RECOVERY OF HYPERBOLIC CONSERVATION LAWS BY SPACE-TIME OPTIMIZATION

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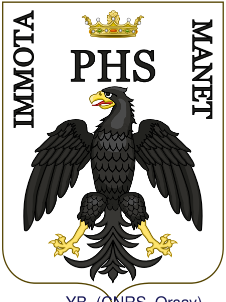
PIERO URBI ET ORBI



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More recently, in Y.B. CRAS '22, we extended this method to the EINSTEIN equation in vacuum, but, since there is (apparently) no convex entropy, we can no longer perform convex minimization. So, we just recover smooth solutions as critical points of a suitable functional which is surprisingly very reminiscent of classical continuum mechanics (BURGERS/EULER).

Let us start with Einstein's equation (Y.B. '22,'23+)

by introducing the variational principle: find 4×4 (non symmetric) matrix-valued fields $(C, M)(x, \xi)$ over the tangent bundle $(x, \xi) \in (\mathbb{R}^4)^2$ of $\mathbb{R}^4$, critical points of

$$\int \mathrm{trace}(MC^{-1}M)(x, \xi) dx d\xi$$

s.t. $\nabla_x \cdot C + \nabla_\xi \cdot M = 0$, and $C = \nabla_\xi A - (\nabla_\xi \cdot A)\, \mathbb{I}_4$, for some vector potential $A = A(x, \xi) \in \mathbb{R}^4$.

N.B. Here ∇_x, ∇_ξ are just FLAT Euclidean gradients. More precisely, in coordinates,

$$C_k^j = \partial_{\xi^k} A^j - \partial_{\xi^\gamma} A^\gamma\ \delta_k^j, \quad \partial_{x^j} C_k^j + \partial_{\xi^j} M_k^j = 0, \quad \mathrm{trace}(MC^{-1}M) = M_k^j (C^{-1})_q^k M_j^q.$$

Theorem (Y.B. CRAS '22) english version:

https://www.lmo.universite-paris-saclay.fr/

∼yann.brenier/GROT-note-english2022.pdf

Let g be a smooth solution to the Einstein equations in vacuum, with Christoffel symbols $\Gamma = "g^{-1}\partial g"$. Set

$$A^j = \xi^j \, \mathrm{det} g(x) \;\cos(\frac{g_{kq}(x)\xi^k\xi^q}{2}), \quad V^j_k = -\Gamma^j_{kq}(x)\xi^q,$$

$$C^j_k = \partial_{\xi^k}A^j - \partial_{\xi^q}A^q \;\delta^j_k, \quad M^j_k = C^j_q V^q_k + V^j_q C^q_k.$$

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Then (C, M) satisfies our variational principle and

$$g^{ij}(x)\sqrt{-\mathrm{det} g(x)} = cst \;\int (\xi^i A^j + \xi^j A^i)(x,\xi) d\xi.$$

Our trick: write everything on the "tangent bundle"

$$(x,\xi) \in \mathbb{R}^4 \times \mathbb{R}^4,\ \ V^j_k(x,\xi) = -\Gamma^j_{k\gamma}(x)\xi^\gamma \quad (j,k,\gamma \in \{0,1,2,3\})$$

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so that the Riemann and the Ricci curvatures just read

$$R^n_{jkm}(x)\xi^m = \Big((\partial_{x^k} + V^\gamma_k \partial_{\xi^\gamma})V^n_j - (\partial_{x^j} + V^\gamma_j \partial_{\xi^\gamma})V^n_k\Big)(x,\xi)$$

$$= \partial_{x^k} V^n_j + \partial_{\xi^j}(V^\gamma_k V^n_\gamma) - \partial_{x^j} V^n_k - \partial_{\xi^k}(V^\gamma_i V^n_\gamma),$$

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and treat this matrix-valued Burgers operator more or less as we did for entropic conservation laws in 2018.

Let us move back to entropic conservation laws

$$\partial_t U + \nabla \cdot (F(U)) = 0, \; U = U(t,x) \in \mathcal{W} \subset \mathbb{R}^m, \; x \in \mathbb{T}^d,$$

with a strictly convex "entropy" $\mathcal{E} : \mathcal{W} \to \mathbb{R}$ (where $\mathcal{W}$ is convex) and an "entropy flux" $\mathcal{Z} \in \mathcal{W} \to \mathbb{R}^d$, such that each smooth solution U satisfies the extra conservation law $\partial_t(\mathcal{E}(U)) + \nabla \cdot (\mathcal{Z}(U)) = 0.$

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The problem is not trivial since there may be many weak solutions starting from U_0 which are not entropy-preserving (by "convex integration" à la De Lellis-Székelyhidi).

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$$\inf_U \sup_A \geq \sup_A \inf_U$$

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$$= \sup_{A(T,\cdot)=0} \int_0^T \int_D -G(\partial_t A, \nabla A) + \partial_t A \cdot U_0,$$

$$G(E,B) = \sup_{V \in \mathcal{W} \subset \mathbb{R}^m} E \cdot V + B \cdot F(V) - \mathcal{E}(V), \ \forall (E,B) \in \mathbb{R}^{m+d}.$$

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Note that G is automatically convex (even without $\mathcal{E}$!)

Main results (Y.B. CMP '18)

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Theorem 1: If U is a smooth solution to the IVP and T is not too large (*), then U can be recovered from the concave maximization problem which admits $A(t,x) = (t - T)\mathcal{E}'(U(t,x))$ as solution.

Theorem 2: For the Burgers equation, all entropy solutions can be recovered, for arbitrarily large T.

(*) more precisely if, $\forall\ t, x, V \in \mathcal{W}$, $\mathcal{E}''(V) - (T-t)F''(V) \cdot \nabla(\mathcal{E}'(U(t,x))) > 0$.

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$\partial_t u + \partial_x (u^2/2) = 0.$ Then, we get the convex problem

$$\inf_{(\rho,m)} \{ \int_{[0,T]\times\mathbb{T}} \rho^{-1} m^2 + 2 m u_0 \mid \partial_t \rho + \partial_x m = 0,\ \rho_{|t=T} = 1 \}.$$

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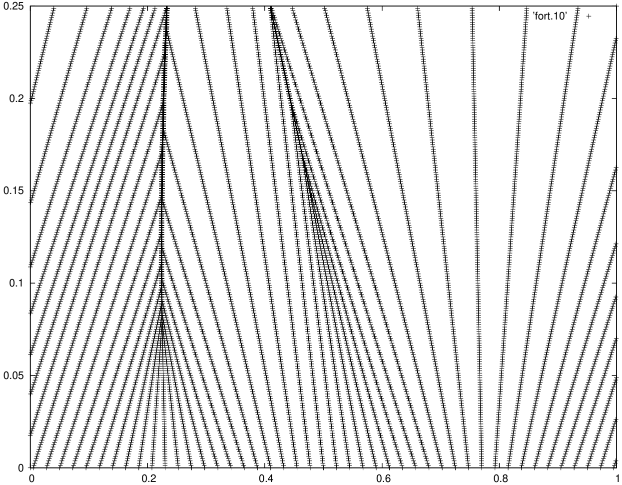
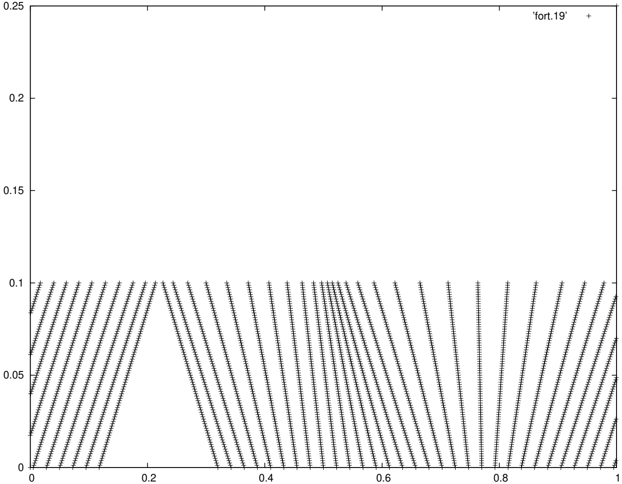
As mentioned, for arbitrarily large T, we may recover, through this problem, the correct "entropy solution" à la Kruzhkov, but only at time T and (surprisingly enough) not for $t < T$, once shocks have formed!



Inviscid Burgers equation : $\partial_t u + \partial_x(u^2/2) = 0$, $u = u(t, x)$, $x \in \mathbb{R}/\mathbb{Z}$, $t \geq 0$.
Formation of two shock waves. (Vertical axis: $t \in [0, 1/4]$, horizontal axis: $x \in \mathbb{T}$.)



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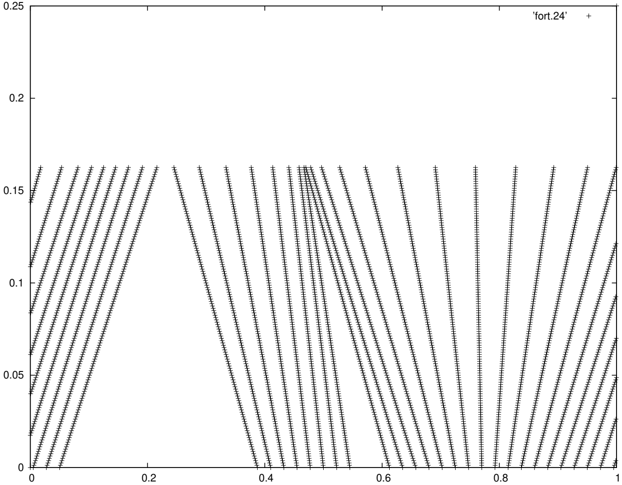
Recovery of the solution at time T=0.1 by convex optimization.
Observe the formation of a first vacuum zone as the first shock has formed.



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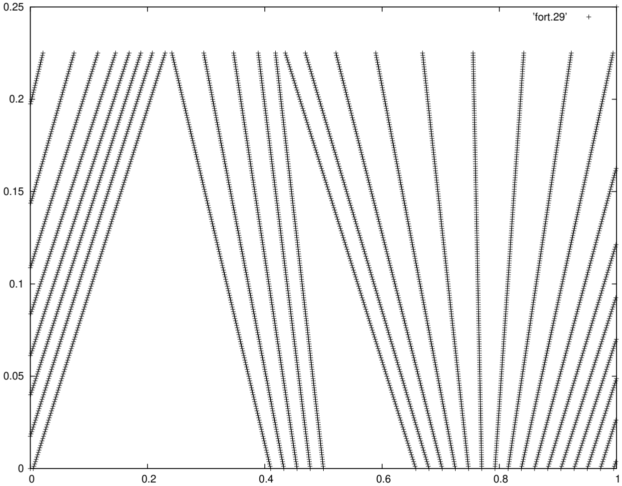
Recovery of the solution at time T=0.16 by convex optimisation.
Observe the formation of a second vacuum zone as the second shock has formed.



Inviscid Burgers equation : $\partial_t u + \partial_x(u^2/2) = 0$, $u = u(t,x)$, $x \in \mathbb{R}/\mathbb{Z}$, $t \geq 0$.
Recovery of the solution at time T=0.225 by convex optimisation.
Observe the extension of the two vacuum zones.

Conclusion: EINSTEIN so close to BURGERS!

$$\mathrm{Inf}_{(\rho,m)} \int_{[0,T]\times\mathbb{T}} (m\rho^{-1}m)(t,x) + 2m(t,x)u_0(x)$$

$$(t,x) \in [0,T]\times\mathbb{T} \to (\rho,m)(t,x) \in \mathbb{R}_+ \times \mathbb{R} \quad \text{s.t.}$$

$$\partial_t \rho + \partial_x m = 0, \ \ \rho_{|t=T} = 1$$

$$\mathrm{Crit}_{(C,M)} \int \mathrm{trace}(MC^{-1}M)(x,\xi)dxd\xi$$

$$(x,\xi) \in \mathbb{R}^{4+4} \to (C,M)(x,\xi) \in \mathbb{R}^{4\times4}\times\mathbb{R}^{4\times4} \quad \text{s.t.}$$

$$\nabla_x \cdot C + \nabla_\xi \cdot M = 0, \ \ \exists A, \ \ C = \nabla_\xi A - (\nabla_\xi \cdot A)\,\mathbb{I}_4.$$

VIVA L'AQUILA!

TANTI AUGURI, PIERO!