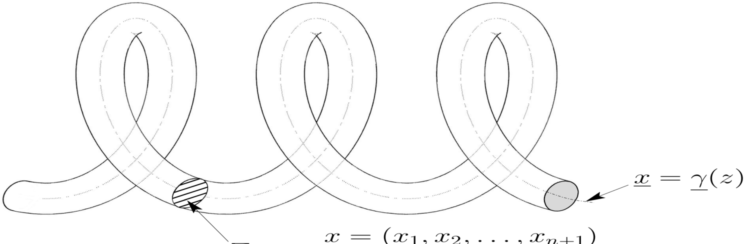
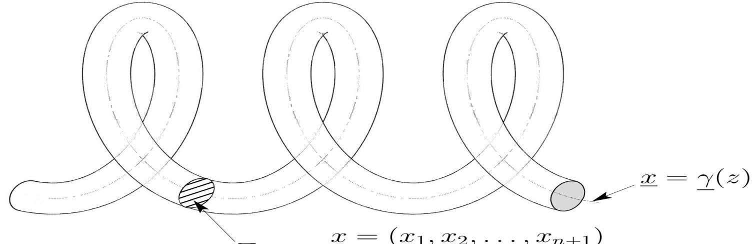


fig. 2

This situation fails inside that previously considered in these notes by setting $z = x_{n+1}$. In this case the surface Σ_z denotes the (non circular) $z-$section of the pipe, orthogonal to the $z = x_{n+1}$ axis. However, the result obtained merely guarantees L-periodicity with respect to the "linear" variable x_{n+1}. In a curved-axis approach the velocity should be independent of the parameter z , in the sense that it should be translational invariant along the bent pipe (along the n coordinate lines parallel to the helix).



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