Approximated Fractional Differential Equations and Numerical Linear Algebra

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In the past two decades we observed an active and still growing activity of models based on fractional derivatives and numerical methods to solve them. As for non-Euclidean geometries, the topic started long ago and there was skepticism from the mainstreams in Mathematics. However, time did and does its job and after a few centuries the work on fractional problems of any kind is still exploding. We will focus our attention on the linear systems arising from the numerical approximation of fractional differential equations (FDEs) and in this direction few items can be indicated:

1. When approximating on a uniform gridding, as for standard partial differential equations (PDEs), the arising matrix structures are of Toeplitz type: we observe d-level Toeplitz matrices for d-dimensional problems, block type Toeplitz matrices if systems of FDEs are considered or if Discontinuous Galerkin or high order Finite Elements are employed.
2. Opposite to the approximation of PDEs, even when using local approximation methods, the nonlocal nature of the fractional derivatives leads to dense structure and this issue calls for the interest of the Numerical Linear Algebra community.

In the current Tutorial we discuss more in detail the consequences of 1), 2) and we propose a general way for studying the spectral features of the involved matrices and matrix sequences, using the Toeplitz and Generalized Locally Toeplitz technology.

Finally we discuss fast numerical solvers based on the spectral information in the framework of preconditioned Krylov methods and multigrid techniques or combinations of the two.

Keywords: Approximated FDEs, eigenvalues, singular values, Weyl distributions, preconditioned Krylov methods, multigrid, multi-iterative solvers

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