A novel approach to solve the Generalized Hydrodynamics equation

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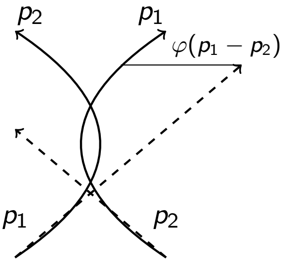
Generalized Hydrodynamics (GHD)

- System of quasi-particles
- Bare velocity $v(p)$
- Scattering: Position shift $\varphi(p_1 - p_2)$
  - Hard rods: $\varphi(p) = -a$
- Quasi-particle density $\rho(t, x, p)$



- GHD equation (conservation form)

$$\partial_t \rho(t, x, p) + \partial_x (v^{\text{eff}}(t, x, p)\rho(t, x, p)) = 0$$

- Effective velocity equation

$$v^{\text{eff}}(p) = v(p) + \int \mathrm{d}q\, \varphi(p - q)\rho(q)(v^{\text{eff}}(q) - v^{\text{eff}}(p))$$

GHD equation

- GHD equation is a highly non-linear PDE
- Efficient algorithms to solve it?
- Existence and Uniqueness of solutions?
- Shock formation?
- Can we simplify the GHD equation?
  - Yes, using natural ideas/objects from physics
  - Many different systems (classical, quantum, soliton gases)
  - Different derivations of GHD $\rightarrow$ variety of approaches/algorithms

The Cauchy-problem of the GHD equation

- Cauchy problem: Given $\rho(0, x, p)$, compute $\rho(t, x, p)$
- Coordinate transformation to free particles (Doyon, Spohn, and Yoshimura 2018):
  - $\partial_t \tilde{\rho} + \partial_{\tilde{x}}(p\tilde{\rho}) = 0$
  - Allows to solve for $\rho(t, \cdot, \cdot)$ at arbitrary t.
- New contribution: Given t and x solve for $\rho(t, x, \cdot)$
  - 'Ideal' to study properties of solutions
  - Efficient and precise solution algorithm

GHD equation in transport form

- Define $\rho_{\mathrm{s}}(t,x,p) = \frac{1+\int \mathrm{d}q\varphi(p-q)\rho(t,x,q)}{2\pi} = \frac{1}{2\pi}1^{\mathrm{dr}}(t,x,p)$
- Occupation function $n(t,x,p) = \frac{\rho(t,x,p)}{\rho_{\mathrm{s}}(t,x,p)}$
- Dressing operation $f(p) \to f^{\mathrm{dr}}(p)$:

$$f^{\mathrm{dr}}(p) = f(p) + \int \frac{\mathrm{d}q}{2\pi}\varphi(p-q)n(q)f^{\mathrm{dr}}(q)$$

- Effective velocity $v^{\mathrm{eff}}(p) = \frac{v^{\mathrm{dr}}(p)}{1^{\mathrm{dr}}(p)}$
- GHD equation in transport form (Castro-Alvaredo, Doyon, and Yoshimura 2016)

$$\partial_t n(t,x,p) + v^{\mathrm{eff}}(t,x,p)\partial_x n(t,x,p) = 0$$

Physical interpretation of occupation function and dressing

- Occupation function and dressing come from quantum mechanics
- Appear naturally in the derivation of GHD via thermodynamic Bethe ansatz (TBA)
- Occupation function $n(p)$: average occupation of quantum states
- Dressing: response of the system to a perturbation
- Lieb-Liniger model (and similarly for other QM models):
  - $\varphi(p) = \frac{2c}{c^2+p^2}$ $\qquad \int \mathrm{d}p\, \varphi(p) = 2\pi$
  - 'Pauli-principle': no double occupation of quantum states $0 < n(p) < 1$

$$\hat{\mathbf{H}}_{\mathrm{LL}} = \sum_i \frac{\hat{\mathbf{p}}_i^2}{2} + c \sum_{i<j} \delta(x_i - x_j)$$

How to solve the transport equation

- GHD equation in transport form

$$\partial_t n(t,x,p) + v^{\mathrm{eff}}(t,x,p)\partial_x n(t,x,p) = 0$$

- Solutions by characteristics

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = v^{\mathrm{eff}}(t, x(t), p)$$

- Occupation function constant along characteristics

$$n(t, x(t), p) = n(0, x(0), p)$$

- Idea for solution algorithm
  1. Choose initial guess for solution $n(t,x,p)$
  2. Compute characteristics from $v^{\mathrm{eff}}(t,x,p)$
  3. Update $n(t,x,p)$ by transporting it along characteristics
  4. Repeat steps 2 and 3 until convergence

Comparison of both GHD equations

- GHD equation in conservation form $\rho = \frac{1}{2\pi} 1^{\mathrm{dr}} n$ and $v^{\mathrm{eff}} \rho = \frac{1}{2\pi} v^{\mathrm{dr}} n$:

$$\partial_t (1^{\mathrm{dr}}(t,x,p) n(t,x,p)) + \partial_x (v^{\mathrm{dr}}(t,x,p) n(t,x,p)) = 0$$

- GHD equation in transport form:

$$1^{\mathrm{dr}}(t,x,p) \partial_t n(t,x,p) + v^{\mathrm{dr}}(t,x,p) \partial_x n(t,x,p) = 0$$

- Difference of both equations:

$$\partial_t 1^{\mathrm{dr}}(t,x,p) + \partial_x v^{\mathrm{dr}}(t,x,p) = 0$$

- Define 'potential' $K(t,x,p)$:

$$\partial_x K(t,x,p) = 1^{\mathrm{dr}}(t,x,p) \qquad \partial_t K(t,x,p) = -v^{\mathrm{dr}}(t,x,p)$$

The function $K(t, x, p)$

- Reinsert into GHD equation in transport form:

$$\partial_x K(t,x,p)\partial_t n(t,x,p) - \partial_t K(t,x,p)\partial_x n(t,x,p) = 0$$

- Define gradient $\boldsymbol{\nabla} = (\partial_x \;\; \partial_t)$: $\boldsymbol{\nabla} K \times \boldsymbol{\nabla} n = 0 \Rightarrow \boldsymbol{\nabla} K \sim \boldsymbol{\nabla} n$
- Solution:

$$n(t,x,p) = \tilde{n}(K(t,x,p),p)$$

- $\tilde{n}(k,p)$ can be determined from the initial condition
- Only need to determine $K(t,x,p)$

Functional fixed-point equation

- Write $K(t,x,p) = K(0,x,p) - v(p)t + L_{t,x}(p)$
- Functional fixed-point equation:

$$L_{t,x}(p) = G_{t,x}[L_{t,x}](p) = \int \frac{\mathrm{d}q}{2\pi}\varphi(p-q) \cdot \Big[N(K(0,x,q) - v(q)t + L_{t,x}(q), q) - N(K(0,x,q),q)\Big]$$

- $K(0,x,p)$ and $N(k,p) = \int_0^k \mathrm{d}k' \, \tilde{n}(k',p)$ from initial data
- Fix-point equation independent for each t and x
- $n(t,x,p) = \tilde{n}(K(0,x,p) - v(p)t + L_{t,x}(p), p)$

Lieb-Liniger model

- Recall $\int \mathrm{d}p \, \varphi(p) = 2\pi$ and $0 < n(p) < 1$
- $N(k,p) = \int_0^k \mathrm{d}k \, \tilde{n}(k,p)$ is Lipschitz: $|N(k_1,p) - N(k_2,p)| < |k_1 - k_2|$

$$\|G_{t,x}[L_1] - G_{t,x}[L_2]\|_\infty < \left(\int \frac{\mathrm{d}q}{2\pi} \varphi(p-q) \right) \|L_1 - L_2\|_\infty$$
$$= \|L_1 - L_2\|_\infty$$

- Fixed point map is contracting
- Banach fixed point theorem: Fixed point always exists and is unique

Lieb-Liniger model

- Solution to the GHD equation exists and is unique for any initial state for all times t (under mild assumptions)
- Smooth initial conditions give rise to smooth solutions
  - No shock formation
- Furthermore:
  - Fixed-point iteration converges exponentially fast
  - New efficient algorithm

Summary

- New approach to study the GHD equation
- Functional fixed point equation to find $\rho(t, x, \cdot)$ at fixed t, x
- Lieb-Liniger model:
    - Fixed point equation is contracting
    - Existence of global unique solution
    - No shock formation

Outlook

- Different formulations of fixed point equation
- What about other (non-quantum) models?:
  - Fixed point equation not necessarily contracting
  - Blow-up solutions in some models
- GHD equation from a optimization problem (convex)
- Impurity problem in GHD (stationary GHD equation with localized external potential)
  - Similar fixed point equation
  - 'Explicit' solutions in simple cases

Thank you

Thank you

How to determine $K(t, x, p)$?

- Dressing equation for 1^{dr} and v^{dr}:

$$\nabla K(t,x,p) = \nabla(x - v(p)t) + \int \frac{\mathrm{d}q}{2\pi}\varphi(p-q)n(t,x,q)\nabla K(t,x,q)$$

- Define $N(k,p) = \int_0^k \mathrm{d}k\, \tilde{n}(k,p)$
- Using $\nabla(N(K(t,x,p),p)) = n(t,x,p)\nabla K(t,x,p)$:

$$\nabla K(t,x,p) = \nabla(x - v(p)t) + \int \frac{\mathrm{d}q}{2\pi}\varphi(p-q)\nabla(N(K(t,x,q),q))$$

- Integrate along any space-time path from $(t = 0, x = 0)$ to t, x.