Computing closest singular matrix-valued functions

Given a set of matrices $A_i \in \mathbb{C}^{n\times n}$ and a set of analytic functions $f_i : \mathbb{C} \mapsto \mathbb{C}$, we consider a regular matrix-valued function $\mathcal{D}(\lambda) = \sum_{i=0}^{d} f_i(\lambda)A_i$, that is such that $\det(\mathcal{D}(\lambda))$ is not identically zero for $\lambda \in \mathbb{C}$. This class of functions includes for example characteristic functions associated to linear systems of delay differential equations with constant delays. An interesting problem consists in the computation of the nearest singular function $\widetilde{\mathcal{D}}(\lambda) = \sum_{i=0}^{d} f_i(\lambda)(A_i + \Delta A_i)$, where the distance is measured in the Frobenius norm. For example, for a system of delay differential equations

$$\mathrm{A}_0\dot{x}(t) = A_1x(t) + \sum_{i=2}^{d} A_dx(t-\tau_d)$$

the characteristic equation is $\det\left(-\lambda A_0 + A_1 + e^{-\tau_2\lambda}A_2 + \ldots + e^{-\tau_d\lambda}A_d\right) = 0$. Differently from the case of matrix polynomials, in the general case of entire functions $f_i(\lambda)$ like the one above, we have to take into account that the function $\det(\mathcal{D}(\lambda))$ may have an infinite number of roots. This represents a delicate feature of the problem and requires an appropriate analysis in the construction of the numerical method for the computation of the distance to singularity. The condition of singularity is associated to the property that the function $\det\left(\widetilde{\mathcal{D}}(\lambda)\right)$ vanishes on a suitable closed complex curve Γ.

We propose a two level procedure, following the idea introduced in [1], and impose that the determinant vanishes on a finite set of prescribed complex points $\{\mu_j\}_{j=1}^{m}$, suitably sampled on the curve Γ. This can be translated into the minimization of the functional $\mathrm{F}_\varepsilon(\Delta A_0, \ldots, \Delta A_d) = \frac{1}{2}\sum_{j=1}^{m}\sigma_{\min}^2\left(\widetilde{\mathcal{D}}(\mu_j)\right), where \sigma_{\min}$ denotes the smallest singular value and $\varepsilon = \|[\Delta A_0, \ldots, \Delta A_d]\|_F$ is the norm of the perturbation, and in finding the smallest value $\varepsilon^\star$ such that the functional vanishes.
This approach can be extended to situations where the matrix-valued functions present a certain structure. For instance, we can include in the method additional constraints, such as a certain sparsity pattern determined by the original matrices or even structures involving the whole functions, like palindromic properties.

[1] M. Gnazzo, N. Guglielmi, Computing the closest singular matrix polynomial, arXiv preprint arXiv:2301.06335, 2023.

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