Isentropic Approximation

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Goals:

In continuum physics, an isentropic model is often used when entropy s is near a constant state $\bar{s}$ in the process, such an approximation $s \to \bar{s}$, is called the **Isentropic Approximation**.

We want to discuss Isentropic Approximation with mathematical perspectives. Because, when candles are blown out, the song starts:

Happy Birthday, Piero!!!

1. Introduction: (i) Basic Quantities

- Specific Volume : v; Density : ρ
- Velocity: u; Pressure: P
- Absolute Temperature: θ
- Internal Energy: e; Specific Entropy: s

(ii) Basic Model: Navier-Stokes-Fourier system

Basic physical principles in continuum mechanics are expressed in terms of balance laws: mass, momentum and energy.

$$\begin{cases} \rho_t + \mathbf{div}(\rho u) = 0, \ \ x \in \mathbb{R}^3, t > 0, \\ (\rho u)_t + \mathbf{div}(\rho u \otimes u) + \nabla P = \mathbf{div}(\mathbb{S}), \\ (\rho e)_t + \mathbf{div}(\rho e u) = \mathbf{div}(\mathbf{q}) + \mathbb{S} : \nabla \mathbf{u} - \mathbf{P}\mathbf{div}(\mathbf{u}) \\ (\rho, u, \theta)(x, 0) = (\rho_0, u_0, \theta_0)(x), x \in \mathbb{R}^3. \end{cases} \quad (1)$$

(iv) Physical settings

- **Newton's rheological law (Newtonian fluid)**

$$\mathbb{S} = \mu(\nabla u + \nabla^T u - \frac{2}{3}\text{div}(u)\mathbb{I}) + \eta\text{div}(u)\mathbb{I} \tag{2}$$

$\mu \geq 0$ is the Shear Viscosity Coefficient;

$\eta \geq 0$ is the Bulk Viscosity Coefficient.

- **Fourier's Law:**

$$\mathbf{q} = -\kappa\nabla\theta. \tag{3}$$

$\kappa \geq 0$ is the heat conductivity coefficient.

Among thermodynamical state variables ρ, P, θ, e, s, and etc., only two of them are independent. In most cases, (ρ, θ) (or (ρ, s)) are convenient, and we have

$$P = P(\rho, \theta), \; e = e(\rho, \theta), \; s = s(\rho, \theta).$$

According to the Second Law of Thermodynamics (known as entropy law), P, e, s satisfy

- **Gibbs' relation (Maxwell's relation)**

$$\theta ds = de + Pd(\frac{1}{\rho}). \tag{4}$$

This, together with internal energy equation, implies that

- **Entropy balance:**

$$(\rho s)_t + \text{div}(\rho s u) + \text{div}(\frac{\mathbf{q}}{\theta}) = \frac{1}{\theta}(\mathbb{S} : \nabla u + \frac{\kappa}{\theta}|\nabla\theta|^2) \qquad (5)$$

  where

$$\sigma = \frac{1}{\theta}(\mathbb{S} : \nabla u + \frac{\kappa}{\theta}|\nabla\theta|^2)$$

  is called the entropy production rate.

The viscosity coefficients and heat conductivity coefficient are called the Transport coefficients. Like most classical mathematical literatures, we assume them to be positive constants in this lecture. However, a more realistic assumption is temperature-dependent, such as Chapman-Enskog model, and Sutherland's formula.

(v) Compressible Euler equations

When all transport coefficients are assumed to be zero (applicable in inviscid flows), the N-S-F system reduces to compressible Euler equations

$$\begin{cases} \rho_t + \textbf{div}(\rho u) = 0, \;\; x \in \mathbb{R}^3, t > 0, \\ (\rho u)_t + \textbf{div}(\rho u \otimes u) + \nabla P = 0, \\ (\rho e)_t + \textbf{div}(\rho e u) + P\textbf{div}(u) = 0 \\ (\rho, u, \theta)(x, 0) = (\rho_0, u_0, \theta_0)(x), x \in \mathbb{R}^3. \end{cases} \tag{6}$$

(vi) Isentropic Approximation

In both N-S-F and Euler cases, when entropy s is near a constant state $\bar{s}$ in the process, using **Isentropic Approximation**, that is $s \to \bar{s}$, the resulting systems are Isentropic Navier-Stokes equations or Isentropic Euler equations. For $P = P(\rho)$, they are

$$\begin{cases} \rho_t + \mathbf{div}(\rho u) = 0, \ \ x \in \mathbb{R}^3, t > 0, \\ (\rho u)_t + \mathbf{div}(\rho u \otimes u) + \nabla P = \mathbf{div}(\mathbb{S}), \\ (\rho, u)(x, 0) = (\rho_0, u_0)(x), x \in \mathbb{R}^3. \end{cases} \tag{7}$$

$$
\begin{cases}
\rho_t + \mathbf{div}(\rho u) = 0, \;\; x \in \mathbb{R}^3, t > 0, \\
(\rho u)_t + \mathbf{div}(\rho u \otimes u) + \nabla P = 0, \\
(\rho, u)(x, 0) = (\rho_0, u_0)(x), x \in \mathbb{R}^3.
\end{cases} \tag{8}
$$

We want to find a way to justify Isentropic Approximation mathematically.

2. Isentropic Approximation in Euler equations (Jia-Pan)

We want to offer an explicit characterization (mathematical justification) on the isentropic approximation for the compressible inviscid fluid flow. For this purpose, we consider the following Cauchy problem

$$\begin{cases} \rho_t + \mathbf{div}(\rho u) = 0, \ \ x \in \mathbb{R}^d, \\ \rho\left(u_t + u \cdot \nabla u\right) + \nabla p = 0, \\ s_t + u \cdot \nabla s = 0, \\ \rho(x,0) = \rho_0(x) \geq 0, \ \ u(x,0) = u_0(x), \ \ s(x,0) = s_0(x). \end{cases} \tag{9}$$

It is obvious that s is more convenient than θ for this purpose, and we assume $p(\rho, s) = \rho^\gamma e^s$ which is the pressure law for polytropic gas, with the adiabatic exponent $\gamma > 1$. In many applications, if in the thermodynamical process the specific entropy has only very small changes near a constant equilibrium

$\bar{s}$, an isentropic approximation is applied by assuming $s(x,t) = \bar{s}$ which reduces (9) to the isentropic Euler equations

$$\begin{cases} \rho_t + \mathbf{div}(\rho u) = 0, \ \ x \in \mathbb{R}^d \\ \rho\,(u_t + u \cdot \nabla u) + \nabla \tilde{p} = 0, \\ \rho(x,0) = \rho_0(x) \geq 0, \quad \ u(x,0) = u_0(x), \end{cases} \tag{10}$$

where $\tilde{p}(\rho) = \rho^\gamma e^{\bar{s}}$. Now, if one assumes

$$s_0(x) = \bar{s}$$

in (9), the solutions of (9) are expected to equal to the corresponding one of (10) formally. More precisely, we will study the limiting process from solutions of (9) to corresponding solutions of (10) when

$$(s_0(x) - \bar{s}) \to 0.$$

2a: Main results and ideas

Our main results for Euler read as, when the solutions of (9) and (10) are classical, then such an Isentropic approximation can be justified with sharp error estimates. However, when the solutions of Euler equations blow up and singularities are developed, the expected limiting process is not true at least by the measurement of Sobolev norms.

For the definite part, the main idea is the following observation:

In the smooth regime, the solution of isentropic Euler (10) is a special solution of full Euler (9).

When both (9) and (10) admit smooth solutions in $\mathbb{R}^d \times [0, T]$ for some positive T, the justification of isentropic limit as $(s_0(x) - \bar{s}) \to 0$ can be obtained by means of the continuity dependence of initial data for solutions of (9) near the initial data $(\rho_0, u_0, \bar{s})$. This comes along with the local well-posdeness theory for smooth initial data. Therefore, the justification of isentropic limit will be achieved by a careful energy method for the symmetric hyperbolic systems. The results will be established for solutions in the critical Besov space $B^{\frac{d}{2}+1}_{2,1}(\mathbb{R}^d)$. On the other hand, when singularities, say shock waves, occur in the solution, such picture breaks down. We will show this by an explicit example. Therefore, our results are somehow optimal for initial data with lowest possible regularity, $B^{\frac{d}{2}+1}_{2,1}(\mathbb{R}^d)$. This is achieved by establishing the corresponding local wellposedness theory for symmetric hyperbolic system in $B^{\frac{d}{2}+1}_{2,1}(\mathbb{R}^d)$.

2b: Explicit characterization on isentropic approximation

As in T. Makino, S. Ukai, S. Kawashima (1986), we use

$$w = p^{\frac{(\gamma-1)}{2\gamma}} = \rho^{\frac{(\gamma-1)}{2}} e^{\frac{(\gamma-1)s}{2\gamma}} \tag{11}$$

to transform Euler equations (9) into

$$\begin{cases} A_0(U)\partial_t U + \sum_{j=1}^{d} A_j(U)\partial_j U = 0, \\ U(x,0) = U_0(x), \end{cases} \tag{12}$$

for $U = (w, u_1, u_2, u_3, s)$, and the matrices

$$A_0(U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{(\gamma-1)^2}{4\gamma} e^{-\frac{s}{\gamma}} I_3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A_j(U) = \begin{pmatrix} u_j & \frac{\gamma-1}{2} w e_j & 0 \\ \frac{\gamma-1}{2} w e_j^T & \frac{(\gamma-1)^2}{4\gamma} e^{-\frac{s}{\gamma}} u_j I_3 & 0 \\ 0 & 0 & u_j \end{pmatrix} \quad (j = 1, 2, 3).$$

Here, e_j is the $j-$th row of I_3.

Theorem 1.1: For some constants $\bar{\rho} \geq 0$, $\bar{s}$ and $\bar{w} = (\bar{\rho})^{\frac{(\gamma-1)}{2}} e^{\frac{(\gamma-1)\bar{s}}{2\gamma}} \geq 0$, if the initial data $U_0 = (w_0, u_0, s_0)$ satisfies that $U_0 - (\bar{w}, 0, \bar{s}) \in B^{\frac{5}{2}}(\mathbb{R}^3)$, there exists a constant $T > 0$ and a unique solution $U = (w, u, s)$ to the problem (12) such that

$$U - (\bar{w}, 0, \bar{s}) \in C([0, T]; B^{\frac{5}{2}}(\mathbb{R}^3)) \cap C^1([0, T]; B^{\frac{3}{2}}(\mathbb{R}^3)).$$

If in addition $w_0(x) \geq 0$ for all $x \in \mathbb{R}^3$, then $w(x, t) \geq 0$ for all $(x, t) \in \mathbb{R}^3 \times [0, T]$. Equivalently, if $1 < \gamma \leq 3$, $\rho_0 \in C^1(\mathbb{R}^3)$, $\rho_0 \geq 0$ and

$$U_0 - (\bar{w}, 0, \bar{s}) \in B^{\frac{5}{2}}(\mathbb{R}^3),$$

then there exists a positive number T and a unique solution $(\rho, u, s)(x,t) \in C^1([0,T]\times\mathbb{R}^3)$ to problem (9) such that $\rho(x,t) \geq 0$ for all $(x,t) \in \mathbb{R}^3 \times [0,T]$ and

$$U(x,t) = \left(\rho^{\frac{(\gamma-1)}{2}} e^{\frac{(\gamma-1)s}{2\gamma}}, u, s\right)(x,t)$$

is the solution of (12) such that

$$U - (\bar{w}, 0, \bar{s}) \in C([0,T]; B^{\frac{5}{2}}(\mathbb{R}^3) \cap C^1([0,T]; B^{\frac{3}{2}}(\mathbb{R}^3)).$$

Remark: This Theorem includes the cases of initial data with or without vacuum. For the H^s theory with $s > \frac{5}{2}$, the case with initial data including vacuum was given in T. Makino, S. Ukai, S. Kawashima(1986), and the case with initial data away from vacuum was given in A. Majda (1984).

In order to give a precise description on the isentropic approximation for compressible Euler equations, we now assume that

$$s_0(x) - \bar{s} = \varepsilon\phi(x) \tag{13}$$

for $\phi(x) \in B^{\frac{5}{2}}(\mathbb{R}^3)$, and $1 > \varepsilon > 0$ is the controlling parameter. As explained in the introduction, given an initial data (ρ_0, u_0, s_0) or (w_0, u_0, s_0) as in Theorem , the Cauchy problem (9) has a smooth solution $(w, u, s)(x, t)$ defined on $\mathbb{R}^3 \times [0, T]$ for some positive T. We denote this solution as $U^\varepsilon(x, t) = (w, u, s)(x, t)$. If one assigns an initial data $(\rho_0, u_0, \bar{s})$ or $(\tilde{w}_0, u_0, \bar{s})$ with $\tilde{w} = w(\rho, \bar{s})$ to (9), the unique solution of (9) on $\mathbb{R}^3 \times [0, T]$ is the same one to (10) with initial data (ρ_0, u_0) or $(\tilde{w}_0, u_0)$, and we denote this solution by $U^I(x, t) = (w^I, u^I, \bar{s})$. We can now apply Theorem 1.1 to obtain the following theorem.

Main theorem 1.2 (Jia-Pan, 2015): Suppose $1 < \gamma \leq 3$, $\varepsilon \in (0,1]$, $\rho_0 \in C^1(\mathbb{R}^3)$, $\rho_0 \geq 0$ and for some $\bar{\rho} \geq 0$,

$$\rho_0^{\frac{(\gamma-1)}{2}} - (\bar{\rho})^{\frac{(\gamma-1)}{2}} \in B^{\frac{5}{2}}(\mathbb{R}^3), \quad u_0 \in B^{\frac{5}{2}}(\mathbb{R}^3), \quad \phi \in B^{\frac{5}{2}}(\mathbb{R}^3),$$

and

$$\rho_0^{\frac{(\gamma-1)}{2}} e^{\frac{(\gamma-1)(\bar{s}+\varepsilon\phi)}{2\gamma}} - (\bar{\rho})^{\frac{(\gamma-1)}{2}} e^{\frac{(\gamma-1)\bar{s}}{2\gamma}} \in B^{\frac{5}{2}}(\mathbb{R}^3).$$

Then, (10) has a unique solution

$$U^I(x,t) = (w^I, u^I, \bar{s})(x,t),$$

and (9) has a unique solution

$$U^\varepsilon(x,t) = (w, u, s)(x,t),$$

both defined on $\mathbb{R}^3 \times [0,T]$. Furthermore, the following estimate holds

$$\sup_{t\in[0,T]} \|U^\varepsilon(\cdot,t) - U^I(\cdot,t)\|_{L^2} \leq C\varepsilon\|\phi\|_{L^2},$$

where C is a positive constant depending on C^1 norms of $U_0(x)$ and T, but not on ε.

Remark: This theorem gives a precise justification with an explicit error estimate on isentropic approximation for compressible Euler equations in the regime of smooth solutions.

2c. Some references

For compressible Euler equations in one space dimension, such a problem was investigated by Saint-Raymond (ARMA, 2000) for BV solutions, where the difference between solutions of isentropic and full Euler equations measured by BV-norm was shown to grow at most linearly in time. For the steady Euler flows, similar results were obtained by Chen-Geng-Zhang (SIMA 2009) and by Liu-Zhang (CPAA 2008) . It remains an interesting open problem on how to offer a (physically and mathematically) sound explanation on the isentropic approximation for physically admissible weak solutions for compressible Euler equations.

3. Failure of Isentropic Approximation

In previous sections, the justification for isentropic limit has been proved for classical solutions of compressible Euler equations. It is well-known that shock waves may develop in finite time even for generic small smooth initial data. When shock forms, the justification of isentropic limit in previous sections breaks down. This can be seen easily by the following example.

Consider the full compressible Euler equations in one space dimension

$$\begin{cases} \rho_t + (\rho u)_x = 0, \ x \in \mathbb{R}, \\ (\rho u)_t + (\rho u^2 + p(\rho, s))_x = 0, \\ (\rho E)_t + (\rho E u + p u)_x = 0 \\ \rho(x,0) = \rho_0(x) \geq 0, \ u(x,0) = u_0(x), \ s(x,0) = \bar{s}, \end{cases} \tag{14}$$

where $E = \frac{1}{2}u^2 + e$ and $\rho e = \frac{c_v}{R}p$ with two positive constants c_v and R, and its isentropic reduction

$$\begin{cases} \rho_t + (\rho u)_x = 0, \ x \in \mathbb{R}, \\ (\rho u)_t + (\rho u^2 + p(\rho, \bar{s}))_x = 0, \\ \rho(x,0) = \rho_0(x) \geq 0, \ u(x,0) = u_0(x). \end{cases} \tag{15}$$

We remark here that unlike (9), we replaced the entropy equation by the energy conservation law in (14), since the entropy equation is no longer valid if singularity occurs in the solutions. It is clear that for any C^1 functions ρ_0 and u_0, both (14) and (15) share exactly the same C^1 solution $(\rho(x,t), u(x,t), \bar{s})$ up to a maximal existence time $T_1 > 0$. However, when this solution blows up at (x_1, T_1) for some $x_1 \in \mathbb{R}$, and shocks appear in the solution, the shock solution for (14) is different from that of (15). Indeed, the Riemann problems of (14) and (15) with the same Riemann data

$$\begin{aligned} \lim_{x \to x_1 -} (\rho, u, s)(x, T_1) = (\rho_-, u_-, \bar{s}); \\ \lim_{x \to x_1 +} (\rho, u, s)(x, T_1) = (\rho_+, u_+, \bar{s}), \end{aligned} \tag{16}$$

are different since the former one has a variable s (entropy s must increase across a shock wave, see Smoller (1994)), while the latter has a constant s in the solution.

Therefore, the framework we used in justifying the isentropic limit process in previous sections is no longer valid when singularity occurs in the solutions of Euler equations. New insights and techniques are required to offer possible description of isentropic approximation for entropy weak solutions.

5. Isentropic Approximation: N-S-F system

We now consider the following Cauchy problem of Navier-Stokes-Fourier system

$$\begin{cases} \rho_t + \textbf{div}(\rho u) = 0, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ (\rho u)_t + \textbf{div}(\rho u \otimes u) + \nabla p = \textbf{div}(\mathbb{S}), \\ (\rho e)_t + \textbf{div}(\rho e u) + p\textbf{div}(u) = \mathbb{S} : \nabla u + \kappa \Delta \theta, \\ \lim_{|x|\to\infty} \rho = \bar{\rho}, \ \lim_{|x|\to\infty} u = 0, \ \lim_{|x|\to\infty} \theta = \bar{\theta}, \\ (\rho, u, \theta)\Big|_{t=0} = (\rho_0, u_0, \theta_0), \end{cases} \tag{17}$$

where we rewrite the stress tensor $\mathbb{S} = \mu(\nabla u + (\nabla u)^t) + \nu(\text{div}(u))I$ with I the identity matrix. We assume that the constant viscosity coefficients $\mu > 0$ and ν satisfy $\nu + \frac{2}{3}\mu > 0$, and the constant $\kappa > 0$ is the coefficient of heat conductivity. We also assume that the fluid is polytropic ideal fluids, there exist two positive constants R, C_ν such that

$$p(\rho,\theta) = R\rho\theta, \quad e = C_\nu\theta, \quad p(\rho,S) = Ae^{\frac{S}{C\nu}}\rho^\gamma, \tag{18}$$

where $A > 0$ is a constant, $\gamma > 1$ is the adiabatic exponent, S is the entropy, and $C_\nu = \frac{R}{\gamma-1}$.

We are interested in the relationship between the solutions of (17) and the following Cauchy problem of the corresponding isentropic Navier-Stokes equations:

$$\begin{cases} \tilde{\rho}_t + \textbf{div}(\tilde{\rho}\tilde{u}) = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ (\tilde{\rho}\tilde{u})_t + \textbf{div}(\tilde{\rho}\tilde{u} \otimes \tilde{u}) + \nabla \tilde{p} = \textbf{div}(\tilde{\mathbb{S}}), \\ \lim\limits_{|x|\to\infty} \tilde{\rho} = \bar{\rho}, \lim\limits_{|x|\to\infty} \tilde{u} = 0 \\ (\tilde{\rho}, \tilde{u})\Big|_{t=0} = (\rho_0, u_0), \end{cases} \tag{19}$$

with the pressure $\tilde{p} = Ae^{\frac{\bar{S}}{C_\nu}}\tilde{\rho}^\gamma$, for a positive constant $\bar{S}$.

5.1 Perspectives: time asymptotics

Because:

- 1. Isentropic NS (19) does not satisfy the 2nd law of Thermodynamics. From the entropy equations of (17) that the nontrivial solution of (19) is not a solution of (17) with initial constant entropy $\bar{S}$. Therefore, it is not appropriate to use the perspective of continuous dependence of initial data to justify the isentropic approximation when $S(x,t) \to \bar{S}$.

- Isentropic NS (19) does not satisfy the 1st law of Thermodynamics. Only an energy inequality holds.

Therefore:

Instead, we try to compare them **time asymptotically**. We will offer a possible explanation time asymptotically in the sense that

the isentropic solution is a better approximation to the solution of N-S-F than the constant equilibrium when time is large.

Therefore, it is necessary to find out optimal lower and upper bounds of decay rates for the isentropic solutions, the N-S-F solutions, and their difference.

5.2 Outline of main steps

- Global existences of both isentropic NS and NSF, energy method, by A. Matsumura and T. Nishida (1979)

- Optimal decay rates (lower and upper bounds) for both isentropic NS and NSF. This includes linear decay plus faster decay of nonlinear remainders

- Compare the isentropic NS with NSF time asymptotically.

5.3 Small data theory for NSF: Global regularity

For multi-dimensional Navier-Stokes-Fourier system (17), the $H^s (s \geq 3)$ global existence with the initial perturbation small in $H^s \cap L^1$ are obtained in whole space first by A. Matsumura and T. Nishida in 1979. This is obtained by the local theory and uniform a priori estimates. We will show how to obtain the uniform estimates by basic energy method.

Letting $n = \rho - \bar{\rho}$, $u = u - 0$, and $q = \theta - \bar{\theta}$, we rewrite (17) in the perturbation form as

$$
\begin{cases}
n_t + \bar{\rho}\,\mathbf{div}(u) = -n\,\mathbf{div}(u) - u \cdot \nabla n, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
u_t + R_1 \nabla n + R \nabla q - \bar{\mu}\Delta u - (\bar{\mu} + \bar{\nu})\nabla \mathbf{div}(u) = f, \\
q_t + R_2\,\mathbf{div}(u) - \bar{\kappa}\Delta q = g, \\
\lim\limits_{|x|\to\infty} n = 0, \ \lim\limits_{|x|\to\infty} q = 0, \\
(n,u,q)\Big|_{t=0} = (\rho_0 - \bar{\rho}, u_0, \theta_0 - \bar{\theta}),
\end{cases}
\tag{20}
$$

where $\bar{\mu} = \frac{\mu}{\bar{\rho}}$, $\bar{\nu} = \frac{\nu}{\bar{\rho}}$, $\bar{\kappa} = \frac{\kappa}{C_\nu \bar{\rho}}$, $R_1 = \frac{R\bar{\theta}}{\bar{\rho}}$, $R_2 = \frac{R\bar{\theta}}{C_\nu}$, and

$$
\begin{aligned}
f &= -u \cdot \nabla u - \bar{\mu}\frac{n}{n+\bar{\rho}}\Delta u - (\bar{\mu} + \bar{\nu})\frac{n}{n+\bar{\rho}}\nabla \mathrm{div}(u) - R\frac{q}{n+\bar{\rho}}\nabla n + R_1\frac{n}{n+\bar{\rho}}\nabla n, \\
g &= -u \cdot \nabla q - \bar{\kappa}\frac{n}{n+\bar{\rho}}\Delta q - \frac{R}{C_\nu} q\,\mathrm{div}(u) + \frac{1}{C_\nu}\frac{\mathbb{S}:\nabla u}{n+\bar{\rho}}.
\end{aligned}
\tag{21}
$$

Using standard energy method, one can prove (c.f. Matsumura-Nishida (1979))

Theorem 5.1 *Assume that* $(n_0, u_0, q_0) \in H^3(\mathbb{R}^3)$, *then there exists constant* $\delta_0 > 0$ *such that if*

$$\|n_0\|_{H^3} + \|u_0\|_{H^3} + \|q_0\|_{H^3} \le \delta_0, \tag{22}$$

then the problem (20) *admits a unique global solution* $(n(t), u(t), q(t))$ *satisfying that for all* $t \ge 0$,

$$\begin{aligned} \|(n,u,q)(t)\|_{H^3}^2 &+ \int_0^t \left(\|\nabla n(\tau)\|_{H^2}^2 + \|(\nabla u, \nabla q)(\tau)\|_{H^3}^2 \right) d\tau \\ &\le C\|(n_0, u_0, q_0)\|_{H^3}^2, \end{aligned} \tag{23}$$

where $C > 0$ *is a positive constant independent of time.*

Similar results hold for isentropic NS equations (19).

5.4: Optimal Decay rates of Isentropic NS equations

We now consider the perturbation problem for (19). Define $\tilde{n} = \tilde{\rho} - \bar{\rho}$, $\tilde{m} = \tilde{\rho}\tilde{u}$, then we rewrite (19) as

$$\begin{cases} (\tilde{n})_t + \mathbf{div}(\tilde{m}) = 0, \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ (\tilde{m})_t + \alpha\nabla\tilde{n} - \bar{\mu}\Delta\tilde{m} - (\bar{\mu} + \bar{\nu})\nabla\mathbf{div}(\tilde{m}) = \tilde{F}, \\ \lim_{|x|\to\infty} \tilde{n} = 0, \lim_{|x|\to\infty} \tilde{m} = 0, \\ (\tilde{n}, \tilde{m})\Big|_{t=0} = (\rho_0 - \bar{\rho}, \rho_0 u_0), \end{cases} \tag{24}$$

where $\bar{\mu} = \frac{\mu}{\bar{\rho}}$, $\bar{\nu} = \frac{\nu}{\bar{\rho}}$, $\alpha = \tilde{p}'(\bar{\rho}) = \gamma A e^{\frac{\bar{S}}{C_\nu}} \bar{\rho}^{\gamma-1}$, and

$$\begin{aligned} \tilde{F} = \ & \text{div}\Big(\Big[\frac{\tilde{m}\otimes\tilde{m}}{\tilde{n}+\bar{\rho}} + \bar{\mu}\nabla\Big(\frac{\tilde{n}\tilde{m}}{\tilde{n}+\bar{\rho}}\Big)\Big]\Big) \\ & -\nabla\Big[(\bar{\mu}+\bar{\nu})\text{div}\Big(\frac{\tilde{n}\tilde{m}}{\tilde{n}+\bar{\rho}}\Big) + (\tilde{p}(\tilde{n}+\bar{\rho}) - \tilde{p}(\bar{\rho}) - \alpha\tilde{n})\Big]. \end{aligned} \tag{25}$$

Theorem 5.2: Assume that $(n_0, m_0) \in L^1(\mathbb{R}^3) \cap H^3(\mathbb{R}^3)$ with $\delta_0 =: \|(n_0, m_0)\|_{L^1(\mathbb{R}^3)\cap H^3(\mathbb{R}^3)}$ is sufficiently small. Then there is a unique global classical solution $\tilde{U} = (\tilde{n}, \tilde{m}) \in \mathcal{C}([0,\infty); H^3(\mathbb{R}^3))$ of the isentropic system (24). Furthermore if in addition (n_0, m_0) satisfies

$$\int_{\mathbb{R}^3} n_0(x)dx \neq 0, \ \ or \quad \int_{\mathbb{R}^3} m_0(x)dx \neq 0, \tag{26}$$

then for $k = 0, 1, 2$, it holds that

$$C^{-1}(1+t)^{-\frac{3}{4}-\frac{k}{2}} \leq \|\partial_x^k(\tilde{n}, \tilde{m})(t)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \tag{27}$$

where C, C^{-1} are positive constants independent of time.

Remark: M. Schonbek (1986, 1991) for incompressible NS; H. Li, A. Matsumura, G. Zhang (2010) for NS-Poisson.

5.5 Optimal decay rates for compressible N-S-F system

Similar to the Isentropic case with much more complicated calculations, we have

Theorem 5.3: *Assume that initial data* $(n_0, m_0, q_0) \in L^1(\mathbb{R}^3) \cap H^3(\mathbb{R}^3)$ *with* $\delta_0 =: \|(n_0, m_0, q_0)\|_{L^1(\mathbb{R}^3)\cap H^3(\mathbb{R}^3)}$ *is sufficiently small. Then there is a unique global classical solution* $U = (n, m, q) \in \mathcal{C}([0,\infty); H^3(\mathbb{R}^3))$ *of the non-isentropic system* (17). *If in addition* (n_0, m_0, q_0) *satisfies*

$$\int_{\mathbb{R}^3} n_0(x)dx \neq 0, \quad \int_{\mathbb{R}^3} m_0(x)dx \neq 0, \quad or \int_{\mathbb{R}^3} q_0(x)dx \neq 0, \quad (28)$$

then for $k = 0, 1, 2$, *it holds that*

$$C^{-1}(1+t)^{-\frac{3}{4}-\frac{k}{2}} \leq \|\partial_x^k (n, m, q)(t)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad (29)$$

where C, C^{-1} *are positive constants independent of time.*

5.6 Justification of Isentropic Approximation

We now compare the solutions (n, m, q) and thus (n, m, S) of (17), and $(\tilde{n}, \tilde{m})$ of (19). Recall,

$$(\rho s)_t + \mathrm{div}(\rho s u) + \mathrm{div}(\frac{\mathbf{q}}{\theta}) = \frac{1}{\theta}(\mathbb{S} : \nabla u + \frac{\kappa}{\theta}|\nabla\theta|^2) \qquad (30)$$

Indeed, we are able to derive the following estimates

Theorem 5.4: *Assume that initial value* $(n_0, m_0) \in L^1(\mathbb{R}^3) \cap H^3(\mathbb{R}^3)$ *with* $\delta_0 =: \|(n_0, m_0)\|_{L^1(\mathbb{R}^3)\cap H^3(\mathbb{R}^3)}$ *is sufficiently small and* $S_0 = \bar{S}$ *with* $\bar{S} = C_\nu \ln \frac{R\bar{\theta}}{A\bar{\rho}^{\gamma-1}}$. *If the initial data are subject to* (26) *and* (28), *then* $(n_e, m_e) = (n - \tilde{n}, m - \tilde{m}) \in \mathcal{C}([0, \infty); H^3(\mathbb{R}^3))$, *satisfies for* $k = 0, 1, 2$, *that*

$$\begin{aligned} \|\partial_x^k n_e(t)\|_{L^2(\mathbb{R}^3)} &< \|(\partial_x^k n, \partial_x^k \tilde{n})(t)\|_{L^2(\mathbb{R}^3)}, \\ \|\partial_x^k m_e(t)\|_{L^2(\mathbb{R}^3)} &< \|(\partial_x^k m, \partial_x^k \tilde{m})(t)\|_{L^2(\mathbb{R}^3)}. \end{aligned} \qquad (31)$$

More precisely, for $s(x,t) = S(x,t) - \bar{S}$*, it holds for* $k = 0, 1, 2$ *that*

$$\begin{aligned} \|\partial_x^k s(t)\|_{L^2(\mathbb{R}^3)} &\lesssim \delta_0^2(1+t)^{-\frac{3}{4}-\frac{k}{2}} + \delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \\ \|\partial_x^3 s(t)\|_{L^2(\mathbb{R}^3)} &\lesssim \delta_0(1+t)^{-\frac{7}{4}}, \end{aligned} \tag{32}$$

and for some sufficiently large $N_0 > 0$ *and arbitrary small* $\varepsilon > 0$*, it holds that*

$$\begin{aligned} &\|\partial_x^k (n_e, m_e)(t)\|_{L^2(\mathbb{R}^3)} \\ \lesssim\ & (\delta_0 + \tfrac{1}{1+N_0} + \tfrac{N_0}{(1+N_0)^{\frac{7}{4}}})\delta_0(1+t)^{-\frac{3}{4}-\frac{k}{2}} \\ & +\delta_0(1+t)^{-\frac{5}{4}-\frac{k}{2}+2\varepsilon}. \end{aligned} \tag{33}$$

Theorem 5.4 clearly shows that the isentropic solution $(\tilde{n}, \tilde{m})$ is a better approximation to (n, m, S) of the compressible Navier-Stokes-Fourier system (17) than the constant equilibrium $(\bar{\rho}, \bar{m}, \bar{S})$ for large time.

6. Summary

- For Euler, in the regime of smooth solutions, the isentropic approximation is justified with the perspective of continuous dependence of initial data. The weak solution case remains open.

- For N-S-F, when initial data is small smooth near a constant equilibrium, the isentropic approximation is justified time asymptotically in the sense that the isentropic solutions are better approximations to the NSF solutions than the constant equilibrium time asymptotically. Joint with Y. Cao and Y. Li, we justified the case with temperature-dependent transport coefficients. All other cases are open.

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