From Euler flows with friction to gradient flows

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in honor of Pierro Marcati

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Euler flows generated by an energy functional

- Hamiltonian Systems driven by an energy $\mathcal{E}(\rho)$

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$
$$\rho \frac{Du}{Dt} = -\rho \nabla_x \frac{\delta \mathcal{E}}{\delta \rho}$$

  - $\mathcal{E}[\rho]$ is an energy functional, e.g. $\mathcal{E}(\rho) = \int h(\rho) + \kappa(\rho) |\nabla(\rho)|^2 dx$

- Gradient flows in Wasserstein

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$
$$\rho u = -\rho \nabla_x \frac{\delta \mathcal{E}}{\delta \rho}$$

Part I, Euler flows generated by an energy functional

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$

$$\rho \frac{Du}{Dt} = \rho\big(\partial_t u + (u \cdot \nabla) u\big) = -\rho \nabla_x \frac{\delta \mathcal{E}}{\delta \rho}$$

where $\mathcal{E}[\rho]$ is a functional

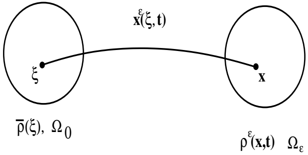
Hamiltonian

$$\mathcal{H}(\rho, u) = \mathcal{E}(\rho) + \int \tfrac{1}{2}\rho |u|^2 dx$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ u\end{pmatrix} = \begin{pmatrix} 0 & -\operatorname{div} \\ -\nabla & 0 \end{pmatrix}\begin{pmatrix}\frac{\delta \mathcal{H}}{\delta \rho} \\ \frac{\delta \mathcal{H}}{\delta u}\end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{\rho}\frac{\delta \mathcal{H}}{\delta u} \times \operatorname{curl}_x\left(\frac{1}{\rho}\frac{\delta \mathcal{H}}{\delta u}\right)\end{pmatrix} .$$

$$\frac{d}{dt}\mathcal{H}(\rho, u) = 0$$

why this structure ?



Family of maps

$$x^\varepsilon(\xi, t) \longrightarrow \begin{cases} u^\varepsilon(x, t) \\ \rho^\varepsilon(x, t) \end{cases}$$

$$\rho^\varepsilon = x^\varepsilon_{\#}\bar\rho\,, \qquad \partial_t \rho^\varepsilon + \operatorname{div}_x(\rho^\varepsilon u^\varepsilon) = 0$$

Variation

$$x^\varepsilon(\xi, t) = x(\xi, t) + \varepsilon \delta x(\xi, t)$$

Find extrema of the action $\mathcal{L}$ over x^ε such that $\rho^\varepsilon(\cdot, t_1) = \rho_1,\ \rho^\varepsilon(\cdot, t_2) = \rho_2$

$$\mathcal{L}[x^\varepsilon] = \int_{t_1}^{t_2} \int_{\Omega_\varepsilon = x^\varepsilon(\Omega_0)} \tfrac{1}{2}\rho^\varepsilon |u^\varepsilon|^2 dxdt - \int_{t_1}^{t_2} \mathcal{E}\big[\rho^\varepsilon(\cdot, t)\big] dt$$

It turns out

$$x^{\varepsilon}(\xi,t) = x(\xi,t) + \varepsilon\delta x(\xi,t)$$
$$\delta x(\xi,t) = \delta\phi(x(\xi,t),t)$$
$$\frac{d\rho^{\varepsilon}}{d\varepsilon}\Big|_{\varepsilon=0} = -\mathrm{div}_x(\rho\delta\varphi)$$

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big(\int_{t_1}^{t_2}\mathcal{E}\big[\rho^{\varepsilon}(\cdot,t)\big]\,dt\Big) = \int_{t_1}^{t_2}\Big\langle\frac{\delta\mathcal{E}}{\delta\rho},\frac{d\rho^{\varepsilon}}{d\varepsilon}\Big|_{\varepsilon=0}\Big\rangle\,d\tau$$
$$= \int_{t_1}^{t_2}\Big\langle\rho\nabla_x\frac{\delta\mathcal{E}}{\delta\rho},\delta\varphi\Big\rangle\,d\tau$$

Obtain the equations:

$$\partial_t\rho + \mathrm{div}\,(\rho u) = 0$$
$$\rho\frac{Du}{Dt} = -\rho\nabla_x\frac{\delta\mathcal{E}}{\delta\rho}$$

$$\mathcal{E}(\rho|\bar\rho) := \mathcal{E}(\rho) - \mathcal{E}(\bar\rho) - \left\langle \frac{\delta \mathcal{E}}{\delta \rho}(\bar\rho), \rho - \bar\rho \right\rangle$$

under hypothesis that $\mathcal{E}(\rho)$ is convex in ρ

Relative energy calculation

$$\frac{d}{dt}\left(\int \tfrac{1}{2}\rho|u-\bar u|^2\, dx + \mathcal{E}(\rho|\bar\rho) \right) = \int -\rho \nabla_x \bar u : (u-\bar u)\otimes(u-\bar u) + \int \nabla \bar u : S(\rho|\bar\rho)\, dx$$

where

$$S(\rho|\bar\rho) := S(\rho) - S(\bar\rho) - \left\langle \frac{\delta S}{\delta \rho}(\bar\rho), \rho - \bar\rho \right\rangle$$

abstract relative energy computation based on

Hypothesis : $\mathcal{E}(\rho)$ satisfies for some functional $S[\rho]$

(*) $$-\rho \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} = \nabla_x \cdot S[\rho]$$

Formula (*)

- gives meaning to weak solutions
- serves as the basis for the relative energy calculation
- Invariance of $\mathcal{E}(\rho)$ under translations $\rho(\cdot) \to \rho(\cdot + h)$ plus Noether's theorem implies (*)

Applications

Example: the quantum hydrodynamics system

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = -\nabla p(\rho) + 2\rho\nabla\left(\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}\right) + \rho\nabla c$$

$$-\triangle c = \rho - \bar{\rho}$$

generated by the energy

$$\mathcal{E}(\rho) = \int h(\rho) + \tfrac{1}{2}\frac{1}{\rho}|\nabla\rho|^2 + \tfrac{1}{2}\rho\, c$$

with $\rho h''(\rho) = p'(\rho)$

Application to the quantum hydrodynamics system

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$
$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = 2\rho \nabla \left(\frac{\triangle \sqrt{\rho}}{\sqrt{\rho}} \right)$$

Thm If (ρ, u) is a weak conservative solution and $(\bar{\rho}, \bar{u})$ smooth conservative soln of QHD then

$$\Psi(t) = \int \tfrac{1}{2}\rho|u - \bar{u}|^2 + h(\rho|\bar{\rho}) + \tfrac{1}{2}\rho\Big|\frac{\nabla \rho}{\rho} - \frac{\nabla \bar{\rho}}{\rho}\Big|^2 dx$$

satisfies the stability estimate

$$\Psi(t) \le \Psi(0) + O(|\nabla \bar{u}|) \int_0^T \Psi(\tau) d\tau$$

Gieselmann - Lattanzio - T 16, Bresch-Gisclon-Violet 18, Basaric-Tang 22

Part II, From Euler flows to gradient flows

- Euler flow with high-friction

$$
\begin{aligned}
&\partial_t \rho + \mathrm{div}\,(\rho u) = 0 \\
&\partial_t (\rho u) + \mathrm{div}\,(\rho u \otimes u) = -\rho \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} - \frac{1}{\varepsilon} \rho u
\end{aligned}
$$

- as $\varepsilon \to 0$ we get trivial dynamics $\partial_t \rho = 0$
- long-time dynamics $t \to \frac{t}{\varepsilon}$, $u \to \varepsilon u$, $\rho \to \rho$, obtain small-mass approximation

$$
\begin{aligned}
&\partial_t \rho + \mathrm{div}\,(\rho u) = 0 \\
&\partial_t (\rho u) + \mathrm{div}\,(\rho u \otimes u) = -\frac{1}{\varepsilon^2} \rho \big(u + \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \big)
\end{aligned}
$$

with energy dissipation identity

$$
\partial_t \Big(\mathcal{E}[\rho] + \int \frac{\varepsilon^2}{2} \rho |u|^2 dx \Big) + \int \rho |u|^2 dx = 0
$$

- Limit Diffusion theory

$$\partial_t \rho + \mathrm{div}\,(\rho u) = 0 \qquad u = -\nabla_x \frac{\delta \mathcal{E}}{\delta \rho}$$

$\mathcal{E}[\rho]$ is a convex functional

$$\partial_t \mathcal{E}[\rho] + \int \rho \Big| \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \Big|^2 dx = 0 \qquad \text{energy dissipation}$$

- Diffusive Relaxation limits — Marcati, Natalini, ...
- Gradient flows in Wasserstein

  Examples: porous media, generalized Keller-Segel models, Cahn-Hilliard equation fit under this framework for various choices of $\mathcal{E}[\rho]$

  Otto, Carillo-Toscani, Villani, Westdickenberg, Ambrosio-Gigli-Savare ...

An Italian perspective on gradient flows in Wasserstein:

Ambrosio-Gigli-Savare

- $$\partial_t \rho + \operatorname{div}(\rho u) = 0 \qquad u = -\nabla_x \frac{\delta \mathcal{E}}{\delta \rho}$$

  Energy dissipation

  $$\partial_t \mathcal{E}[\rho] + \int \rho \left| \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \right|^2 dx = 0 \qquad (*)$$

- Let (ρ, u) be a flow that satisfies $\partial_t \rho + \operatorname{div}(\rho u) = 0$.
  Then

  $$\partial_t \mathcal{E}[\rho] = \langle \frac{\delta \mathcal{E}}{\delta \rho}, \rho_t \rangle = \langle \rho u, \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \rangle \geq -\frac{1}{2} \int \rho |u|^2 + \rho |\nabla_x \frac{\delta \mathcal{E}}{\delta \rho}|^2 dx$$

  with equality iff

  $$u = -\nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \qquad \text{and then } (*) \text{ holds}$$

maximal energy dissipation

Suppose the flow (ρ, u) satisfies $\partial_t \rho + \operatorname{div}(\rho u) = 0$

$$
\begin{aligned}
\frac{d}{dt}\Big(\int_\Omega \varepsilon \tfrac{1}{2}\rho |u|^2 \, dx + \mathcal{E}[\rho] \Big) &= \langle \rho u, \varepsilon \frac{Du}{Dt} + \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \rangle \\
&= \int_\Omega \tfrac{1}{2}\rho \left| \varepsilon \frac{Du}{Dt} + \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} + u \right|^2 - \int_\Omega \tfrac{1}{2}\rho \left| \varepsilon \frac{Du}{Dt} + \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \right|^2 - \int_\Omega \tfrac{1}{2}\rho |u|^2 dx \\
&\geq - \int_\Omega \tfrac{1}{2}\rho \left| \varepsilon \frac{Du}{Dt} + \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \right|^2 - \int_\Omega \tfrac{1}{2}\rho |u|^2 dx
\end{aligned}
$$

with equality if and only if

$$
\varepsilon \rho \frac{Du}{Dt} = -\rho \nabla \frac{\delta \mathcal{E}}{\delta \rho} - \rho u
$$

Relative entropy for the relaxation system and the limiting diffusion theory

Let (ρ, u) be an entropy weak solution and $(\bar\rho, \bar u)$ a strong conservative solution of the Euler relaxation system

$$\frac{d}{dt}\left(\mathcal{E}(\rho\,|\bar\rho) + \int \frac{\varepsilon^2}{2}|u-\bar u|^2 dx\right) + \int \rho|u-\bar u|^2 dx$$
$$= -\int \left(\varepsilon^2 \rho \nabla_x \bar u : (u-\bar u)\otimes(u-\bar u) + \nabla \bar u : S(\rho\,|\bar\rho)\right) dx$$

used to compare (ρ, u) and $(\bar\rho, \bar u)$ and to establish convergence results from relaxation system to diffusion theory

Lattanzio - AT 17

Applications in several contexts:
From gas dynamics to porous media equation
From Euler-Poisson w. attracting potentials to Keller-Segel systems,
From Euler-Korteweg with nonmonotone pressure to Cahn-Hilliard

Lattanzio, Gieselmann, Spirito, Antonelli, Cianfarani Carnevale, ...

example from Euler-Korteweg to Cahn-Hilliard

Let (ρ, u) be a dissipative solution of the Euler-Korteweg system (with $\kappa = 1$) with friction and $\bar\rho$ a smooth solution of the Cahn-Hilliard equation

$$\rho_t = \operatorname{div}\left(\nabla p(\rho) - C_\kappa \rho \nabla \triangle \rho\right)$$

Thm If $p'(\rho) > 0$, $h(\rho) = \frac{1}{\gamma-1}\rho^\gamma + o(\rho^\gamma)$ and $|p''(\rho)| \le A\frac{p'(\rho)}{\rho}$, and

$$\gamma \ge 2 \qquad \text{or} \qquad \left\{\ 1 < \gamma < 2 \quad \text{and} \quad \int \rho_0 = \int \bar\rho_0\ \right\}$$

then

$$\Psi(t) = \int_{\mathbb{T}^d} \tfrac{1}{2}\rho|u - \bar u|^2 + h(\rho|\bar\rho) + \frac{1}{2}|\nabla(\rho - \bar\rho)|^2 dx \to 0$$

as $\varepsilon \to 0$ where $\frac{1}{\varepsilon}$ is the friction coefficient.

example Bipolar Euler-Poisson model, two electrically charged fluids

$$
\begin{aligned}
\rho_t + \nabla \cdot (\rho u) &= 0 \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla p_1(\rho) &= -\rho \nabla \phi - \frac{1}{\tau} \rho u \\
n_t + \nabla \cdot (n v) &= 0 \\
(n v)_t + \nabla \cdot (n v \otimes v) + \nabla p_2(n) &= n \nabla \phi - \frac{1}{\tau} n v \\
-\Delta \phi &= \rho - n
\end{aligned}
$$

Energy identity

$$
\frac{d}{dt} \left(\int \tfrac{1}{2} \rho |u|^2 + \tfrac{1}{2} n |v|^2 \, dx + \mathcal{E}(\rho, n) \right) + \frac{1}{\tau} \int \rho |u|^2 + n |v|^2 \, dx
$$

$$
\begin{aligned}
\mathcal{E}(\rho, n) &= \int_\Omega h_1(\rho) + h_2(n) + \frac{1}{2} |\nabla \phi|^2 dx, \\
-\Delta \phi &= \rho - n
\end{aligned}
$$

Alves - AT, 20

from bipolar Euler-Poisson to drift-diffusion

Convergence as $\tau \to 0$ to the bipolar drift-diffusion model used in the analysis of semiconductors (after scaling $t \to \frac{t}{\tau}$, $u \to \tau u$, $v \to \tau v$)

$$\begin{cases} \rho_t = \nabla \cdot \big(\nabla p_1(\rho) + \rho \nabla \phi \big) \\ n_t = \nabla \cdot \big(\nabla p_2(n) - n \nabla \phi \big) \\ -\Delta \phi = \rho - n. \end{cases}$$

Relaxation from Euler-Poisson (unipolar) to Drift-Diffusion — Lattanzio-Marcati 99

Bipolar Euler-Poisson (1d) to Drift-Diffusion — Natalini 96

Alves - AT, 20

ZEM and QN limit

$$
\begin{aligned}
\rho_t + \nabla \cdot (\rho u) &= 0 \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla p_1(\rho) &= -\rho \nabla \phi \\
n_t + \nabla \cdot (n v) &= 0 \\
\varepsilon\Big((nv)_t + \nabla \cdot (nv \otimes v) \Big) + \nabla p_2(n) &= n \nabla \phi \\
-\delta \Delta \phi &= \rho - n
\end{aligned}
$$

Existence of global solutions for irrotational velocities and densities near equilibrium for Euler-Poisson / Euler-Maxwell

Yan Guo, B. Pausader 11 / Yan Guo, Ionescu, B. Pausader 16

Zero-electron mass limit $\varepsilon \to 0$ from weak solutions to smooth solutions

$$
\begin{aligned}
\rho_t + \nabla \cdot (\rho u) &= 0 \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla p_1(\rho) + \rho \nabla h_2'(n) &= 0 \\
-\delta \Delta h_2'(n) &= \rho - n
\end{aligned}
$$

ZEM limit $\varepsilon \to 0$ plus Quasi-neutral limit $\delta \to 0$ from weak solutions to smooth solutions

$$
\begin{aligned}
\rho_t + \nabla \cdot (\rho u) &= 0 \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla \big(p_1(\rho) + p_2(\rho) \big) &= 0
\end{aligned}
$$

Alves - AT, 23

Part III Maxwell-Stefan system via relaxation

$$
(MS) \qquad \begin{cases} \partial_t \rho_i + \operatorname{div}(\rho_i u_i) = 0 \qquad i = 1, \dots, n \\[1em] \text{where } \rho_i u_i \text{ is determined by } \begin{cases} -\sum_j b_{ij} \rho_i \rho_j (u_i - u_j) = \nabla \rho_i \\[1em] \sum_i \rho_i u_i = 0 \end{cases} \end{cases}
$$

$\sum \rho_i = 1$ is a constraint propagating from data.

More generally

$$
\begin{cases} -\sum_j b_{ij} \rho_j (u_i - u_j) = \nabla \mu_i \qquad \text{chemical potential } \mu_i = \frac{\partial}{\partial \rho_i}(\rho \Psi) \\[1em] \sum_i \rho_i u_i = 0 \end{cases}
$$

MS as high-friction limit for isothermal multi-component flows

$(\rho_1, \dots, \rho_n, v_1, \dots, v_n)$ the densities/velocities of a multi-component fluid

$$(MCF) \quad \begin{cases} \partial_t \rho_i + \operatorname{div}(\rho_i v_i) = 0 \\ \partial_t(\rho_i v_i) + \operatorname{div}(\rho_i v_i \otimes v_i) = -\rho_i \nabla \frac{\delta \mathcal{E}}{\delta \rho_i} - \frac{1}{\varepsilon} \sum_j b_{ij} \rho_i \rho_j (v_i - v_j) \end{cases}$$

$$\frac{d}{dt}\left(\mathcal{E}(\rho) + \int \sum_i \tfrac{1}{2} \rho_i |v_i|^2 \right) + \tfrac{1}{2\varepsilon} \int \sum_{i,j} b_{ij} \rho_i \rho_j |v_1 - v_j|^2 = 0$$

$b_{ij} = b_{ji} > 0$

We are interested in the high friction limit $\varepsilon \to 0$

Moments

$$\rho = \sum \rho_i \quad \rho v = \sum_i \rho_i v_i \quad u_i := v_i - v$$

ρ total density (conserved); v barycentric velocity; u_i relative velocities

The Chapman-Enskog expansion calculates the (formal) effective equation for $(\rho_1, \dots, \rho_n)$, v, $(u_1, \dots, u_n)$

which up to order $O(\varepsilon^2)$ is

$$\begin{cases} \partial_t \rho_i + \operatorname{div}(\rho_i v) = -div(\rho_i u_i) \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) = -\sum_i \rho_i \nabla \frac{\delta \mathcal{E}}{\delta \rho_i} \end{cases}$$

$$-\sum_j b_{ij} \rho_i \rho_j (u_i - u_j) = \varepsilon \left(\rho_i \nabla \frac{\delta \mathcal{E}}{\delta \rho_i} - \frac{\rho_i}{\rho} \sum_j \rho_j \nabla \frac{\delta \mathcal{E}}{\delta \rho_j} \right)$$

$$\sum_i \rho_i u_i = 0$$

It can be expressed in the form

$$(HP) \qquad \begin{cases} \partial_t \rho_i + \operatorname{div}(\rho_i v) = \varepsilon \operatorname{div}\Big(\sum_{j=1}^n D_{ij} \nabla \frac{\delta \mathcal{E}}{\delta \rho_j} \Big) \\ \partial_t (\rho v) + \operatorname{div}(\rho v \otimes v) = - \sum_{i=1}^n \rho_i \nabla \frac{\delta \mathcal{E}}{\delta \rho_i} \end{cases}$$

where

$$D^{n \times n} = G Q^{-1}(\rho) \tau^{-1} Q^{-1}(\rho) G^T \qquad D \geq 0$$

(HP) is a Hyperbolic - Parabolic system

$$\partial_t \sum \rho_i + \operatorname{div}\big(\sum \rho_i v\big) = 0$$

$$\partial_t \Big(\mathcal{E}[\rho] + \int \tfrac{1}{2} \big(\sum_i \rho_i\big) |v|^2 \, dx \Big) + \varepsilon \int \sum_{ij} \nabla \frac{\delta \mathcal{E}}{\delta \rho_i} D_{ij} \nabla \frac{\delta \mathcal{E}}{\delta \rho_j} = 0$$

Relation between (MCF) and (HP)

Thm Under hypothesis

$$(H) \quad \text{Hypothesis} \qquad -\rho_i \nabla \frac{\delta \mathcal{E}}{\delta \rho_i} = \operatorname{div} S_i(\rho) \quad i = 1, \dots, n$$

(ρ_i, v_i) is solution of the relaxation system (MCF)
$(\hat{\rho}_i, \hat{v}_i)$ solution of the Chapman-Enskog expansion (HP)

the relative entropy

$$\chi(t) = \int \sum_i \tfrac{1}{2}\rho_i |v_i - \hat{v}_i|^2 + |\rho_i - \hat{\rho}_i|^2 + \tfrac{1}{2}\kappa(\rho_i)|k_i(\rho_i)\nabla\rho_i - k_i(\hat{\rho}_i)\nabla\hat{\rho}_i|^2$$

satisfies

$$\chi(t) \leq C(\chi(0) + \varepsilon^2)$$

Huo - Juengel - T. 2019

Validates Chapman-Enskog, convergence to zero-relaxation limit for gas-dynamics and Euler-Korteweg models

Mathematical Properties of the classical Maxwell-Stefan system

$$\mathcal{E}(\rho) = \int \sum_i \rho_i \ln \rho_i \qquad \sum_i \rho_i = 1$$

Maxwell-Stefan diffusion

$$(MS) \qquad \begin{cases} \partial_t \rho_i + \operatorname{div}(\rho_i u_i) = 0 \\ \text{where } \rho_i u_i \text{ is} \begin{cases} -\sum_j b_{ij}\rho_i\rho_j(u_i - u_j) = \nabla \rho_i \\ \sum_i \rho_i u_i = 0 \end{cases} \end{cases}$$

$\sum \rho_i = 1$ is a constraint propagating from data.

Existence

strong solutions Bothe 2011 weak solutions Juengel - Stelzer 2013

Weak solutions are constructed in the energy space

$$\int \sum_i \rho_i \ln \rho_i \, dx + \beta \int_0^t \int \sum_i |\nabla \sqrt{\rho_i}|^2 \, dx \le \int \sum_i \rho_{0i} \ln \rho_{0i} \, dx$$

weak-strong Uniqueness

Thm Suppose that $0 < \rho_i < 1$ $\sum \rho_i = 1$, $0 < \bar{\rho}_i < 1$, $\sum \bar{\rho}_i = 1$ two weak solutions of (MS) system

If $\bar{\rho}_i$ does not present anomalous diffusion, and $\bar{v}_i \in L^\infty$ then we have (weak-strong) uniqueness.

Huo - Juengel - T. 22

Maxwell-Stefan diffusion

compare two solutions

$0 < \rho_i < 1,\ u_i$

$0 < \bar{\rho}_i < 1,\ \bar{u}_i$

of (MS) system

$$\begin{cases} \partial_t \rho_i + \operatorname{div}(\rho_i u_i) = 0 \\ \begin{cases} -\sum_j b_{ij} \rho_j (u_i - u_j) = \nabla \ln \rho_i \\ \sum_i \rho_i u_i = 0 \end{cases} \end{cases}$$

$$\frac{d}{dt} \int \sum_i \left(\rho_i \ln \rho_i - \bar{\rho}_i \ln \bar{\rho}_i - (1 + \ln \bar{\rho}_i)(\rho_i - \bar{\rho}_i) \right)$$

$$= \sum_i \int \nabla_x (\ln \rho_i - \ln \bar{\rho}_i) \cdot \rho_i (u_i - \bar{u}_i) dx$$

$$\frac{d}{dt} \mathcal{E}(\rho|\bar{\rho}) + \int \underbrace{\sum_{i,j} \frac{1}{2} b_{ij} \rho_i \rho_j \left| (u_i - u_j) - (\bar{u}_i - \bar{u}_j) \right|^2}_{Q(\rho)}$$

$$= - \int \sum_i \rho_i (u_i - \bar{u}_i) \cdot \sum_j b_{ij} (\rho_j - \bar{\rho}_j)(\bar{u}_i - \bar{u}_j)$$

the frictional dissipation

$$
\begin{aligned}
Q(\rho) &= \sum_{i,j} \frac{1}{2} b_{ij}\rho_i\rho_j \Big| (u_i - u_j) - (\bar{u}_i - \bar{u}_j) \Big|^2 \geq 0 \\
&= \sum_{ij} \sqrt{\rho_i}(u_i - \bar{u}_i) \cdot \Big(B_S(\rho) \Big)_{ij} \sqrt{\rho_j}(u_j - \bar{u}_j)
\end{aligned}
$$

$$
\text{where} \quad \Big(B_S(\rho) \Big)_{ij} = \begin{cases} \sum_{k \neq i} b_{ik}\rho_k & i = j \\ -b_{ij}\sqrt{\rho_i}\sqrt{\rho_j} & i \neq j \end{cases}
$$

$$
B = \begin{pmatrix} \sum_{k\neq 1} b_{1k} & -b_{12} & -b_{13} & -b_{14} \\ -b_{21} & \sum_{k\neq 2} b_{2k} & -b_{23} & -b_{24} \\ -b_{31} & -b_{32} & \sum_{k\neq 3} b_{3k} & -b_{34} \\ -b_{41} & -b_{42} & -b_{43} & \sum_{k\neq 4} b_{4k} \end{pmatrix}
$$

b_{ij} symmetric, irreducible, for simplicity $b_{ij} > 0$

Ingredients $B_S(\rho)$ symmetric, positive semidefinite

$$\mathcal{N}(B_S) = \text{span}\underbrace{\left(\sqrt{\rho_1}, \dots, \sqrt{\rho_n}\right)^T}_{\sqrt{\rho}} \quad \mathcal{R}(B_S) = \mathcal{N}(B_S)^{\perp} = \{x \cdot \sqrt{\rho} = 0\}$$

$\mathcal{N}(B_S)$ has dimension 1 and positive eigenvector

$\sigma(B_S(\rho)) \subset \{0\} \cup [d, \Delta]$ where $\delta = \min b_{ij}$, $\Delta = \max b_{ij}$

$$\Delta |PX|^2 \le Q(\rho) := X \cdot B_S(\rho) X \ge \delta |PX|^2$$

where P projection on $\mathcal{R}(B_S)$
$X \in \mathbb{R}^n = \mathcal{N}(B_S) \oplus \mathcal{R}(B_S)$

$$\frac{d}{dt}\mathcal{E}(\rho|\bar{\rho}) + \delta \int |P\sqrt{\rho}(u - \bar{u})|^2$$
$$\le \frac{\delta}{2} \int |P\sqrt{\rho}(u - \bar{u})|^2 dx + C_\delta \int \sum_i |\rho_i - \bar{\rho}_i|^2$$

Part IV, Variational approximation of anisotropic diffusion

- $$\partial_t \rho + \text{div}\,(\rho u) = 0 \qquad u = -\nabla_x \frac{\delta \mathcal{E}}{\delta \rho}$$

- Diffusion theory arises by variational minimization based on Wasserstein distance, Jordan-Kinderlehrer-Otto scheme

$$\rho^{n+1} \text{ is the minimizer of the problem} \quad \min \left\{ \frac{1}{2\tau} d_W(\rho, \rho^n)^2 + \mathcal{E}[\rho] \right\}$$

- Brenier-Benamou formula

$$d_W(\rho_0, \rho_1)^2 = \inf_{(\rho,u)} \left\{ \tau \int_0^\tau \int \rho |u|^2 \, dxdt \;\middle|\; \begin{array}{r} \partial_t \rho + \text{div}\,\rho u = 0 \\ \rho(0) = \rho_0\,, \;\; \rho(\tau) = \rho_1 \end{array} \right\}$$

$$\partial_t \rho = \operatorname{div}\Big(A(x) \nabla \rho \Big) \qquad\qquad A(x) > 0 \text{ and symmetric}$$

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$$= \operatorname{div}\Big(\rho A(x) \nabla \frac{\delta \mathcal{E}}{\delta \rho} \Big) \qquad \mathcal{E}(\rho) = \int \rho \ln \rho \, dx$$

Visualize this diffusion as

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \qquad u = -A(x) \nabla \frac{\delta \mathcal{E}}{\delta \rho}$$

small mass approximation of the Euler system

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$

$$\varepsilon^2 \rho \Big(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \Big) = -\Big(\rho B(x) u + \rho \nabla \frac{\delta \mathcal{E}}{\delta \rho} \Big) \qquad B(x) = A^{-1}(x) > 0$$

$$\partial_t \Big(\mathcal{E}[\rho] + \int \frac{\varepsilon^2}{2} \rho |u|^2 dx \Big) + \int \rho u \cdot B(x) u dx = 0$$

Analog of the Brenier-Benamou formula

$$W_A(\rho_0, \rho_1)^2 = \inf_{(\rho, v)} \left\{ \tau \int_0^\tau \int v \cdot B(x) v \, \rho dx ds \;\middle|\; \begin{array}{c} \partial_s \rho + \operatorname{div} \rho v = 0 \\ \rho(0) = \rho_0 \, , \; \rho(\tau) = \rho_1 \end{array} \right\}$$

- minimum is achieved
- $B(x) = (\nabla_x b)^T (\nabla_x b)$ $\quad\quad b : (\mathbb{R}^d, B) \to (R^N, Euclidean)$
  secured by isometric embedding theorem of Nash-Kuiper
- defines a 2-Wasserstein distance associated to the friction matrix $A(x)$ (or the mobility matrix $B(x)$)

H. Liu - AT 22

Analog of the Jordan-Kinderlehrer-Otto scheme

$$\rho^{n+1} \text{ is the minimizer of the problem} \quad \min_{\rho \in K} \left\{ \frac{1}{2\tau} W_A(\rho, \rho^n)^2 + \int \rho \ln \rho \, dx \right\}$$

Variational scheme approximates implicit Euler Scheme of the form

$$\frac{\rho^{n+1} - \rho^n}{\tau} = \operatorname{div}_x \Big(\rho^{n+1} A(x) \nabla_x \frac{\delta \mathcal{E}}{\delta \rho}(\rho^{n+1}) \Big)$$

and as $\tau \to 0$, $\rho^\tau(x,t) \to \rho(x,t)$ with

$$\partial_t \rho = \operatorname{div} \Big(\rho A(x) \nabla \ln \rho \Big)$$

Lisini 09

To PIERO MARCATI with friendship and admiration

with best wishes for many, healthy and creative years