Thermalization and hydrodynamics in an interacting intergrable system: the case of hard rods

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April 18, 2023

Thermalization

Phase space density ρ.

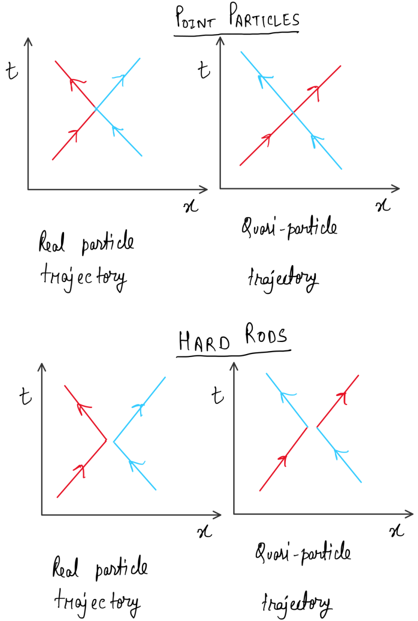
$$\frac{\partial \rho}{\partial t} = \{H, \rho\} \tag{1}$$

In equilibrium, $\frac{\partial \rho}{\partial t} = 0 \implies \{H, \rho\} = 0$
$\implies \rho$ is a function of the constants of motion of the system $I_1, I_2, \ldots$
Non-integrable systems: H is the only constant. So, $\rho \propto e^{-\beta H}$ (Gibbs ensemble)
Integrable systems: $\rho \propto e^{-(\beta_1 I_1 + \beta_2 I_2 + \ldots)}$ (Generalised Gibbs ensemble)

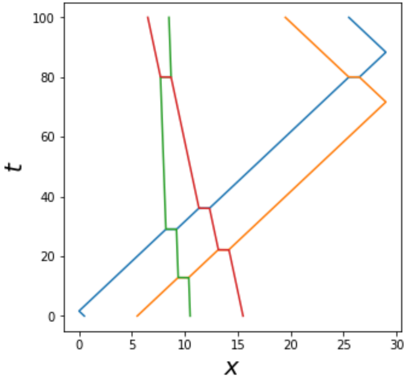
Integrable systems: examples

- 1D point particle gas
- 1D hard rod gas
- ....

Point particle vs hard rods: interacting vs non-interacting



Quasiparticle trajectories



Conserved quantities for hard rods

$$I_n = \sum_{i=1}^{N} p_i^n \tag{2}$$

Thus, if $f(x, p, t)$ is the single particle phase space distribution, the GGE is given by:

$$f(x, p, t) \propto e^{-(\beta_1 p + \beta_2 p^2 + \ldots)} \tag{3}$$

Thus in GGE, $f(x, p, t)$ for a given p does not depend on x

Checking thermalization to GGE

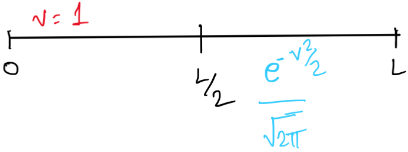
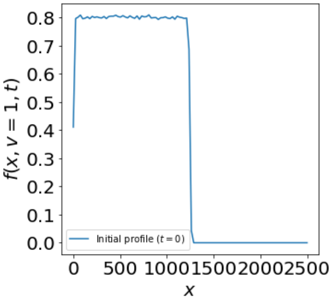
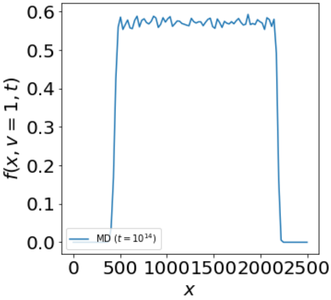


Figure 1: Initial condition.

Checking thermalization to GGE





Thermalization to GGE

Question: Why doesn't it reach GGE?
Answer: Because of finite system size (True thermodynamic limit is not reached)
GGE is reached in an infinite box, as we now show...

Lebowitz, Percus, Syke (LPS) initial condition

LPS like initial condition: $f(x, v, t = 0) = \delta(x)\delta(v - 1) + \rho_0 h(v)$,
where $h(v) = \frac{e^{-v^2/2}}{\sqrt{2\pi}}$.
Solution at other times is of the form
$f(x, v, t) = \delta\rho(x, t)\delta(v - 1) + \rho_0 h(v)$.

LPS initial condition

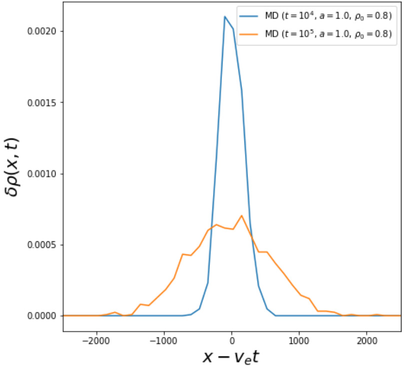


Figure 3: We have taken $N = 2 \times 10^6$, $L = 2.5 \times 10^6$, $a = 1.0$. For MD, ensemble averaging has been done over 1000 realizations. Note that the times are much before the pulse hits the walls of the box, and thus the system is effectively infinite for the times considered.

Euler equation for hard rods

$f(x, v, t)$ is the density of a conserved quantity for a given v.

$$\partial_t f(x, v, t) + \partial_x (v_{eff} f) = 0 \tag{4}$$

where,

$$v_{eff} = \frac{v - a\rho u}{1 - a\rho} \tag{5}$$

and,

$$\rho = \int f(x, v, t) dv \tag{6}$$

$$u = \frac{1}{\rho} \int v f(x, v, t) dv \tag{7}$$

Solution of the Euler equation (Percus)

$$f^0(x', v, t) = \frac{f(x, v, t)}{1 - a\rho(x, t)} \tag{8}$$

where, $x' = x - aF(x, t)$ and $F(x, t) = \int_B^x \rho(y, t) dy$.
$f^0(x', v, t)$ satisfies the free particle equation:

$$\partial_t f^0 + v \partial_{x'} f^0 = 0 \tag{9}$$

This is just a wave equation and can be solved exactly.

Inverse mapping

$$f(x, v, t) = \frac{f^0(x', v, t)}{1 + a\rho^0(x', t)}, \tag{10}$$

where $x = x' + aF^0(x', t)$, $\rho^0(x', t) = \int f^0(x', v, t)dv$ and $F^0(x', t)$ is the cumulative density corresponding to $\rho^0(x', t)$.

Euler vs Newtonian dynamics for the free expansion problem

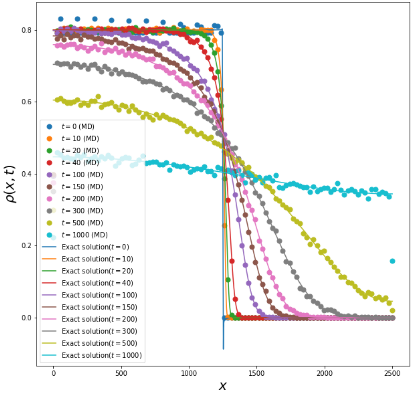


Figure 4: Plot comparing Euler vs MD for $\rho(x, t)$.

Euler vs Newtonian dynamics

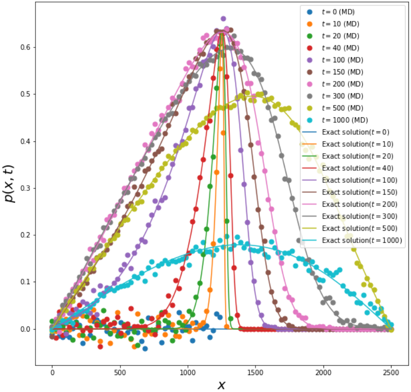


Figure 5: Plot comparing Euler vs MD for $p(x, t)$.

Navier-Stokes vs Newtonian dynamics

$$\partial_t f + \partial_x(v_{eff} f) = \partial_x \mathcal{N} \tag{11}$$

where

$$\mathcal{N} = \frac{a^2}{2(1-a\rho)} \int dw |v-w| (f(w)\partial_x f(v) - f(v)\partial_x f(w)) \tag{12}$$

(Spohn and Doyon)
We again consider LPS-like initial condition:

$$f(x,v,t) = \delta\rho(x,t)\delta(v-1) + \rho_0 h(v) \tag{13}$$

Navier-Stokes

In the long time limit, $\delta\rho(x,t) << \rho_0$. $\rho \approx \rho_0$ and $u \approx 0$. We use this in the Navier-Stokes equation to get:

$$\partial_t(\delta\rho) + v_e \partial_x(\delta\rho) = \frac{n\mu(v')a^2}{2}\partial_x^2(\delta\rho), \tag{14}$$

where $v_e = \frac{v'}{1-a\rho_0}$, $n = \frac{\rho_0}{1-a\rho_0}$, $\mu(v') = \int dv|v - v'|h(v)$. Can be solved exactly:

$$\delta\rho(x,t) = \frac{1}{\sqrt{2\pi n a^2 \mu(v')t}} e^{-\frac{(x-v_e t)}{2a^2 n\mu(v')t}} \tag{15}$$

Navier-Stokes vs Newtonian dynamics

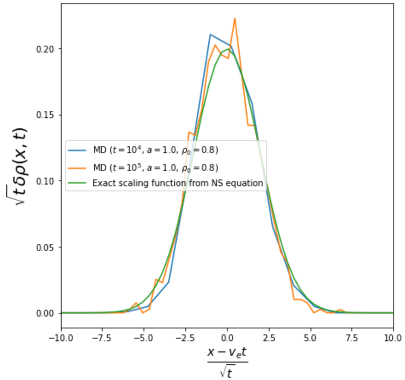


Figure 6: $N = 2 \times 10^6$, $L = 2.5 \times 10^6$.

Conclusion

- No dissipation and thermalization to GGE observed in finite systems.
- True thermodynamic limit is necessary to observe dissipation and thermalization in integrable systems
- Euler equation agrees with Newtonian dynamics in a finite box
- Navier-Stokes equation agrees with Newtonian dynamics in an infinite box in the long time limit.

The end