On the Compressible Euler-Poisson Equations & Related Nonlinear Partial Differential Equations

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Dedicated to Piero Marcati on the 70th Birthday

International Conference on Partial Differential Equations and Applications

Gran Sasso Science Institute, L'Aquila, Italy

Multidimensional Euler-Poisson Equations for Compressible Fluids

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \\ (\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla P + \rho \nabla \Phi = 0, \\ \Delta \Phi = \kappa \rho. \end{cases}$$

$\nabla = (\partial_{x_1}, \dots, \partial_{x_d})$ — Gradient with respect to $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$

$\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2$ — Laplace operator with respect to $\mathbf{x} \in \mathbb{R}^d$

ρ — Density, $\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{R}^d$ — Velocity

$P = P(\rho) = \rho^2 e'(\rho)$ — Pressure with internal energy $e(\rho)$

For a polytropic perfect gas: $P(\rho) = a\rho^\gamma,\ e(\rho) = \frac{a}{\gamma-1}\rho^{\gamma-1},\ \gamma > 1$

Φ — Gravitational potential of gaseous stars for $\kappa = 4\pi g > 0$ $(d = 3)$

Plasma electric field potential if $\kappa < 0$

Multidimensional Euler Equations for Compressible Fluids – Nonlocal Effect

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \\ (\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla P = -\kappa \rho \nabla K * \rho. \end{cases}$$

$K = K(\mathbf{x}, \mathbf{y})$ — Green function kernel $\sim$ fundamental solution

$\nabla = (\partial_{x_1}, \dots, \partial_{x_d})$ — Gradient with respect to $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$

$\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2$ — Laplace operator with respect to $\mathbf{x} \in \mathbb{R}^d$

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Nonlinear Hyperbolic Systems of Balanced Laws

$$\partial_t U + \nabla \cdot \mathbf{F}(U) = G(U, \nabla K * U, \cdots)$$

$U = (u_1, \cdots, u_m)^\top$, $\mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $\nabla_{\mathbf{x}} = (\partial_{x_1}, \cdots, \partial_{x_d})$

$\mathbf{F} = (F_1, \cdots, F_d) : \mathbb{R}^m \to (\mathbb{R}^m)^d$ is a nonlinear mapping

Hyperbolicity in $\mathcal{D}$: For any $\omega \in S^{d-1}$, $U \in \mathcal{D}$,

$$(\nabla_U \mathbf{F}(U) \cdot \omega)_{m\times m} \, \mathbf{r}_j(U, \omega) = \lambda_j(U, \omega) \, \mathbf{r}_j(U, \omega), \; 1 \le j \le m$$

$\lambda_j(U, \omega)$ **are real**

Connections and Applications:

- Relaxation Theory for Hyperbolic Conservation Laws
- Combustion Theory, MHD Theory, Damping/Coriolis/Quantum Effects, $\cdots$
- Differential Geometry: Isometric Embeddings, Nonsmooth Manifolds...
- **Nonlocal Effects & Geometric Effects**
  Self-gravitational potential field (gaseous stars, ...)
  Self-consistent electric potential field (plasma, semiconductor, ...)
  Solutions with geometric structure
- $\cdots\cdots$

Multidimensional Euler-Poisson Equations for Compressible Fluids with Spherical Symmetry

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \\ (\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla P + \rho \nabla \Phi = 0, \\ \Delta \Phi = \kappa \rho. \end{cases}$$

Spherically Symmetric Solutions:

$$\rho(t, \mathbf{x}) = \rho(t, r), \quad \mathbf{v}(t, \mathbf{x}) = v(t, r)\frac{\mathbf{x}}{r}, \quad \Phi(t, \mathbf{x}) = \Phi(t, r), \qquad r = |\mathbf{x}|.$$

Then the functions $(\rho, m) = (\rho, \rho v)$ **are governed by**

$$\begin{cases} \rho_t + m_r = -\dfrac{d-1}{r} m, \\ m_t + \left(\dfrac{m^2}{\rho} + P(\rho)\right)_r = -\dfrac{\kappa \rho}{r^{d-1}} \displaystyle\int_0^r \rho(t, y)\, y^{d-1} \mathrm{d}y - \dfrac{d-1}{r} \dfrac{m^2}{\rho}. \end{cases}$$

Nonlinear Hyperbolic Systems of Balanced Laws

$$\partial_t U + \nabla \cdot \mathbf{F}(U) = G(U, \nabla K * U, \cdots)$$

$U = (u_1, \cdots, u_m)^\top$, $\mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$, $\nabla_{\mathbf{x}} = (\partial_{x_1}, \cdots, \partial_{x_d})$

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$$(\nabla_U \mathbf{F}(U) \cdot \omega)_{m \times m} \mathbf{r}_j(U, \omega) = \lambda_j(U, \omega) \mathbf{r}_j(U, \omega), \ 1 \le j \le m$$

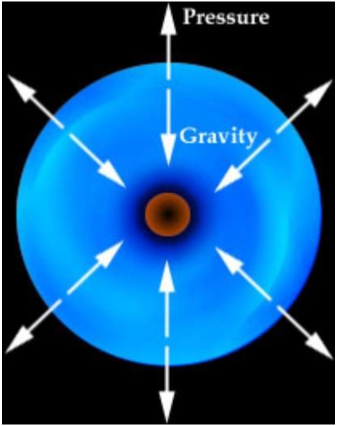
$\lambda_j(U, \omega)$ **are real**

Challenges: **Singularities** $\longrightarrow$ Discontinuous/Wild/Singular Solutions

- **Shocks, Vortex Sheets, Vorticity Waves, Entropy Waves, ...**
- **Compactness/Oscillation** $\Longleftrightarrow$ **Weak Continuity & Uniqueness??**
- **Cavitation/Decavitation** $\Longrightarrow$ **Degeneracy,** $\cdots$
- **Concentration/Deconcentration** $\Longrightarrow$ ∞**–Propagation Speed,**$\cdots$
- $\cdots\cdots$

The Compressible Euler-Poisson Equations for Self-Gravitating Newtonian Gaseous Stars

A gaseous star is modeled as a compactly supported gaseous fluid surrounded by vacuum subject to self-gravitation.



Euler-Poisson Equations with $\kappa > 0$ Self-Gravitational Gaseous Stars: Smooth Solutions

- Chandrasekhar 1938:
  - $\gamma > \frac{2d}{d+2}$ (*e.g.* $\gamma > \frac{6}{5}$ for $d = 3$) is necessary to ensure the global existence of finite-energy solutions with finite mass, which corresponds to the one for the **Lane-Emden solutions**.
  - There no exist steady **white dwarf star** with total mass larger than the **Chandrasekhar limit** $M_{\rm ch}$ when $\gamma \in (\frac{6}{5}, \frac{4}{3}]$ for $d = 3$.
- Goldreich-Webber 1980 (see also Deng-Xiang-Yang 2003, Fu-Lin 1998, Makino 1992): There exist homologous self-similar collapsing solutions when $\gamma = \frac{4}{3}$ for $d = 3$.
- **Guo-Hadzic-Jang** (ARMA 2021): $\exists$ **∞–D family of collapsing solutions.**
  $\gamma \in (1, \frac{4}{3})$ (mass supercritical) & Mach number $\gg 1 \Longrightarrow$ **Concentration**

  Lei-Gu 2016, Luo-Xin-Zeng 2014, Makino 1986, $\cdots\cdots$.

Weak Solutions outside a solid ball $|\mathbf{x}| \geq 1$: Makino 1997, Xiao 2016, $\cdots$

Open Problem: **?** $\exists$ **Global Weak Entropy Solutions including the Origin**??
Even under Self-Gravitation?

Stationary Self-Gravitating Gaseous Stars Ω: $\kappa > 0$

$$\begin{cases} \nabla P(\rho) = -\rho \nabla \Phi, \quad \Delta \Phi = \kappa \rho \qquad \text{in } \Omega, \\ \rho|_{\partial \Omega} = 0. \end{cases}$$

Then $Q(\rho) = \rho^{\gamma - 1}$ is determined by the elliptic problem:

$$\begin{cases} \Delta Q = -A Q^{\frac{1}{\gamma - 1}}, \\ Q|_{\partial \Omega} = 0, \end{cases} \qquad A = \frac{(\gamma - 1)\kappa}{\gamma a} > 0, \ \gamma > 1.$$

Theorem (Deng-Liu-Yang-Yao: ARMA 2002)

- $\frac{6}{5} < \gamma < 2$: *There is a positive solution on* Ω
- $1 < \gamma \le \frac{6}{5}$ *and* Ω *is a ball: There is no positive solution*

The total energy: $E = \frac{4 - 3\gamma}{\gamma - 1} \int_{\Omega} P(\rho) \, \mathrm{d}\mathbf{x}$

- $\gamma > \frac{4}{3}$: the gas may expand to infinity and become a gas cloud.
- $\gamma \le \frac{4}{3}$: the gas may collapse into a single point in finite time and may eventually become a black hole.

Euler-Poisson Equations for Plasma with $\kappa < 0$

Theorem (Existence Theory of Smooth Solutions)

There exist global smooth solutions around a constant neutral background under irrotational, smooth, and localized perturbation of the background with small amplitude.

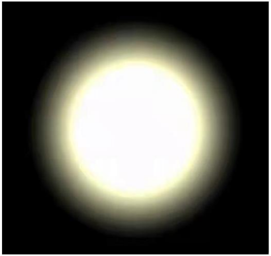
- Guo: CMP 1998
- Guo-Pausader: CMP 2011
- Ionescu-Pausader: IMRN 2013
- Guo-Ionescu-Pausader: Ann. Math. 2016
- ......

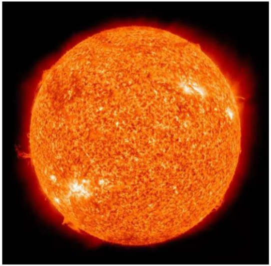
Chen-Wang (JDE 1998): Smooth initial data with large C^1 – norm

Development of Singularities $\Longrightarrow$ Global weak solutions??

Spherically Symmetric Solutions

- **The study of spherically symmetric solutions can date back to the 1950s and has been motivated by many important physical problems such as stellar dynamics including gaseous stars and supernova formation.**
- **Open Question: Could concentration be formed at the origin (the density becomes a Dirac measure at the origin), especially when a focusing spherical shock is moving inward the origin?**





Multidimensional Isentropic Euler Equations: $\kappa = 0$

$$\begin{cases} \rho_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v}) = 0, \\ (\rho \mathbf{v})_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla_{\mathbf{x}} P = 0. \end{cases}$$

$\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$, $\nabla_{\mathbf{x}}$ — Gradient w.r.t. $\mathbf{x} \in \mathbb{R}^d$
ρ — Density, $\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{R}^d$ — Velocity,
$P = P(\rho) = \rho^2 e'(\rho)$ — Pressure with internal energy $e(\rho)$

For a polytropic perfect gas: $P(\rho) = a\rho^\gamma,\ e(\rho) = \frac{a}{\gamma-1}\rho^{\gamma-1},\ \gamma > 1$

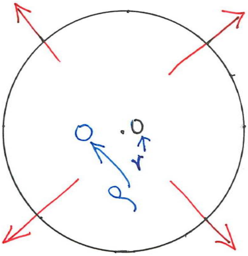
Spherically Symmetric Solutions:

$$\rho(t, \mathbf{x}) = \rho(t, r), \qquad \mathbf{v}(t, \mathbf{x}) = v(t, r)\frac{\mathbf{x}}{r}, \qquad r = |\mathbf{x}|.$$

Then the functions $(\rho, m) = (\rho, \rho v)$ **are governed by**

$$\begin{cases} \rho_t + m_r = -\frac{d-1}{r} m, \\ m_t + \left(\frac{m^2}{\rho} + P(\rho)\right)_r = -\frac{d-1}{r}\frac{m^2}{\rho}. \end{cases}$$

Defocusing: Expanding Spherically Symmetric Solns



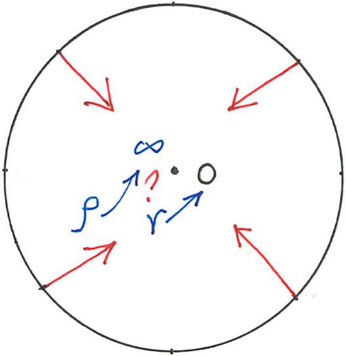
G.-Q. Chen: Proc. Royal Soc. Edinburgh, 127A (1997), 243–259.

$$0 \leq \int_0^{\rho_0(r)} \frac{\sqrt{P'(s)}}{s}\,\mathrm{d}s \leq v_0(r) \leq C < \infty$$

$\Longrightarrow$ **Formulation of Cavitation near the origin via Finite Difference Scheme....**

* M. Slemrod: PRSE, 1996: Spherical Self-Similar Piston Problem
* F. Huang, T.-H. Li & D. Yuan 2019, $\cdots\cdots$

Focusing: Imploding Spherically Symmetric Solns



Guderley 1942, Courant-Fridrichs 1945, $\cdots$

Merle-Raphaël-Ronianski-Szeftel 2022: Singularity of Self-Similar Solutions

Rauch 1986: No BV or L^∞ Bounds

Longstanding Problem: Does the concentration occur generically?

$\Longleftrightarrow$ **Does the density develop into a measure at the origin generically?**

Spherically Symmetric Solutions for the Euler Equations via Navier-Stokes Viscosity Limits

Theorem (Chen-Wang: ARMA 2022, Chen-Schrecker: ARMA 2018
Chen-Perepelitsa: CMP 2015)

Let the initial functions (ρ_0, m_0) satisfy the relative finite-energy conditions with $\bar{\rho} := \lim_{r\to\infty} \rho_0(r) \geq 0$.

$\Longrightarrow$ **There exists a sequence of Navier-Stokes-type approximate solutions $(\rho^\varepsilon, m^\varepsilon), m^\varepsilon = \rho^\varepsilon v^\varepsilon$, for $\varepsilon > 0$ such that, when $\varepsilon \to 0$, there exists a subsequence of $(\rho^\varepsilon, m^\varepsilon)$ that converges strongly almost everywhere to a finite-energy spherically symmetric entropy solution (ρ, m) with**

$$\rho(t,\mathbf{x}) = \rho(t,|\mathbf{x}|), \quad (\rho\mathbf{v})(t,\mathbf{x}) = m(t,|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|} \qquad \text{for all } \gamma > 1.$$

***There EXIST entropy solutions (as zero viscosity limits) even $\bar{\rho} > 0$**
with ∞–propagation speed,
but without concentration at the origin!!

Entropy Analysis I

$$\partial_t U + \partial_x F(U) = G(\cdots), \qquad U \in \mathbb{R}^2$$

Entropy-Entropy Flux Pair (η, q) if they satisfy the 2×2 hyperbolic system:

$$\nabla q(U) = \nabla \eta(U) \nabla F(U).$$

For smooth solution U, $\partial_t \eta(U) + \partial_x q(U) = \nabla \eta(U) G(\cdots)$.

If the system is endowed with globally defined Riemann invariants $w_i(U), 1 \leq i \leq 2$, satisfying $\nabla w_i(U) \cdot \nabla F(U) = \lambda_i(U) \nabla w_i(U)$ so that

$$q_{w_i} = \lambda_i \eta_{w_i}, \qquad\qquad i = 1, 2.$$

That is, the entropy function η is determined by

$$\eta_{w_1 w_2} + \frac{\lambda_{2 w_1}}{\lambda_2 - \lambda_1} \eta_{w_2} - \frac{\lambda_{1 w_2}}{\lambda_2 - \lambda_1} \eta_{w_1} = 0.$$

For the Euler system, η is determined by the **Euler-Poisson-Darboux equation**:

$$\eta_{w_1 w_2} + \frac{\alpha}{w_2 - w_1} (\eta_{w_2} - \eta_{w_1}) = 0, \qquad \alpha = \frac{3 - \gamma}{2(\gamma - 1)}.$$

Entropy Analysis - II

$$\begin{cases} \rho_t + m_x = -\frac{d-1}{r} m, & (m = \rho v) \\ m_t + (\frac{m^2}{\rho} + P(\rho))_x = -\frac{d-1}{r} \frac{m^2}{\rho}. \end{cases}$$

Strict Hyperbolicity – fails: $\lambda_2 - \lambda_1 = 2\sqrt{P'(\rho)} \to 0$ when $\rho \to 0$ (vacuum)

Entropy Pair (η, q): $\nabla q(U) = \nabla \eta(U) \nabla F(U)$ for $U = (\rho, m)^\top$

Convex Entropy: $\nabla^2 \eta(U) > 0$ **Weak Entropy**: $\eta(\rho, \rho v)|_{\rho=0} = 0$

Weak entropy pairs are represented as

$$\eta^\psi(\rho, \rho v) = \int_{\mathbb{R}} \chi(s)\psi(s)\,\mathrm{d}s, \quad q^\psi(\rho, \rho v) = \int_{\mathbb{R}} (\theta s + (1-\theta)v)\chi(s)\psi(s)\,\mathrm{d}s$$

by C^2-functions $\psi(s)$, where $\chi(s)$ is the weak entropy kernel:

$$\chi(s) := [\rho^{2\theta} - (v-s)^2]_+^\lambda, \qquad \theta = \frac{\gamma-1}{2}, \lambda = \frac{3-\gamma}{2(\gamma-1)}$$

Physical Convex Entropy: **Mechanical energy-energy flux pair** (η_*, q_*)**:**

$$\eta_*(\rho, m) = \frac{1}{2}\frac{m^2}{\rho} + \rho e(\rho), \qquad q_*(\rho, m) = \frac{1}{2}\frac{m^3}{\rho^2} + m(e(\rho) + \frac{P}{\rho})$$

Entropy Analysis - III: L^p–Compactness Framework

Theorem (L^p-Compensated Compactness Framework)

Let a function sequence $(\rho^\varepsilon, m^\varepsilon)(t,r)$ defined on a compact domain $\Omega \Subset \mathbb{R}_+ \times \mathbb{R}_+$ satisfy

- *There exists a constant $C > 0$, independent of $\varepsilon > 0$, such that*

$$\|\rho^\varepsilon\|_{L^{\max\{\gamma+1,\gamma+\theta\}}(\Omega)} + \|\rho^\varepsilon(u^\varepsilon)^3\|_{L^1(\Omega)} \le C,$$

- *For any weak entropy pair generated by $\psi \in C_c^2(\mathbb{R})$ such that the corresponding sequence of entropy dissipation measures*

$$\partial_t \eta^\psi(\rho^\varepsilon, m^\varepsilon) + \partial_r q^\psi(\rho^\varepsilon, m^\varepsilon) \qquad \text{is compact in } H^{-1}(\Omega).$$

Then there exist both a subsequence (still denoted) $(\rho^\varepsilon, m^\varepsilon)(t,r)$ and a measurable vector function $(\rho, m)(t,r)$ such that

$$(\rho^\varepsilon, m^\varepsilon)(t,r) \to (\rho, m)(t,r) \qquad \text{a.e. as } \varepsilon \to 0.$$

L^p–Framework for General $\gamma > 1$: Chen-Perepelitsa, CPAM 2010

* DiPerna, Ding-Luo-Chen, Lions-Perthame-Souganidis-Tadmor, Chen-LeFloch, LeFloch-Westdickenberg, $\cdots$

Multidimensional Euler-Poisson Equations

$$\begin{cases} \rho_t + \nabla \cdot \mathcal{M} = 0, \\ \mathcal{M}_t + \nabla \cdot \left(\frac{\mathcal{M}\otimes\mathcal{M}}{\rho}\right) + \nabla P + \rho\nabla\Phi = 0, \\ \Delta\Phi = \kappa\rho, \qquad\qquad \mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d. \end{cases}$$

ρ — Density, $\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{R}^d$ — Velocity, $\nabla_{\mathbf{x}}$ — Gradient w.r.t. $\mathbf{x} \in \mathbb{R}^d$

Φ — Gravitational potential of gaseous stars if $\kappa = 4\pi g > 0$ when $d = 3$

& plasma electric field potential if $\kappa < 0$

Spherically Symmetric Solutions:

$$\rho(t,\mathbf{x}) = \rho(t,r), \quad \mathbf{v}(t,\mathbf{x}) = v(t,r)\frac{\mathbf{x}}{r}, \quad \Phi(t,\mathbf{x}) = \Phi(t,r), \qquad\qquad r = |\mathbf{x}|.$$

Then the functions $(\rho, m) = (\rho, \rho v)$ **are governed by**

$$\begin{cases} \rho_t + m_r = -\frac{d-1}{r} m, \\ m_t + \left(\frac{m^2}{\rho} + P(\rho)\right)_r = -\rho\Phi_r - \frac{d-1}{r}\frac{m^2}{\rho}, \\ \Phi_{rr} + \frac{d-1}{r}\Phi_r = \kappa\rho. \end{cases}$$

Finite Initial Total-Energy and Total-Mass

Initial Condition:

$$(\rho, \mathcal{M})|_{t=0} = (\rho_0(\mathbf{x}), \mathcal{M}_0(\mathbf{x})) = (\rho_0(|\mathbf{x}|), m_0(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}) \longrightarrow (0, \mathbf{0}) \quad \text{as } |\mathbf{x}| \to \infty.$$

Asymptotic Condition:

$$\Phi(t, \mathbf{x}) = \Phi(t, |\mathbf{x}|) \longrightarrow 0 \qquad \text{as } |\mathbf{x}| \to \infty.$$

Finite initial total-energy:

$$E_0 := \begin{cases} \int_{\mathbb{R}^d} \Big(\frac{1}{2}\big|\frac{\mathcal{M}_0}{\sqrt{\rho_0}}\big|^2 + \rho_0 e(\rho_0) + \frac{1}{2}|\nabla_{\mathbf{x}}\Phi_0|^2\Big)(\mathbf{x})\,\mathrm{d}\mathbf{x} < \infty & \text{for } \kappa < 0 \text{ (plasmas)}, \\ \int_{\mathbb{R}^d} \Big(\frac{1}{2}\big|\frac{\mathcal{M}_0}{\sqrt{\rho_0}}\big|^2 + \rho_0 e(\rho_0)\Big)(\mathbf{x})\,\mathrm{d}\mathbf{x} < \infty & \text{for } \kappa > 0 \text{ (gaseous stars)}. \end{cases}$$

Finite initial total-mass: $M := \int_{\mathbb{R}^d} \rho_0(\mathbf{x})\,\mathrm{d}\mathbf{x} = \omega_d \int_0^\infty \rho_0(r)\, r^{d-1}\mathrm{d}r < \infty.$

$e(\rho) := \frac{a_0}{\gamma-1}\rho^{\gamma-1}$ — internal energy

$\omega_d := \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ — surface area of the unit sphere in $\mathbb{R}^d$

Spherically Symmetric Solutions for the Euler-Poisson Equations via inviscid Navier-Stokes-Poisson-type Limits

Theorem (Chen-He-Wang-Yuan (CPAM 2023))

Let $(\rho_0, m_0)(|\mathbf{x}|)$ **satisfy the finite-energy and finite-mass conditions.**
$\Longrightarrow$ **There exist Navier-Stokes-Poisson-type viscosity solutions** $(\rho^\varepsilon, m^\varepsilon, \Phi^\varepsilon)$ **for** $\varepsilon > 0$ **such that, when** $\varepsilon \to 0$, **there exists a subsequence of** $(\rho^\varepsilon, m^\varepsilon, \Phi^\varepsilon)$ **that converges strongly a.e. to a finite-energy spherically symmetric entropy solution** $(\rho, m, \Phi)(t, r)$ **with**

$$\rho(t,\mathbf{x}) = \rho(t,|\mathbf{x}|), \quad \mathcal{M}(t,\mathbf{x}) = m(t,|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}, \quad \Phi(t,\mathbf{x}) = \Phi(t,|\mathbf{x}|)$$

when (i) $\gamma > 1$ **and** $\rho_0 \in L^{\frac{2d}{d+2}}(\mathbb{R}^d)$ **for** $\kappa < 0$ **(plasma);**

(ii) $\gamma > \frac{2(d-1)}{d}$ **or** $\gamma \in (\frac{2d}{d+2}, \frac{2(d-1)}{d}]$ **with the critical mass** $M_c(\gamma)$ **for** $\kappa > 0$ **(gaseous stars).**

There exist entropy solutions (as inviscid Navier-Stokes limits) without concentration at the origin even under self-gravitation!!

Remarks I

- The results provide the global-in-time solutions of the M-D CEPEs with large initial data.
  - For $\kappa > 0$ (gaseous stars), condition: $\gamma > \frac{2d}{d+2}$ (*i.e.*, $\gamma > \frac{6}{5}$ for $d = 3$) is necessary to ensure the global existence of finite-energy solutions with finite total mass, which corresponds to the one for the Lane-Emden solutions.
  - Chandrasekhar (1938) showed that there is no spherically symmetric steady solution of gaseous stars for the 3-D CEPEs with $\gamma \in (1, \frac{6}{5})$ with finite total mass (also see S. Lin, SIMA 1997). Thus, the conjecture is that there is no global-in-time solution even in the weak sense in general.
- For the Poisson equation, the initial condition is not needed since $\nabla\Phi_0$ is indeed determined by the initial density ρ_0.
  - When $\kappa < 0$ (plasma) and $\gamma \in (1, \frac{2d}{d+2})$, the additional condition: $\rho_0 \in L^{\frac{2d}{d+2}}(\mathbb{R}^d)$ is required to make the Poisson equations solvable.
  - For case $\kappa > 0$ (gaseous stars), this condition is not required since $\gamma > \frac{2d}{d+2}$ (necessary for the existence).

Remarks II

- For the steady gaseous star problem, Chandrasekhar(1938) observed that there no exist steady white dwarf star with total mass larger than the Chandrasekhar limit $M_{\rm ch}$ when $\gamma \in (\frac{6}{5}, \frac{4}{3}]$ for $d = 3$.

  In our results for the 3-D time-dependent problem with $\gamma \in (\frac{6}{5}, \frac{4}{3}]$, the restriction on the total initial-mass $M < M_{\rm c}(\gamma)$ is also required, which is consistent with the Chandrasekhar phenomenon.

- A further fundamental question is whether concentration (the delta measure) could be formed at some time when $M > M_{\rm ch}$.

  Indeed, for the case that $\gamma \in (1, \frac{4}{3})$ and the Mach number $\gg 1$: Guo-Hadzic-Jang (2021) constructed an infinite-D family of collapsing spherically symmetric solutions of the 3-D CEPEs: The gaseous star continuously shrinks to be one point (*i.e.*, the delta measure).

Main Strategies

- Design an appropriate free boundary problem with
  - appropriate approximate initial data
  - stress-free boundary condition

  to construct the approximate solutions (involving the initial location $b > 0$ of the free boundary – a large parameter, besides the small parameter $\varepsilon > 0$) for CNSPEs.
- Obtain the trace estimates in the energy estimates
  & adopt the Bresch-Desjardins entropy
  to make uniform estimates of the approximate solutions, independent of $\varepsilon > 0$ and $b > 0$.
- Prove that the Navier-Stokes-Poisson viscosity solutions satisfy the L^p–compensated compactness framework after first taking $b \to \infty$, which then ensures the strong convergence of the viscosity solutions as $\varepsilon \to 0$.
- Verify that the strong limit functions are finite-energy global solutions of the compressible Euler-Poisson equations with large initial data of spherical symmetry.

Navier-Stokes-Poisson Approximate Solutions

Consider the following approximate free boundary problem for CNSPEs:

$$\begin{cases} \rho_t + (\rho v)_r + \frac{d-1}{r}\rho v = 0, \\ (\rho v)_t + (\rho v^2 + P)_r + \frac{d-1}{r}\rho v^2 + \frac{\kappa\rho}{r^{d-1}}\int_{b^{-1}}^{r} \rho(t,y)\, y^{d-1}\mathrm{d}y \\ \qquad\qquad = \varepsilon\big(\rho(v_r + \frac{d-1}{r} v)\big)_r - \varepsilon\frac{d-1}{r} v\rho_r, \end{cases}$$

for $(t,r) \in \Omega_T := \{(t,r)\ :\ b^{-1} \le r \le b(t),\, 0 \le t \le T\}$ (moving domain), with $\{r = b(t) : 0 < t \le T\}$ as a free boundary:

$$b'(t) = v(t, b(t)) \text{ for } t > 0, \qquad b(0) = b \gg 1,$$

- On the free boundary $r = b(t)$, the stress-free boundary condition:
$$\big(P(\rho) - \epsilon\rho(v_r + \frac{d-1}{r} v)\big)(t, b(t)) = 0 \qquad \text{for } t > 0.$$
- On the fixed boundary $r = b^{-1}$, the Dirichlet boundary condition:
$$v\big|_{r=b^{-1}} = 0 \qquad \text{for } t > 0.$$
- The initial condition: $(\rho, \rho v)|_{t=0} = (\rho_0^{\epsilon,b}, \rho_0^{\epsilon,b} v_0^{\epsilon,b})(r)$ for $r \in [b^{-1}, b]$. $(\rho_0^{\epsilon,b}, v_0^{\epsilon,b})(r)$ are smooth/compatible and $0 < C_{\epsilon,b}^{-1} \le \rho_0^{\epsilon,b}(r) \le C_{\epsilon,b} < \infty$.

*Duan-Li, JDE 2015: $\kappa > 0$ with $\gamma \in (\frac{6}{5}, \frac{4}{3}] \Longrightarrow$ General as needed for $d \ge 2$.

*Donatelli-Marcati, Nonlinearity 2008: Navier–Stokes–Poisson system with large data

Basic Energy Estimates for the Approx. Solutions I

The approximate solution $(\rho, v)(t,r) := (\rho^{\epsilon,b}, v^{\epsilon,b})(t,r)$ satisfies the following energy identity:

$$\int_{b^{-1}}^{b(t)} \big(\frac{1}{2}\rho v^2 + \rho e(\rho)\big)(t,r)\, r^{d-1}\mathrm{d}r - \frac{\kappa}{2}\int_{b^{-1}}^{b(t)} \frac{1}{r^{d-1}}\big(\int_{b^{-1}}^{r} \rho(t,y)\, y^{d-1}\mathrm{d}y\big)^2\mathrm{d}r$$
$$+ \epsilon\int_0^t \int_{b^{-1}}^{b(s)} \big(\rho v_r^2 + (d-1)\rho\frac{v^2}{r^2}\big)(t,r)\, r^{d-1}\mathrm{d}r\mathrm{d}s$$
$$+ (d-1)\epsilon\int_0^t (\rho v^2)(s,b(s))b(s)^{d-2}\,\mathrm{d}s$$
$$= \int_{b^{-1}}^{b} \Big(\big(\frac{1}{2}\rho_0 v_0^2 + \rho_0 e(\rho_0)\big)(r) - \frac{\kappa}{2}\frac{1}{r^{2(d-1)}}\big(\int_{b^{-1}}^{r} \rho_0(t,y)y^{d-1}\mathrm{d}y\big)^2\Big)\, r^{d-1}\mathrm{d}r,$$

where $\rho(t,r)$ is understood to be 0 for $r \in [0, b^{-1}] \cup (b,\infty)$ in the 2^{nd} term of the right-hand side (RHS) and the 2^{nd} term of the left-hand side (LHS).

There are the **three cases**:

Case 1: If $\kappa < 0$ (plasmas) with $\gamma > 1$, then

$$\int_{b^{-1}}^{b(t)} \big(\frac{1}{2}\rho v^2 + \rho e(\rho)\big)(t,r)\, r^{d-1}\mathrm{d}r + \frac{|\kappa|}{2}\int_{b^{-1}}^{b(t)} \frac{1}{r^{d-1}}\Big(\int_{b^{-1}}^{r} \rho(t,y)\, y^{d-1}\mathrm{d}y\Big)^2$$

$$+ \epsilon \int_0^t \int_{b^{-1}}^{b(s)} \big(\rho v_r^2 + (d-1)\rho \frac{v^2}{r^2}\big)(s,r)\, r^{d-1}\mathrm{d}r\mathrm{d}s$$

$$+ (d-1)\epsilon \int_0^t (\rho u^2)(s,b(s))\, b^{d-2}(s)\,\mathrm{d}s$$

$$= \int_{b^{-1}}^{b} \Big(\big(\frac{1}{2}\rho_0 v_0^2 + \rho_0 e(\rho_0)\big)(r) + \frac{|\kappa|}{2r^{2(d-1)}}\big(\int_{b^{-1}}^{r} \rho_0(y)\, y^{d-1}\mathrm{d}y\big)^2\Big)\, r^{d-1}\mathrm{d}r.$$

Basic Energy Estimates for the Approx. Solutions IV

Case 2: If $\kappa > 0$ (gaseous stars) with $\gamma > \frac{2(d-1)}{d}$, then

$$\begin{aligned}
&\frac{1}{2}\int_{b^{-1}}^{b(t)} \big(\rho v^2 + \rho e(\rho)\big)(t,r)\, r^{d-1}\mathrm{d}r \\
&\quad + \epsilon\int_0^t \int_{b^{-1}}^{b(s)} \big(\rho v_r^2 + (d-1)\rho\frac{v^2}{r^2}\big)(s,r)\, r^{d-1}\mathrm{d}r\mathrm{d}s \\
&\quad + (d-1)\epsilon\int_0^t (\rho v^2)(s,b(s))\, b^{d-2}(s)\,\mathrm{d}s \\
&\le C(M,E_0),
\end{aligned}$$

where $C(M,E_0) > 0$ is some positive constant depending only on the total initial-mass M and initial-energy E_0.

Basic Energy Estimates for the Approx. Solutions III

Case 3: If $\kappa > 0$ (gaseous stars)

with $\gamma \in (\frac{2d}{d+2}, \frac{2(d-1)}{d}]$ and $M < M_{\mathrm{c}}^{\epsilon,b}(\gamma)$, then

$$
\begin{aligned}
&\int_{b^{-1}}^{b(t)} \big(\frac{1}{2}\rho v^2 + C_{d,\gamma}\rho e(\rho)\big)(t,r)\, r^{d-1}\mathrm{d}r \\
&\quad + \epsilon \int_0^t \int_{b^{-1}}^{b(s)} \big(\rho v_r^2 + (d-1)\rho \frac{v^2}{r^2}\big)(s,r)\, r^{d-1}\mathrm{d}r\mathrm{d}s \\
&\quad + (d-1)\epsilon \int_0^t (\rho u^2)(s,b(s))\, b^{d-2}(s)\,\mathrm{d}s \\
&\le \int_{b^{-1}}^{b} \big(\frac{1}{2}\rho_0 v_0^2 + \rho_0 e(\rho_0)\big)(r)\, r^{d-1}\mathrm{d}r,
\end{aligned}
$$

where $C_{d,\gamma} > 0$ is some positive constant depending only on d and γ.

Uniform Estimates for the Approx. Solutions

- The basic energy estimates lead to the following estimates:

$$|r^{d-1}\Phi_r(t,r)| \le \frac{M}{\omega_d} \qquad \text{for } (t,r) \in [0,\infty)\times[0,\infty),$$

$$\|\Phi(t)\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} + \|\nabla\Phi(t)\|_{L^2(\mathbb{R}^d)} \le C(M,E_0) \qquad \text{for } t \ge 0.$$

- BD-type entropy estimate: Given any fixed $T > 0$, then

$$\epsilon^2 \int_{b^{-1}}^{b(t)} \frac{|\rho(t,r)_r|^2}{\rho(t,r)}\, r^{d-1}\mathrm{d}r + \epsilon \int_0^t \int_{b^{-1}}^{b(s)} |(\rho^{\frac{\gamma}{2}})_r|^2\, r^{d-1}\mathrm{d}r\mathrm{d}s$$
$$+ P(\rho(t,b(t)))\, b^d(t) + \frac{1}{\epsilon}\int_0^t P(\rho(s,b(s)))P'(\rho(s,b(s)))\, b^d(s)\,\mathrm{d}s$$
$$\le C(E_0,M,T) \qquad \text{for all } t\in[0,T].$$

- Higher integrability on the density and the velocity:

$$\int_0^T \int_K \big(\rho|v|^3 + \rho^{\max\{\gamma+1,\gamma+\theta\}}\big)(t,r)\,\mathrm{d}r\mathrm{d}t \le C(K,M,E_0,T)$$

for any $K \Subset [a, b(t)]$ and any $t\in[0,T]$.

Expanding of Domain Ω_T with Free Boundary

Given $T > 0$ and $\epsilon \in (0, \epsilon_0]$, there exists a positive constant $B(M, E_0, T, \epsilon) > 0$ such that, if $b \geq B(M, E_0, T, \epsilon)$,

$$b(t) \geq \frac{b}{2} \qquad \text{for } t \in [0, T].$$

* For the free boundary problem, a follow-up point is whether the free boundary domain Ω_T will expand to the whole space as $b \to \infty$; otherwise, it would not be a good approximation to the original Cauchy problem.

* We solve this difficulty by proving that

$$b(t) \geq \frac{b}{2} \qquad \text{for } t \in [0, T].$$

provided $b \gg 1$ for any given T.

Existence of Global Weak Solutions of CNSPEs

- Similar to the compactness arguments of Mellet-Vasseur (CPDE, 2007) based on these uniform estimates just presented, we take the limit, $b \to \infty$, to obtain the global weak viscosity solutions of CNSPEs.
- Let (η, q) be a weak entropy pair for any smooth compact supported function $\psi(s)$ on $\mathbb{R}$. Then, for $\epsilon \in (0, \epsilon_0]$, the Navier-Stokes-Poisson viscosity solutions $(\rho^\epsilon, m^\epsilon)$ satisfy that

$$\partial_t \eta(\rho^\epsilon, m^\epsilon) + \partial_r q(\rho^\epsilon, m^\epsilon) \qquad \text{is compact in } H^{-1}_{\mathrm{loc}}(\mathbb{R}^2_+).$$

- Given any $T \in (0, \infty)$, the following uniform bounds hold for all $t \in [0, T]$:

$$\int_0^\infty \rho^\epsilon(t,r)\, r^{d-1} \mathrm{d}r = \int_0^\infty \rho_0^\epsilon(r)\, r^{d-1} \mathrm{d}r = M,$$

$$\int_0^\infty \eta^*(\rho^\epsilon, m^\epsilon)(t,r)\, r^{d-1} \mathrm{d}r + \epsilon \int_{\mathbb{R}^2_+} (\rho^\epsilon |u^\epsilon|^2)(t,r)\, r^{d-3} \mathrm{d}r \mathrm{d}t + \|\Phi^\epsilon(t)\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}$$

$$+ \int_0^\infty \Big(\int_0^r \rho^\epsilon(t,y)\, y^{d-1} \mathrm{d}z \Big) \rho^\epsilon(t,r)\, r \mathrm{d}r + \|\nabla \Phi^\epsilon(t)\|_{L^2(\mathbb{R}^d)} \le C(M, E_0),$$

$$\epsilon^2 \int_0^\infty \big|(\sqrt{\rho^\epsilon(t,r)})_r\big|^2 r^{d-1} \mathrm{d}r + \epsilon \int_0^T \int_0^\infty \big|((\rho^\epsilon)^{\frac{\gamma}{2}})_r\big|^2 r^{d-1} \mathrm{d}r \mathrm{d}t \le C(M, E_0, T).$$

Entropy Analysis: L^p–Compactness Framework

Theorem (L^p-Compensated Compactness Framework)

Let a function sequence $(\rho^\varepsilon, m^\varepsilon)(t,r)$ *defined on a compact domain* $\Omega \Subset \mathbb{R}_+ \times \mathbb{R}_+$ *satisfy*

- *There exists a constant* $C > 0$, *independent of* $\varepsilon > 0$, *such that*

$$\|\rho^\varepsilon\|_{L^{\max\{\gamma+1,\gamma+\theta\}}(\Omega)} + \|\rho^\varepsilon (u^\varepsilon)^3\|_{L^1(\Omega)} \le C.$$

- *For any weak entropy pair generated by* $\psi \in C_c^2(\mathbb{R})$ *such that the corresponding sequence of entropy dissipation measures*

$$\partial_t \eta^\psi(\rho^\varepsilon, m^\varepsilon) + \partial_r q^\psi(\rho^\varepsilon, m^\varepsilon) \qquad \textit{is compact in } H^{-1}(\Omega).$$

Then there exist both a subsequence (still denoted) $(\rho^\varepsilon, m^\varepsilon)(t,r)$ *and a measurable vector function* $(\rho, m)(t,r)$ *such that*

$$(\rho^\varepsilon, m^\varepsilon)(t,r) \to (\rho, m)(t,r) \qquad \textit{a.e. as } \varepsilon \to 0.$$

L^p**–Framework for General** $\gamma > 1$: Chen-Perepelitsa, CPAM 2010

* DiPerna, Ding-Luo-Chen, Lions-Perthame-Souganidis-Tadmor, Chen-LeFloch, LeFloch-Westdickenberg, $\cdots$

M-D Euler-Poisson Equations for White Dwarf Stars

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \\ (\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla P + \rho \nabla \Phi = 0, \\ \Delta \Phi = \kappa \rho. \end{cases}$$

ρ – Density, $\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{R}^d$ – Velocity

Φ – Self-consistent electric field potential, $\kappa > 0$.

$P = P(\rho) = \rho^2 e'(\rho)$ – General pressure with internal energy $e(\rho)$

For a white dwarf star (Chandrasekhar 1938),

$$P(\rho) = A \int_0^{B\rho^{\frac{1}{3}}} \frac{\sigma^4}{\sqrt{D + \sigma^2}} \, \mathrm{d}\sigma \qquad \text{for } \rho > 0,$$

where A, B and D are positive constants.

$\Longrightarrow \qquad P(\rho) \cong \rho^{\frac{5}{3}}$ as $\rho \to 0$, $\qquad P(\rho) \cong \rho^{\frac{4}{3}}$ as $\rho \to \infty$.

*G.-Q. Chen, F. Huang, T.-H. Li, W. Wang, and Y. Wang:

Global Finite-Energy Solutions of the Compressible Euler-Poisson Equations with Spherical Symmetry for White Dwarf Stars, Preprint 2023.

L^p–Compactness Framework for General Pressure Laws I: $P(\rho)$

(i) $P(\rho) \in C^1([0,\infty)) \cap C^4(\mathbb{R}_+)$ and satisfies the hyperbolic and genuinely nonlinear conditions:

$$P'(\rho) > 0, \quad 2P'(\rho) + \rho P''(\rho) > 0 \qquad \text{for } \rho > 0.$$

(ii) There exist constants $\gamma_1 \in (1,3)$ and $a_1 > 0$ such that

$$P(\rho) \sim a_1 \rho^{\gamma_1} \qquad \text{as } \rho \sim 0.$$

(iii) There exist constants $\gamma_2 \in (\frac{6}{5}, \gamma_1]$ and $a_2 > 0$ such that

$$P(\rho) \sim a_2 \rho^{\gamma_2} \qquad \text{as } \rho \sim \infty.$$

***Examples**: White dwarf stars, $\cdots$.

L^p–Compactness Framework for General Pressure Laws II: $p(\rho)$

Theorem (G.-Q. Chen, F. Huang, T.-H. Li, W. Wang & Y. Wang 2023)

Let a function sequence $(\rho^\varepsilon, m^\varepsilon)(t,r)$ defined on a compact domain $\Omega \Subset \mathbb{R}_+ \times \mathbb{R}_+$ satisfy

- **There exists a constant $C > 0$, independent of $\varepsilon > 0$, such that**

$$\|\rho^\varepsilon\|_{L^{\gamma_2+1}(\Omega)} + \Big\|\frac{(m^\varepsilon)^3}{(\rho^\varepsilon)^2}\Big\|_{L^1(\Omega)} \le C.$$

- **For any weak entropy pair generated by compactly supported test function $\psi \in C_c^2(\mathbb{R})$ such that the corresponding sequence of entropy dissipation measures**

$$\partial_t \eta^\psi(\rho^\varepsilon, m^\varepsilon) + \partial_r q^\psi(\rho^\varepsilon, m^\varepsilon) \qquad \text{is compact in } W^{-1,1}(\Omega).$$

Then there exist both a subsequence (still denoted) $(\rho^\varepsilon, m^\varepsilon)(t,r)$ and a measurable vector function $(\rho, m)(t,r)$ such that

$$(\rho^\varepsilon, m^\varepsilon)(t,r) \to (\rho, m)(t,r) \qquad a.e. \text{ as } \varepsilon \to 0.$$

*Schrecker-Schultz 2019–20

Multidimensional Euler-Poisson Equations with Doping Profile for Plasma

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \\ (\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla P + \rho \nabla \Phi = 0, \\ \Delta \Phi = \kappa(\rho - b(\mathbf{x})). \end{cases}$$

$\nabla = (\partial_{x_1}, \ldots, \partial_{x_d})$ — Gradient with respect to $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$
$\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2$ — Laplace operator with respect to $\mathbf{x} \in \mathbb{R}^d$

ρ — Density, $\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{R}^d$ — Velocity

$P = P(\rho) = \rho^2 e'(\rho)$ — Pressure with internal energy $e(\rho)$

Φ — Self-consistent electric field potential

$b(\mathbf{x})$ — Doping profile with $\lim_{|\mathbf{x}| \to \infty} b(\mathbf{x}) = \rho_* > 0$.

*G.-Q. Chen, L. He, Y. Wang and D. Yuan: Global Solutions of the Compressible Euler-Poisson Equations with Doping Profile and Large Data of Spherical Symmetry for Plasma Dynamics, Preprint 2023.

Nonlinear Hyperbolic Systems of Balance Laws

$$\partial_t U + \nabla_{\mathbf{x}} \cdot \mathbf{F}(U) = G(U, \nabla K * U, \cdots)$$

$$U = (u_1, \cdots, u_m)^\top, \ \ \mathbf{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d, \ \ \nabla_{\mathbf{x}} = (\partial_{x_1}, \cdots, \partial_{x_d})$$

$\mathbf{F} = (F_1, \cdots, F_d): \ \mathbb{R}^m \to (\mathbb{R}^m)^d$ is a nonlinear mapping

Hyperbolicity in $\mathcal{D}$: For any $\omega \in S^{d-1}$, $U \in \mathcal{D}$,

$$(\nabla_U \mathbf{F}(U) \cdot \omega)_{m\times m}\, \mathbf{r}_j(U,\omega) = \lambda_j(U,\omega)\, \mathbf{r}_j(U,\omega), \ 1 \le j \le m$$

$\lambda_j(U,\omega)$ **are real**

Connections and Applications:

- Relaxation Theory for Hyperbolic Conservation Laws
- Combustion Theory, MHD Theory, Damping/Coriolis/Quantum Effects, $\cdots$
- Differential Geometry: Isometric Embeddings, Nonsmooth Manifolds...
- **Nonlocal Effects & Geometric Effects**
  Self-gravitational potential field (gaseous stars, ...)
  Self-consistent electric potential field (plasma, semiconductor, ...)
  Solutions with geometric structure
- $\cdots\cdots$

Buon Settantesimo Compleanno

Piero!