Integrable Systems and Random Matrices

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Overview

- Background, and motivations
- Recent results
- What is a β-ensemble?
- (Meta) Theorems
- What are we missing?
- Future developments

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- T. Grava, M. Gisonni, G.Gubbiotti, and G.M. *Discrete integrable systems and random Lax matrices*.
- G. M., and R. Memin: *Large Deviations for Ablowitz-Ladik lattice, and the Schur flow*.
- G. M., and T. Grava: *Generalized Gibbs ensemble of the Ablowitz-Ladik lattice, circular β-ensemble and double confluent Heun equation*.

Integrable systems

Consider a Poisson manifold $(M, \{\ ,\ \})$, such that $\{\ ,\ \}$ is non-degenerate. Let $\boldsymbol{x} = (x_1, \ldots, x_{2N})$ be coordinates on M. The evolution $\boldsymbol{x}(0) \to \boldsymbol{x}(t)$ according to Hamilton equations with Hamiltonian $H(\boldsymbol{x})$

$$\frac{dx_j}{dt} = \dot{x}_j = \{x_j, H\}, \quad j = 1, \ldots, 2N$$

is integrable if there are $H_1 = H, H_2, \ldots H_N$ independent conserved quantities $(\dot{H}_k = 0)$ that Poisson commute: $\{H_j, H_k\} = 0$. (Liouville)

Lax Pair

The integrable Hamilton equations

$$\dot{x}_j = \{x_j, H\}, \quad j = 1, \ldots, 2N$$

admits a Lax pair formulation if there exist two square matrices $L = L(\boldsymbol{x})$ and $A = A(\boldsymbol{x})$ such that

$$\dot{L} = [A, L] := LA - AL \longleftrightarrow \dot{x}_j = \{x_j, H\}, \quad j = 1, \ldots, 2N$$

Then, $\mathrm{Tr}L^k$, k integer, are constant of motions: $\frac{d}{dt}\mathrm{Tr}L^k = 0$

Gibbs measure

Consider the Gibbs measure

$$\mu = \frac{1}{Z(V)} e^{-\mathrm{Tr}(V(L(\mathbf{x})))} \tilde{\mu}\,, \qquad \tilde{\mu} = m(x) dx_1, \ldots dx_{2N}\,,$$

here V is a continuous function, and $\tilde{\mu}$ is invariant for the dynamics, thus also μ is invariant.

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In particular, we consider Gibbs measure of the form

$$\mu = \frac{1}{Z(V,\alpha)} \prod_{j=1}^{N} F(x_j,\alpha) \prod_{j=N}^{2N} G(x_j,\alpha) e^{-\mathrm{Tr}(V(L(\boldsymbol{x})))} d\mathbf{x}$$

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$$\mu \longrightarrow L$$

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- How do the correlation functions look like $S(j,t) = \mathbb{E}(x_j(t)x_\ell(0)) - \mathbb{E}(x_j(t))\mathbb{E}(x_\ell(0))$ behave when $N \to \infty$ and $t \to \infty$?

Why:

Correlation functions $\rightarrow$ Transport properties

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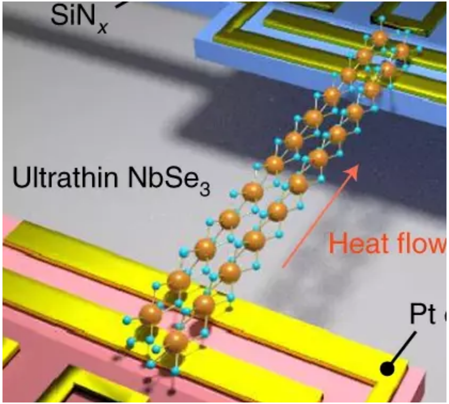
Specific 1D phenomenon: conductivity diverges as the length of the chain grows (Anomalous transport).

Surprisingly, this is **measured** experimentally:



(Nature Nanotechnology 2021)

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For a general dynamical system, the computation of a general correlation function $S(j,t)$ as $t, N \rightarrow \infty$ is a challenging question. Rigorous mathematical results in dimension bigger or equal to 3 (Lukkarinen-Spohn 2011).

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$$S(j,t) \simeq \frac{1}{\lambda t^{\gamma}} f\left(\frac{j - vt}{\lambda t^{\delta}}\right) .$$

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- **Short range harmonic chain**, we can perfectly describe the behaviour of the correlation functions (Mazur;..., M - Grava - McLaughlin - Kriecherbauer).

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  Applying the theory of Generalized Hydrodynamic, he argued that the decay of correlation functions is ballistic. ($\delta = \gamma = 1$)
- A. Guionnet, and R. Memin generalized Spohn results, obtaining a Large deviations principle for the empirical measures with continuous potential.

β-ensemble at high temperature	Integrable System
Gaussian	Toda lattice (Spohn; Guionnet-Memin)
Circular	Defocusing Ablowitz-Ladik lattice (Spohn, Grava-M.; Memin-M.)
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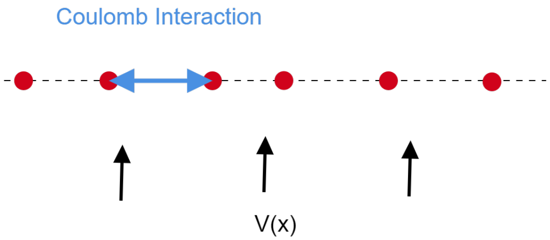
What is a β-ensemble at high temperature?

β - ensembles

Models of Coulomb gas with external potential, and particles constrained to some subset of the real line.



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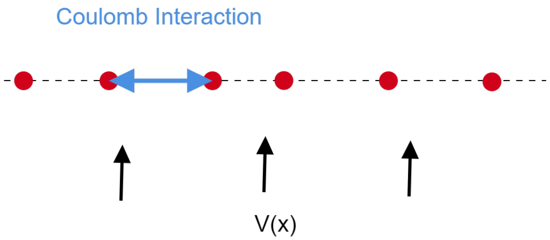

$$H(\mathbf{x}) = \beta \sum_{i<j=1}^{N} \log(|x_j - x_i|) + \sum_{j=1}^{N} V(x_j)\,,$$

$$\nu \sim \exp(-H(\mathbf{x}))d\mathbf{x}$$

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- The density of states of these matrices can be characterized as the minimizer of

$$\mathcal{F}(V,\beta) = \int_{\mathcal{I}} V(x)\sigma(dx) + \beta \int_{\mathcal{I}\times\mathcal{I}\setminus\{x=y\}} \log(|x-y|)\sigma(dx)\sigma(dy)\,.$$

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The distribution of the entries of the matrices reads

$$\nu = \frac{1}{\mathcal{Z}(V,\alpha)} \prod_{j=1}^{N} F\left(x_j, \alpha\left(1-\frac{j}{N}\right)\right) \prod_{j=N}^{2N} G\left(x_j, \alpha\left(1-\frac{j}{N}\right)\right) e^{-\mathrm{Tr}(V(\mathcal{L}(\mathbf{x})))} d\mathbf{x}$$

They look similar...

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$$\mu = \frac{1}{Z(V,\alpha)} \prod_{j=1}^{N} F(x_j, \alpha) \prod_{j=N}^{2N-1} G(x_j, \alpha) e^{-\mathrm{Tr}(V(L(\mathbf{x})))} d\mathbf{x}$$

ramp vs. slide

Metatheorem

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- L is just the periodic version of $\mathcal{L}$;

$$L = \begin{pmatrix} a_1 & b_1 & 0 & \dots & b_N \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_{N-1} \\ b_N & \dots & 0 & b_{N-1} & a_N \end{pmatrix},$$

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then:

$$\partial_\alpha \alpha \mathcal{F}(V, \alpha) = F(V, \alpha)$$

$$\mathcal{F}(V, \alpha) = -\lim_{N \to \infty} \frac{1}{N} \mathcal{Z}(V, \alpha)$$

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then:

$$\partial_\alpha \alpha \nu_{\mathcal{L}}(x) = \mu_L(x)$$

$$\frac{1}{N}\sum_{j=1}^{N} \delta_{\lambda_j^{(L)}} \rightharpoonup \mu_L$$

Why is it relevant?

$$\frac{1}{N}\mathbb{E}\left[\mathrm{Tr}\left(L^k\right)\right] \longrightarrow \partial_\alpha \partial_t \alpha \mathcal{F}(V + itx^k, \alpha)_{|_{t=0}}$$

$$\frac{1}{N}\left(\mathbb{E}\left[\mathrm{Tr}\left(L^k\right)\mathrm{Tr}\left(L^\ell\right)\right] - \mathbb{E}[\mathrm{Tr}\left(L^k\right)]\mathbb{E}[\mathrm{Tr}\left(L^\ell\right)]\right) \to \partial_\alpha \partial_{t_1} \partial_{t_2} \alpha \mathcal{F}(V + it_1 x^k + it_2 x^\ell, \alpha)_{|_{t_1=t_2=0}}$$

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These quantities are relevant for the Theory of generalized hydrodynamics, indeed they are one of the building block of this theory.

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Soon (hopefully): generalization for polynomial potential.

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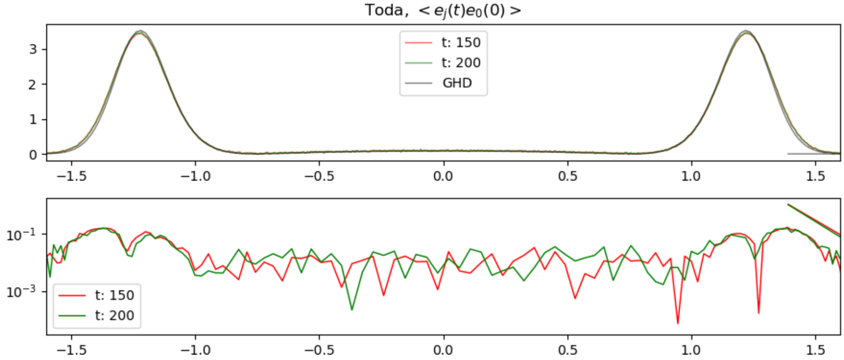
we are interested in

$$\mathbb{E}[J_j^{[k]}] .$$

Toda numerics

H. Spohn was able to compute such quantities for the Toda lattice:



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- Numerically explore the behaviour of correlation for short-range integrable systems,

$$\dot{a}_j = a_j\left(\sum_{k=1}^{\ell} a_{k+j} - \sum_{k=1}^{\ell} a_{-k+j}\right), \quad \dot{a}_j = a_j\left(\prod_{k=1}^{\ell} a_{k+j} - \prod_{k=1}^{\ell} a_{-k+j}\right)$$

Thank you for the attention!