The two-dimensional plasma-vacuum interface problem in ideal MHD

Paolo Secchi

UNIVERSITY OF BRESCIA

Joint work with A. Morando, Y. Trakhinin, P. Trebeschi & D. Yuan

Int. Conf. on PDEs and Applications in honor of the 70th birthday of Pierangelo Marcati, GSSI L'Aquila, June 22, 2023 manuscripta math. 60, 49 - 69 (1988)

manuscripta mathematica

Singular Convergence of Weak Solutions for a Quasilinear Nonhomogeneous Hyperbolic System.

Pierangelo Marcati
Albert J. Milani
Paolo Secchi

SUMMARY: We show that the weak solutions of the nonlinear hyperbolic system

$$\begin{cases} \varepsilon\, u^\varepsilon_t + p(v^\varepsilon)_x = -u^\varepsilon \\ v^\varepsilon_t - u^\varepsilon_x = 0 \end{cases}$$

converge, as ε tends to zero, to the solutions of the reduced problem

$$\begin{cases} u + p(v)_x = 0 \\ v_t - u_x = 0. \end{cases}$$

Then they satisfy the nonlinear parabolic equation

$$v_t + p(v)_{xx} = 0.$$

The limiting procedure is carried out using the techniques of "Compensated Compactness". Some connections with the theory of nonlinear heat conduction and the theory of nonlinear diffusion in a porous medium are suggested. The main result is stated in th. (2.9).

Plan

1. Introduction
   - The MHD equations for compressible fluids
   - 3D problem, stability condition
2. 2D Plasma-vacuum interface problem
   - Stability condition
   - Linearized problem
     - Hyperbolic approximation
     - Secondary symmetrization
   - Nonlinear problem, main result

The MHD equations for compressible fluids

Ideal Compressible Magneto-hydrodynamics (MHD)

$$\begin{array}{l} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0\,, \\ \partial_t(\rho \mathbf{u}) + \mathrm{div}(\rho\, \mathbf{u} \otimes \mathbf{u} - \mathbf{H} \otimes \mathbf{H}) + \nabla(p + \frac{1}{2}|\mathbf{H}|^2) = 0\,, \\ \partial_t \mathbf{H} - \nabla \times (\mathbf{u} \times \mathbf{H}) = 0\,, \\ \partial_t(\rho E + \frac{1}{2}|\mathbf{H}|^2) + \mathrm{div}((\rho E + p)\mathbf{u} + \mathbf{H} \times (\mathbf{u} \times \mathbf{H})) = 0\,, \end{array} \qquad (1)$$

ρ density, $\mathbf{u} \in \mathbb{R}^3$ plasma velocity, $\mathbf{H} \in \mathbb{R}^3$ magnetic field, p pressure , S entropy, $E = e + \frac{1}{2}|\mathbf{u}|^2$ total energy, e internal energy.

The MHD equations for compressible fluids

Ideal Compressible Magneto-hydrodynamics (MHD)

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho\mathbf{u}) = 0\,, \\ \partial_t(\rho\mathbf{u}) + \mathrm{div}(\rho\,\mathbf{u}\otimes\mathbf{u} - \mathbf{H}\otimes\mathbf{H}) + \nabla(p + \frac{1}{2}|\mathbf{H}|^2) = 0\,, \\ \partial_t\mathbf{H} - \nabla\times(\mathbf{u}\times\mathbf{H}) = 0\,, \\ \partial_t(\rho E + \frac{1}{2}|\mathbf{H}|^2) + \mathrm{div}((\rho E + p)\mathbf{u} + \mathbf{H}\times(\mathbf{u}\times\mathbf{H})) = 0\,, \end{cases} \tag{1}$$

ρ density, $\mathbf{u}\in\mathbb{R}^3$ plasma velocity, $\mathbf{H}\in\mathbb{R}^3$ magnetic field, p pressure , S entropy, $E = e + \frac{1}{2}|\mathbf{u}|^2$ total energy, e internal energy.

Under a state equation $\rho = \rho(p, S)$ (such that $\rho_p > 0$) and the 1-st principle of thermodynamics, (1) becomes a closed system for the unknowns $(p, \mathbf{u}, \mathbf{H}, S)$.

The MHD equations for compressible fluids

Ideal Compressible Magneto-hydrodynamics (MHD)

$$\begin{array}{l} \partial_t\rho + \mathrm{div}(\rho\mathbf{u}) = 0\,, \\ \partial_t(\rho\mathbf{u}) + \mathrm{div}(\rho\,\mathbf{u}\otimes\mathbf{u} - \mathbf{H}\otimes\mathbf{H}) + \nabla(p + \frac{1}{2}|\mathbf{H}|^2) = 0\,, \\ \partial_t\mathbf{H} - \nabla\times(\mathbf{u}\times\mathbf{H}) = 0\,, \\ \partial_t(\rho E + \frac{1}{2}|\mathbf{H}|^2) + \mathrm{div}((\rho E + p)\mathbf{u} + \mathbf{H}\times(\mathbf{u}\times\mathbf{H})) = 0\,, \end{array} \tag{1}$$

ρ density, $\mathbf{u}\in\mathbb{R}^3$ plasma velocity, $\mathbf{H}\in\mathbb{R}^3$ magnetic field, p pressure , S entropy, $E = e + \frac{1}{2}|\mathbf{u}|^2$ total energy, e internal energy.

Under a state equation $\rho = \rho(p, S)$ (such that $\rho_p > 0$) and the 1-st principle of thermodynamics, (1) becomes a closed system for the unknowns $(p, \mathbf{u}, \mathbf{H}, S)$.

(1) is supplemented by the divergence constraint on the initial data

$$\mathrm{div}\,\mathbf{H} = 0\,. \tag{2}$$

3D Plasma-vacuum interface problem

In terms of $\mathbf{U} = (q, \mathbf{u}, \mathbf{H}, S)^T$ (where $q = p + \frac{1}{2}|\mathbf{H}|^2$ is the total pressure) system (1) admits the symmetrization

$$\begin{pmatrix} \rho_p/\rho & \underline{0}^T & -(\rho_p/\rho)\mathbf{H} & 0 \\ \underline{0} & \rho I_3 & 0_3 & \underline{0} \\ -(\rho_p/\rho)\mathbf{H}^T & 0_3 & I_3 + (\rho_p/\rho)\mathbf{H}\otimes\mathbf{H} & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & 1 \end{pmatrix} \partial_t \begin{pmatrix} q \\ \mathbf{u} \\ \mathbf{H} \\ S \end{pmatrix} +$$

$$\begin{pmatrix} (\rho_p/\rho)\mathbf{u}\cdot\nabla & \nabla\cdot & -(\rho_p/\rho)\mathbf{H}\mathbf{u}\cdot\nabla & 0 \\ \nabla & \rho\mathbf{u}\cdot\nabla I_3 & -\mathbf{H}\cdot\nabla I_3 & \underline{0} \\ -(\rho_p/\rho)\mathbf{H}^T\mathbf{u}\cdot\nabla & -\mathbf{H}\cdot\nabla I_3 & (I_3 + (\rho_p/\rho)\mathbf{H}\otimes\mathbf{H})\mathbf{u}\cdot\nabla & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & \mathbf{u}\cdot\nabla \end{pmatrix} \begin{pmatrix} q \\ \mathbf{u} \\ \mathbf{H} \\ S \end{pmatrix} = 0 \tag{3}$$

where $\underline{0} = (0,0,0)^T$.

Then (3) reads as a symmetric hyperbolic system

$$A_0(\mathbf{U})\partial_t \mathbf{U} + \sum_{j=1}^{3} A_j(\mathbf{U})\partial_j \mathbf{U} = 0\,, \qquad A_j(\mathbf{U}) \text{ symmetric}\,,\ A_0(\mathbf{U}) > 0\,.$$

under the *hyperbolicity* conditions

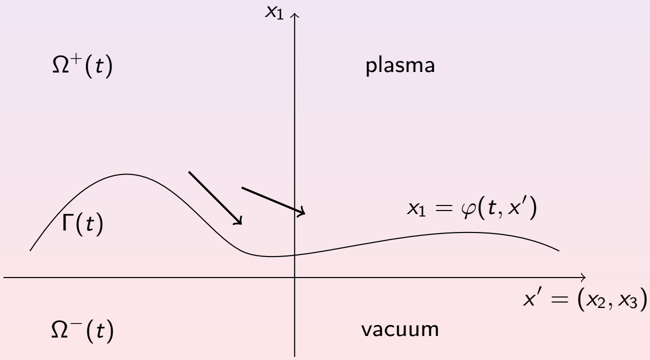
$$\rho > 0\,, \quad \rho_p > 0\,.$$

Given the smooth hypersurface

$$\Gamma(t) = \{x_1 = \varphi(t, x_2, x_3)\} \qquad \text{in } [0, T] \times \mathbb{R}^3,$$

we denote $\Omega^\pm(t) = \mathbb{R}^3 \cap \{x_1 \gtrless \varphi(t, x')\}$ (here $x' = (x_2, x_3)$).

The plasma is governed by equations (3) in the region $\Omega^+(t) = \mathbb{R}^3 \cap \{x_1 > \varphi(t, x')\}$:



The vacuum region is $\Omega^-(t) = \mathbb{R}^3 \cap \{x_1 < \varphi(t,x')\}$, where we assume the so-called *pre-Maxwell dynamics*:

$$\nabla \times \mathcal{H} = 0, \qquad \mathrm{div}\,\mathcal{H} = 0, \tag{4}$$

$$\nabla \times E = -\partial_t \mathcal{H}, \qquad \mathrm{div}\,E = 0, \tag{5}$$

$\mathcal{H}$ denotes the vacuum magnetic field and E the electric field.

As usual in nonrelativistic MHD, we neglect the displacement current $(1/c)\,\partial_t E$, where c is the speed of light.

From (5) the electric field E is a secondary variable that may be computed from the magnetic field $\mathcal{H}$. Hence, in the vacuum only one basic variable is needed, viz. $\mathcal{H}$, satisfying (4).

On the moving interface $\Gamma(t)$ the plasma and the vacuum magnetic fields are related by:

$$\partial_t\varphi = u_N, \quad [q]=0, \quad H_N = 0, \quad \mathcal{H}_N = 0 \quad \text{on } \Gamma(t), \tag{6}$$

where $u_N^\pm := \mathbf{u}^\pm \cdot N\,, \quad H_N := \mathbf{H}\cdot N\,, \quad \mathcal{H}_N := \mathcal{H}\cdot N\,, \quad N := (1, -\partial_2\varphi, -\partial_3\varphi)$ and $[q] = q|_\Gamma - \frac{1}{2}|\mathcal{H}|^2_{|\Gamma}$.

The interface $\Gamma(t)$ moves with the plasma. The total pressure is continuous across $\Gamma(t)$. The magnetic field on both sides is tangent to $\Gamma(t)$.

The function φ describing the interface is one unknown of the problem, i.e. this is a free boundary problem.

System (4) for the vacuum magnetic field $\mathcal{H}$,

$$\nabla \times \mathcal{H} = 0, \qquad \mathrm{div}\, \mathcal{H} = 0, \tag{4}$$

is elliptic. Plasma-vacuum problem (3), (4) is a coupled hyperbolic-elliptic system.

System (4) for the vacuum magnetic field $\mathcal{H}$,

$$\nabla \times \mathcal{H} = 0, \qquad \mathrm{div}\, \mathcal{H} = 0, \tag{4}$$

is elliptic. Plasma-vacuum problem (3), (4) is a coupled hyperbolic-elliptic system.

In (4) time t plays the role of a parameter. Time dependence of $\mathcal{H}$ comes from the coupling with the plasma variables through the boundary conditions (6) at the moving front $\Gamma(t)$.

System (4) for the vacuum magnetic field $\mathcal{H}$,

$$\nabla \times \mathcal{H} = 0, \qquad \mathrm{div}\, \mathcal{H} = 0, \tag{4}$$

is elliptic. Plasma-vacuum problem (3), (4) is a coupled hyperbolic-elliptic system.

In (4) time t plays the role of a parameter. Time dependence of $\mathcal{H}$ comes from the coupling with the plasma variables through the boundary conditions (6) at the moving front $\Gamma(t)$.

System (3), (4), (6) is supplemented with initial conditions

$$\begin{array}{ll} U(0,x) = U_0(x)\,, \quad \mathcal{H}(0,x) = \mathcal{H}_0(x)\,, \quad x \in \Omega^{\pm}(0)\,, \\ \varphi(0,x') = \varphi_0(x')\,, \quad x' \in \Gamma(0), \end{array} \tag{7}$$

where $\mathrm{div}\, H_0 = 0$ in $\Omega^+(0)$, $\mathrm{div}\, \mathcal{H}_0 = 0$ in $\Omega^-(0)$, $H_{0,N} = \mathcal{H}_{0,N} = 0$ on $\Gamma(0)$.

Motivation from astrophysics:

the study of stars or the solar corona

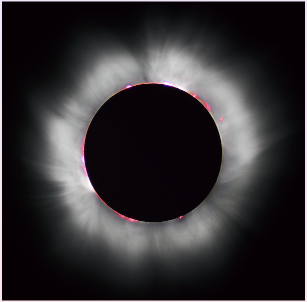
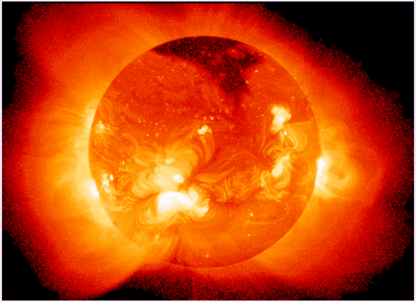
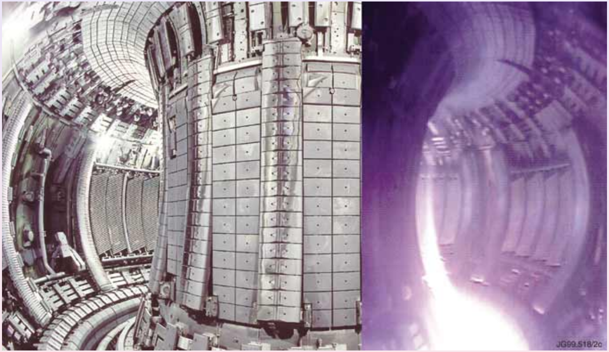


Image by Luc Viatour.

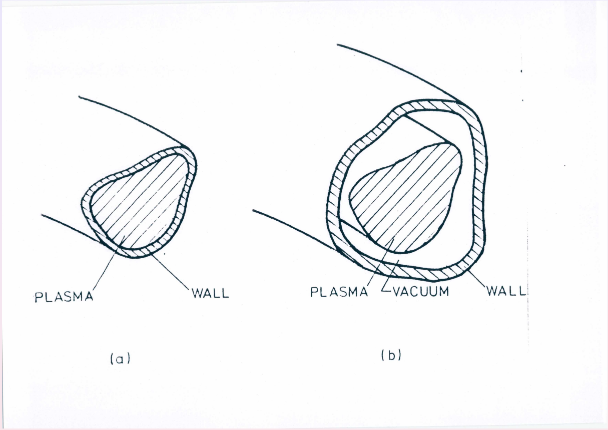


From Yohkoh satellite (Courtesy by JAXA)

Other motivation: the study of magnetic confinement for nuclear fusion



Tunnel at Monte Carlo (Courtesy by JET)



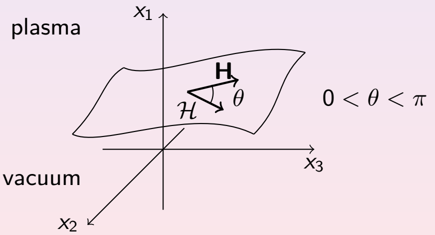
A toroidal plasma configuration: (a) surrounded by a perfectly conducting wall; (b) isolated from a wall by a vacuum region.

The 3D stability condition

We have solved[1] the problem (3), (4), (6), (7) under the stability condition

$$|\mathbf{H} \times \mathcal{H}| > 0 \qquad \text{on } [0,T] \times \Gamma, \tag{8}$$

i.e. the magnetic fields on the two sides of the free-boundary are noncollinear.



[1] [P.S. & Y.Trakhinin, Interfaces FB 2013, Nonlinearity 2014]

2D MHD equations for compressible fluids

$2D$ planar MHD flows means

- the flow is x_3-invariant, i.e $\mathbf{U} = \mathbf{U}(t, x_1, x_2)$
- the velocity and the magnetic field are shearless $(u_3 = H_3 = 0)$.

From the system it follows that for x_3-invariant flows

$$u_3|_{t=0} = H_3|_{t=0} = 0 \ \Rightarrow \ u_3 = H_3 = 0, \ \forall t > 0$$

hence the restriction that the velocity and the magnetic field are shearless at the first moment guarantees that the $2D$-flows are planar.

For $\mathbf{U}=\mathbf{U}(t,x)=(q,\mathbf{u},\mathbf{H},S)$, with $t\in[0,T]$, $x=(x_1,x_2)\in\mathbb{R}^2$, $\mathbf{u}=(u_1,u_2)$, $\mathbf{H}=(H_1,H_2)$, the MHD equations (3) reduce to

$$\begin{pmatrix} \rho_p/\rho & \underline{0}^T & -(\rho_p/\rho)\mathbf{H} & 0 \\ \underline{0} & \rho I_2 & 0_2 & \underline{0} \\ -(\rho_p/\rho)\mathbf{H}^T & 0_2 & I_2+(\rho_p/\rho)\mathbf{H}\otimes\mathbf{H} & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & 1 \end{pmatrix} \partial_t \begin{pmatrix} q \\ \mathbf{u} \\ \mathbf{H} \\ S \end{pmatrix} +$$

$$\begin{pmatrix} (\rho_p/\rho)\mathbf{u}\cdot\nabla & \nabla\cdot & -(\rho_p/\rho)\mathbf{H}\mathbf{u}\cdot\nabla & 0 \\ \nabla & \rho\mathbf{u}\cdot\nabla I_2 & -\mathbf{H}\cdot\nabla I_2 & \underline{0} \\ -(\rho_p/\rho)\mathbf{H}^T\mathbf{u}\cdot\nabla & -\mathbf{H}\cdot\nabla I_2 & (I_2+(\rho_p/\rho)\mathbf{H}\otimes\mathbf{H})\mathbf{u}\cdot\nabla & \underline{0} \\ 0 & \underline{0}^T & \underline{0}^T & \mathbf{u}\cdot\nabla \end{pmatrix} \begin{pmatrix} q \\ \mathbf{u} \\ \mathbf{H} \\ S \end{pmatrix} = 0 \tag{9}$$

where $\underline{0}=(0,0)^T$, that we write again as a symmetric hyperbolic system

$$A_0(\mathbf{U})\partial_t\mathbf{U}+\sum_{j=1}^{2}A_j(\mathbf{U})\partial_j\mathbf{U}=0\,, \qquad A_j(\mathbf{U}) \text{ symmetric}\,,\ \ A_0(\mathbf{U})>0\,.$$

under the *hyperbolicity* conditions

$$\rho>0\,, \quad \rho_p>0\,.$$

2D Plasma-vacuum interface problem

The 2D plasma-vacuum interface problem amounts to find smooth solutions $\mathbf{U} = (q, \mathbf{u}, \mathbf{H}, S)$, $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2)$, φ of the following IBVP:

$$\begin{aligned} A_0(\mathbf{U})\partial_t \mathbf{U} + \sum_{j=1}^{2} A_j(\mathbf{U})\partial_j \mathbf{U} = 0, \qquad & x \in \Omega^+(t)\,, \\ \mathrm{curl}\,\mathcal{H} = 0, \quad \mathrm{div}\,\mathcal{H} = 0, \qquad & x \in \Omega^-(t)\,, \\ \partial_t\varphi - u_N = 0, \quad [q] = 0, \qquad & x \in \Gamma(t)\,, \end{aligned} \tag{10}$$

for given initial data

$$\begin{aligned} &\mathbf{U}(0,x) = \mathbf{U}_0(x), \quad \mathcal{H}(0,x) = \mathcal{H}_0(x) \quad x \in \Omega^\pm(0), \\ &\varphi(0,x_2) = \varphi_0(x_2), \quad x_2 \in \mathbb{R}\,, \end{aligned} \tag{11}$$

such that $\mathrm{div}\,\mathbf{H}_0 = 0$, $\mathrm{div}\,\mathcal{H}_0 = 0$ in $\Omega^\pm(0)$ and $\mathbf{H}_{0,N} = \mathcal{H}_{0,N} = 0$ on $\Gamma(0)$.

$\mathrm{curl}\,\mathcal{H} := \partial_1\mathcal{H}_2 - \partial_2\mathcal{H}_1$

The 2D stability condition

Remark
The 3D stability condition fails (8) when passing to 2D, that is 3D stability cannot be reduced to 2D stability directly!

We solve the problem (10), (11) under the stability condition

$$|\mathbf{H}| + |\mathcal{H}| > 0 \qquad \text{on } [0,T]\times\Gamma, \tag{12}$$

i.e. at least one of the two magnetic fields is non-zero at the free-boundary.

(12) shows again the stabilizing effect of magnetic fields, not necessarily large (different from current-vortex sheets).

Remark

In both 2D and 3D the stability conditions yield the ellipticity of the symbol of the front function:

- In 3D $\quad (N := (1, -\partial_2\varphi, -\partial_3\varphi))$

$$|\mathbf{H} \times \mathcal{H}| > 0, \quad \mathbf{H} \cdot N = \mathcal{H} \cdot N = 0 \quad \Rightarrow \quad H_2\mathcal{H}_3 - H_3\mathcal{H}_2 \neq 0 \quad \text{on } \Gamma,$$

and we may solve for $\nabla_{t,x'}\varphi$ the system of equations

$$\partial_t\varphi - u_N = 0, \quad \mathbf{H} \cdot N = 0, \quad \mathcal{H} \cdot N = 0,$$

that is:

$$\begin{cases} \partial_t\varphi + u_2\partial_2\varphi + u_3\partial_3\varphi = u_1 \\ H_2\partial_2\varphi + H_3\partial_3\varphi = H_1 \\ \mathcal{H}_2\partial_2\varphi + \mathcal{H}_3\partial_3\varphi = \mathcal{H}_1. \end{cases}$$

- In 2D $\quad (N := (1, -\partial_2\varphi))$

$$|\mathbf{H}| + |\mathcal{H}| > 0, \quad \mathbf{H} \cdot N = \mathcal{H} \cdot N = 0 \quad \Rightarrow \quad H_2 \neq 0 \quad \text{or} \quad \mathcal{H}_2 \neq 0 \quad \text{on } \Gamma,$$

and we may solve for $\nabla_{t,x'}\varphi$ the system of equations

$$\partial_t\varphi - u_N = 0, \quad \mathbf{H} \cdot N = 0, \quad \mathcal{H} \cdot N = 0,$$

that is:

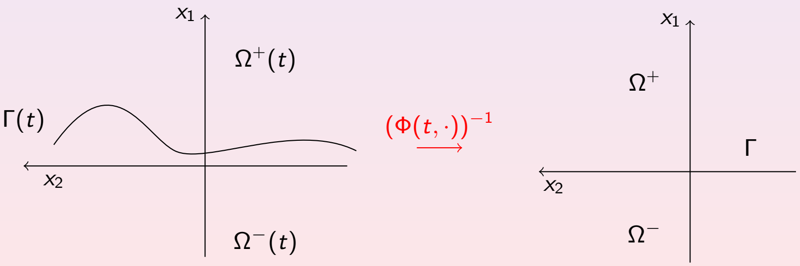
$$\begin{cases} \partial_t\varphi + u_2\partial_2\varphi = u_1 \\ H_2\partial_2\varphi = H_1 \\ \mathcal{H}_2\partial_2\varphi = \mathcal{H}_1. \end{cases}$$

Reduction to a fixed domain

Construct a global diffeomorphism of $\mathbb{R}^2$

$$\Phi(t,x) := (x_1 + \Psi(t,x), x_2), \quad \Psi = \chi(\pm x_1)\varphi(t,x_2)$$

such that $(\Phi(t,\cdot))^{-1}$ maps
$\Omega^\pm(t)$ onto the half-planes $\Omega^\pm := \{x_1 \gtrless 0\}$
$\Gamma(t)$ onto $\Gamma := \{x_1 = 0\}$



Define the new unknowns:

$$\widetilde{\mathbf{U}}(t,x) := \mathbf{U}(t,\Phi^{\pm}(t,x)), \quad \widetilde{\mathcal{H}}(t,x) := \mathcal{H}(t,\Phi(t,x)).$$

Dropping tildes from $\widetilde{\mathbf{U}}, \widetilde{\mathcal{H}}$, the plasma–vacuum problem on the fixed domains $\Omega^{\pm}$ becomes

$$P(\mathbf{U},\Psi)\mathbf{U} = 0 \qquad \text{in } [0,T]\times\Omega^{+}, \qquad (13)$$

$$\operatorname{curl}\mathfrak{H} = 0, \quad \operatorname{div}\mathfrak{h} = 0 \qquad \text{in } [0,T]\times\Omega^{-}, \qquad (14)$$

$$u_N - \partial_t\varphi = 0, \quad [q] := q - \frac{1}{2}|\mathcal{H}|^2 = 0, \quad \mathcal{H}_N = 0 \qquad \text{on } [0,T]\times\Gamma, \qquad (15)$$

where:

$$P(\mathbf{U},\Psi) := A_0(\mathbf{U})\partial_t + \widetilde{A}_1(\mathbf{U},\Psi)\partial_1 + A_2(\mathbf{U})\partial_2\,,$$

$$\widetilde{A}_1(\mathbf{U},\Psi) := \frac{1}{\partial_1\Phi_1}\Big(A_1(\mathbf{U}) - \partial_t\Psi A_0(\mathbf{U}) - \partial_2\Psi A_2(\mathbf{U})\Big)\,,$$

$$\mathfrak{H} = (\mathcal{H}_1\partial_1\Phi_1, \mathcal{H}_\tau), \quad \mathfrak{h} = (\mathcal{H}_N, \mathcal{H}_2\partial_1\Phi_1),$$

$$\mathcal{H}_\tau = \mathcal{H}_1\partial_2\Psi + \mathcal{H}_2, \quad \mathcal{H}_N = \mathcal{H}_1 - \mathcal{H}_2\partial_2\Psi, \quad N := (1, -\partial_2\varphi)\,.$$

Linearized problem

The first step is a careful study of the linearized problem.

1. We choose a suitable basic state $(\widehat{\mathbf{U}}, \widehat{\mathcal{H}}, \hat{\varphi})$ sufficiently smooth;
2. We linearize around the basic state, we pass to Alinhac's "good unknowns" $\dot{\mathbf{U}} = \mathbf{U} - \frac{\Psi}{\partial_1 \hat{\Phi}_1} \partial_1 \widehat{\mathbf{U}}$, $\dot{\mathcal{H}} = \mathcal{H} - \frac{\Psi}{\partial_1 \hat{\Phi}_1} \partial_1 \widehat{\mathcal{H}}$ to get rid of zero-th order terms in Ψ and get the linear problem

$$\begin{aligned} P(\widehat{\mathbf{U}}, \widehat{\Psi})\dot{\mathbf{U}} + (\text{0-th order opt in } \dot{\mathbf{U}}) &= \mathbf{f} && \text{in } \Omega_T^+ , \\ \operatorname{curl} \dot{\mathfrak{H}} = \chi, \quad \operatorname{div} \dot{\mathfrak{h}} &= \Xi && \text{in } \Omega_T^- , \\ \begin{bmatrix} (\partial_t + \hat{u}_2 \partial_2)\varphi - \dot{u}_N - \varphi\, \partial_1 \hat{u}_{\hat{N}} \\ \dot{q} - \widehat{\mathcal{H}} \cdot \dot{\mathcal{H}} + \varphi(\partial_1 \hat{q} + \widehat{\mathcal{H}} \cdot \partial_1 \widehat{\mathcal{H}}) \\ \dot{\mathcal{H}}_N - \partial_2(\widehat{\mathcal{H}}_2 \varphi) \end{bmatrix} &= \mathbf{g} && \text{on } \partial\Omega_T , \qquad (16) \\ (\dot{\mathbf{U}}, \dot{\mathcal{H}}, \varphi) &= 0 && \text{for } t < 0 , \end{aligned}$$

where $\mathbf{f}$, χ, Ξ, $\mathbf{g}$ are given source terms and $\dot{u}_N := \dot{u}_1 - \partial_2 \widehat{\varphi}\, \dot{u}_2$, $\dot{\mathcal{H}}_N := \dot{\mathcal{H}}_1 - \partial_2 \widehat{\varphi}\, \dot{\mathcal{H}}_2$.

Two main ideas for the proof of linear stability

1. Hyperbolic approximation
2. Secondary symmetrization

1. Hyperbolic approximation

To solve the coupled hyperbolic-elliptic (partly homogeneous) problem (16) we re-introduce the displacement current $\partial_t\mathcal{E}$ and accordingly modify the boundary conditions:

$$
\begin{aligned}
&P(\widehat{\mathbf{U}},\widehat{\Psi})\mathbf{U}^{\varepsilon}+(\text{0-th order opt in }\mathbf{U}^{\varepsilon})=\mathbf{f} && \text{in } \Omega_T^+,\\
&\varepsilon\partial_t\mathfrak{e}^{\varepsilon}-\operatorname{curl}\mathfrak{H}^{\varepsilon}=0,\quad \varepsilon\partial_t\mathfrak{h}^{\varepsilon}+\operatorname{Curl}\mathcal{E}^{\varepsilon}=0 && \text{in } \Omega_T^-,\\
&\begin{bmatrix} (\partial_t+\hat{u}_2\partial_2)\varphi^{\varepsilon}-u_N^{\varepsilon}-\varphi^{\varepsilon}\,\partial_1\hat{u}_{\hat{N}} \\ q^{\varepsilon}-\widehat{\mathcal{H}}\cdot\mathcal{H}^{\varepsilon}+\varphi^{\varepsilon}(\partial_1\hat{q}+\widehat{\mathcal{H}}\cdot\partial_1\widehat{\mathcal{H}}) \\ \mathcal{E}^{\varepsilon}+\varepsilon\partial_t(\widehat{\mathcal{H}}_2\varphi^{\varepsilon}) \end{bmatrix}=\begin{bmatrix} g_1\\ g_2\\ 0\end{bmatrix} && \text{on } \partial\Omega_T, \quad (17)\\
&(\mathbf{U}^{\varepsilon},\mathcal{H}^{\varepsilon},\mathcal{E}^{\varepsilon},\varphi^{\varepsilon})=0 && \text{for } t<0,
\end{aligned}
$$

where $\mathfrak{e}^{\varepsilon}=\mathcal{E}^{\varepsilon}\partial_1\widehat{\Phi}_1$, $\epsilon>0$ is a small parameter, $\operatorname{Curl}\mathcal{E}^{\varepsilon}:=(\partial_2\mathcal{E}^{\varepsilon},-\partial_1\mathcal{E}^{\varepsilon})$.

1. Hyperbolic approximation

To solve the coupled hyperbolic-elliptic (partly homogeneous) problem (16) we re-introduce the displacement current $\partial_t\mathcal{E}$ and accordingly modify the boundary conditions:

$$\begin{aligned}
&P(\widehat{\mathbf{U}},\widehat{\Psi})\mathbf{U}^\varepsilon + (\text{0-th order opt in } \mathbf{U}^\varepsilon) = \mathbf{f} && \text{in } \Omega_T^+,\\
&\varepsilon\partial_t\mathfrak{e}^\varepsilon - \operatorname{curl}\mathfrak{H}^\varepsilon = 0, \quad \varepsilon\partial_t\mathfrak{h}^\varepsilon + \operatorname{Curl}\mathcal{E}^\varepsilon = 0 && \text{in } \Omega_T^-,\\
&\begin{bmatrix} (\partial_t + \hat{u}_2\partial_2)\varphi^\varepsilon - u_N^\varepsilon - \varphi^\varepsilon\,\partial_1\hat{u}_{\hat{N}} \\ q^\varepsilon - \widehat{\mathcal{H}}\cdot\mathcal{H}^\varepsilon + \varphi^\varepsilon(\partial_1\hat{q} + \widehat{\mathcal{H}}\cdot\partial_1\widehat{\mathcal{H}}) \\ \mathcal{E}^\varepsilon + \varepsilon\partial_t(\widehat{\mathcal{H}}_2\varphi^\varepsilon) \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ 0 \end{bmatrix} && \text{on } \partial\Omega_T, \qquad (17)\\
&(\mathbf{U}^\varepsilon, \mathcal{H}^\varepsilon, \mathcal{E}^\varepsilon, \varphi^\varepsilon) = 0 && \text{for } t < 0,
\end{aligned}$$

where $\mathfrak{e}^\varepsilon = \mathcal{E}^\varepsilon\partial_1\widehat{\Phi}_1$, $\epsilon > 0$ is a small parameter, $\operatorname{Curl}\mathcal{E}^\varepsilon := (\partial_2\mathcal{E}^\varepsilon, -\partial_1\mathcal{E}^\varepsilon)$.
Solutions to (17) satisfy

$$\begin{aligned}
&\operatorname{div} h^\varepsilon = 0 \quad \text{in } \Omega_T^+, \quad \operatorname{div}\mathfrak{h}^\varepsilon = 0 \quad \text{in } \Omega_T^-, && (18)\\
&h_1^\varepsilon = \widehat{H}_2\partial_2\varphi^\varepsilon - \varphi^\varepsilon\partial_1\widehat{H}_N, \quad \mathcal{H}_N^\varepsilon = \partial_2\big(\widehat{\mathcal{H}}_2\varphi^\varepsilon\big) \quad \text{on } \partial\Omega_T,
\end{aligned}$$

as restrictions on the initial data.

If we look for a standard L^2 energy estimate we get the boundary integral

$$-\int_{\partial\Omega_T} q^\varepsilon u_N^\varepsilon \, dx_2 \, dt.$$

We don't know how to control it !

If we look for a standard L^2 energy estimate we get the boundary integral

$$-\int_{\partial\Omega_T} q^{\varepsilon} u_N^{\varepsilon}\, dx_2\, dt.$$

We don't know how to control it !

We introduce a secondary symmetrization.

Write the modified Maxwell equations

$$\varepsilon\partial_t \mathfrak{e}^{\varepsilon} - \mathrm{curl}\, \mathfrak{H}^{\varepsilon} = 0, \qquad \varepsilon\partial_t \mathfrak{h}^{\varepsilon} + \mathrm{Curl}\, \mathcal{E}^{\varepsilon} = 0 \qquad \text{in } \Omega_T^- , \tag{19}$$

in terms of the "curved" unknown $W^{\varepsilon} = (\mathfrak{H}^{\varepsilon}, \mathcal{E}^{\varepsilon})$ as

$$B_0 \partial_t W^{\varepsilon} + \sum_{j=1}^{2} B_j^{\varepsilon} \partial_j W^{\varepsilon} + B_3 W^{\varepsilon} = 0 .$$

2. Secondary symmetrization

Following Godunov 1972, Trakhinin 2009, we get the secondary symmetrization, for any choice of vector functions $\vec{\nu} = (\nu_1, \nu_2) \neq \vec{0}$:

$$K\mathfrak{B}_0^\varepsilon K^{-1}\left(B_0\partial_t W^\varepsilon + \sum_{j=1}^{2} B_j^\varepsilon \partial_j W^\varepsilon + B_3 W^\varepsilon\right) + \frac{1}{\partial_1\widehat{\Phi}_1} KR \,\mathrm{div}\, \mathfrak{h}^\varepsilon \tag{20}$$
$$:= M_0^\varepsilon \partial_t W^\varepsilon + M_1^\varepsilon \partial_1 W^\varepsilon + M_2^\varepsilon \partial_2 W^\varepsilon + M_3^\varepsilon W^\varepsilon = 0,$$

where

$$K = \begin{bmatrix} 1 & -\partial_2\widehat{\Psi} & 0 \\ 0 & \partial_1\widehat{\Phi}_1 & 0 \\ 0 & 0 & \partial_1\widehat{\Phi}_1 \end{bmatrix}, \quad \mathfrak{B}_0^\varepsilon = \begin{bmatrix} 1 & 0 & -\varepsilon\nu_2 \\ 0 & 1 & \varepsilon\nu_1 \\ -\varepsilon\nu_2 & \varepsilon\nu_1 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} \nu_1 \\ \nu_2 \\ 0 \end{bmatrix}.$$

- The system (20) originates from a linear combination of equations (18), (19), that is

$$\left(\partial_t \mathfrak{h}^{\varepsilon} + \frac{1}{\varepsilon}\,\mathrm{Curl}\,\mathcal{E}^{\varepsilon}\right) - \frac{1}{\partial_1\widehat{\Phi}_1}\left(\varepsilon\partial_t \mathfrak{e}^{\varepsilon} - \mathrm{curl}\,\mathfrak{H}^{\varepsilon}\right)\hat{\eta}\vec{\nu}^{\perp} + \frac{\hat{\eta}\vec{\nu}}{\partial_1\widehat{\Phi}_1}\,\mathrm{div}\,\mathfrak{h}^{\varepsilon} = 0\,,$$

$$\left(\partial_t \mathfrak{e}^{\varepsilon} - \frac{1}{\varepsilon}\,\mathrm{curl}\,\mathfrak{H}^{\varepsilon}\right) + \partial_1\widehat{\Phi}_1\left(\vec{\nu}\wedge\hat{\eta}^{-1}\left(\varepsilon\partial_t \mathfrak{h}^{\varepsilon} + \mathrm{Curl}\,\mathcal{E}^{\varepsilon}\right)\right) = 0\,.$$

- The system (20) originates from a linear combination of equations (18), (19), that is

$$
\begin{aligned}
&\left(\partial_t \mathfrak{h}^\varepsilon + \frac{1}{\varepsilon}\,\mathrm{Curl}\,\mathcal{E}^\varepsilon\right) - \frac{1}{\partial_1\widehat{\Phi}_1}\left(\varepsilon\partial_t \mathfrak{e}^\varepsilon - \mathrm{curl}\,\mathfrak{H}^\varepsilon\right)\hat{\eta}\vec{\nu}^\perp + \frac{\hat{\eta}\vec{\nu}}{\partial_1\widehat{\Phi}_1}\,\mathrm{div}\,\mathfrak{h}^\varepsilon = 0\,,\\
&\left(\partial_t \mathfrak{e}^\varepsilon - \frac{1}{\varepsilon}\,\mathrm{curl}\,\mathfrak{H}^\varepsilon\right) + \partial_1\widehat{\Phi}_1\left(\vec{\nu}\wedge\hat{\eta}^{-1}\left(\varepsilon\partial_t \mathfrak{h}^\varepsilon + \mathrm{Curl}\,\mathcal{E}^\varepsilon\right)\right) = 0\,.
\end{aligned}
$$

- (20) is symmetric hyperbolic provided

$$\varepsilon|\vec{\nu}| < 1.$$

- $$\det(M_1^\varepsilon) = \left(1+(\partial_2\hat{\varphi})^2\right)^2(\nu_1-\nu_2\partial_2\hat{\varphi})^2\left(|\vec{\nu}|^2-1/\varepsilon^2\right)^2.$$

  Thus for small ε the boundary is noncharacteristic provided

$$\nu_1-\nu_2\partial_2\hat{\varphi}\neq 0.$$

Actually, the good choice of the functions ν_j is

$$\nu_1 = \hat{u}_2 \partial_2 \hat{\varphi}, \quad \nu_2 = \hat{u}_2, \tag{21}$$

that makes the boundary characteristic (because $\nu_1 - \nu_2 \partial_2 \hat{\varphi} = 0$).

Now, if we look for a standard L^2 energy estimate we get the boundary integral

$$\mathcal{A}^\varepsilon := \int_{\partial\Omega_T} \Bigg(-q^\varepsilon u_N^\varepsilon \underbrace{- \frac{1}{\varepsilon} \mathfrak{H}_2^\varepsilon \mathcal{E}^\varepsilon + \hat{u}_2 \mathfrak{H}_2^\varepsilon \mathcal{H}_N^\varepsilon}_{\text{added terms}} \Bigg) \, dx_2 \, dt.$$

Actually, the good choice of the functions ν_j is

$$\nu_1 = \hat{u}_2 \partial_2 \hat{\varphi}, \quad \nu_2 = \hat{u}_2, \tag{21}$$

that makes the boundary characteristic (because $\nu_1 - \nu_2 \partial_2 \hat{\varphi} = 0$).

Now, if we look for a standard L^2 energy estimate we get the boundary integral

$$\mathcal{A}^\varepsilon := \int_{\partial\Omega_T} \left(-q^\varepsilon u_N^\varepsilon \underbrace{- \frac{1}{\varepsilon} \mathfrak{H}_2^\varepsilon \mathcal{E}^\varepsilon + \hat{u}_2 \mathfrak{H}_2^\varepsilon \mathcal{H}_N^\varepsilon}_{\text{added terms}} \right) dx_2 \, dt.$$

Inserting the boundary conditions (among all $\mathcal{E}^\varepsilon = -\varepsilon \partial_t (\widehat{\mathcal{H}}_2 \varphi^\varepsilon)$) gives

$$\mathcal{A}^\varepsilon = \int_{\partial\Omega_T} \varphi^\varepsilon \left\{ [\partial_1 \hat{q}] \left(u_N^\varepsilon + \varphi^\varepsilon \partial_1 \hat{u}_N \right) + \partial_1 \hat{u}_N q^\varepsilon + (\partial_t \widehat{\mathcal{H}}_2 + \partial_2 \widehat{\mathcal{H}}_2 \hat{u}_2) \mathfrak{H}_2^\varepsilon \right\} dx_2 \, dt,$$

which is of lower order than before because $\nabla_{t,x_2} \varphi^\varepsilon$ has the regularity of $u_N^\varepsilon, q^\varepsilon, H_N^\varepsilon, \mathcal{H}_N^\varepsilon$ at $\partial\Omega_T$, i.e. φ^ε gains one derivative (consequence of the stability condition!).

Stability result in $H^1_{tan} \times H^1$ for the hyperbolic approximation

Lemma

Let $T > 0$. Let the basic state be sufficiently regular and satisfy the hyperbolicity condition, the divergence-free and boundary constraints. Moreover it satisfies the stability condition (12).
Then for all $\epsilon > 0$ sufficiently small and all $\mathbf{f} \in H^1_{tan}(\Omega_T^+)$, $\mathbf{g} \in H^{1.5}(\partial\Omega_T)$, vanishing in the past, problem (17) has a unique solution $(\mathbf{U}^\varepsilon, \mathfrak{H}^\varepsilon, \mathfrak{E}^\varepsilon, \varphi^\varepsilon) \in H^1_{tan}(\Omega_T^+) \times H^1(\Omega_T^-)^2 \times H^{1.5}(\partial\Omega_T)$ such that

$$\|\mathbf{U}^\varepsilon\|_{H^1_{tan}(\Omega_T^+)} + \|\mathfrak{H}^\varepsilon, \mathfrak{E}^\varepsilon\|_{H^1(\Omega_T^-)} + \|(q^\varepsilon, u_N^\varepsilon, H_N^\varepsilon)|_{\partial\Omega_T}\|_{H^{0.5}(\partial\Omega_T)} + \|\varphi^\varepsilon\|_{H^{1.5}(\partial\Omega_T)} \leq C\big(\|\mathbf{f}\|_{H^1_{tan}(\Omega_T^+)} + \|\mathbf{g}\|_{H^{1.5}(\partial\Omega_T)}\big) \quad (22)$$

where $C = C(T) > 0$ is a constant independent of ϵ and the data $\mathbf{f}, \mathbf{g}$.

- Notice the loss of regularity of the solution from the source terms

Well-posedness of the linearized problem

After passing to the limit as $\varepsilon \to 0$ we obtain

Theorem

Let the basic state be sufficiently regular and satisfy the hyperbolicity condition, the divergence-free and boundary constraints. Moreover it satisfies the stability condition (12).
Then, for all $\mathbf{f} \in H^{m+1}_*(\Omega^+_T)$, $(\chi, \Xi) \in H^{m+1}(\Omega^-_T)$, $\mathbf{g} \in H^{m+1.5}(\partial\Omega_T)$, *all functions vanishing in the past, the linearized problem* (16) *has a solution* $(\dot{\mathbf{U}}, \dot{\mathcal{H}}, \varphi) \in H^m_*(\Omega^+_T) \times H^m(\Omega^-_T) \times H^{m+0.5}(\partial\Omega_T)$ *satisfying the tame estimate*

$$||\dot{\mathbf{U}}||^2_{H^m_*(\Omega^+_T)} + ||\dot{\mathcal{H}}||^2_{H^m(\Omega^-_T)} + ||\varphi|^2_{H^{m+0.5}(\partial\Omega_T)} \le C\Big\{ \Big(||\mathbf{f}||^2_{H^7_*(\Omega^+_T)} + ||\chi, \Xi||^2_{H^{7.5}(\Omega^-_T)} + ||\mathbf{g}||^2_{H^{7.5}(\partial\Omega_T)} \Big) \times \Big(||\hat{\mathbf{U}}||^2_{H^{m+2}_*(\Omega^+_T)} + ||\hat{\mathcal{H}}||^2_{H^{m+2}(\Omega^-_T)} + ||\hat{\varphi}||^2_{H^{m+2.5}(\partial\Omega_T)} \Big) + ||\mathbf{f}||^2_{H^{m+1}_*(\Omega^+_T)} + ||\chi, \Xi||^2_{H^{m+1}(\Omega^-_T)} + ||\mathbf{g}||^2_{H^{m+1.5}(\partial\Omega_T)} \Big\},$$

where the constant $C = C(K, T)$ *is independent of the data* $\mathbf{f}, \chi, \Xi, \mathbf{g}$.

Main result

Theorem (A.Morando, P.S., Y.Trakhinin, P.Trebeschi, D.Yuan – 2023)

Let $m \in \mathbb{N}$ *and* $m \geq 13$. *Suppose that the initial data*

$$(\mathbf{U}_0, \mathcal{H}_0, \varphi_0) \in H^{m+9.5}(\Omega^+) \times H^{m+9.5}(\Omega^+) \times H^{m+10}(\Gamma),$$

satisfy the hyperbolicity condition, the divergence-free and boundary constraints and suitable compatibility conditions up to order $m+9$. *Assume moreover the initial data satisfy the stability condition* (12).
Then, there exists a sufficiently short time $T > 0$ *such that the problem* (10), (11) *has a unique solution on the time interval* $[0, T]$ *satisfying*

$$(\mathbf{U}, \mathcal{H}, \varphi) \in H^m_*([0,T] \times \Omega^+) \times H^m([0,T] \times \Omega^-) \times H^{m+0.5}([0,T] \times \Gamma).$$

- The nonlinear problem is solved by a Nash–Moser iteration scheme

Thank you for your attention!

Thank you for your attention!

Best wishes, Piero!!!