Dynamical corellation of the Gibbs measure in a gas in low density scaling

Corentin Le Bihan
(ENS de Lyon, PhD sudent under the supervision of S. Simonella)

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The system of particles

Look n particles in $\mathbb{T}^d$ with coordinates

$$Z_n := (x_1, v_1, \cdots, x_n, v_n) \in \mathbb{D}^n = \left(\mathbb{T}^d \times \mathbb{R}^d\right)^n$$

They follow the Hamiltonian dynamic

$$\mathcal{H}_n(Z_n) := \frac{1}{2}\|V_n\|^2 + \sum_{1 \leq i < j \leq N} \mathcal{V}\left(\frac{\|x_i - x_j\|}{\varepsilon}\right)$$

$$\frac{d}{dt}x_i = v_i, \quad \frac{d}{dt}v_i = \frac{1}{\varepsilon}\sum_{j \neq i} \nabla\mathcal{V}\left(\frac{x_i - x_j}{\varepsilon}\right).$$

The potential $\mathcal{V}$ is assume

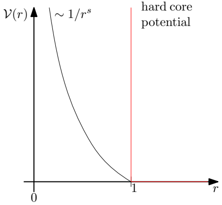


Figure: $\mathcal{V}$ compactly supported interaction potential

Grand canonical ensemble

We don't fix the number of particles:
Define $\mathcal{D} := \bigsqcup_{n\geq 0} \mathbb{D}^n$ grand canonical phase space and $\mathcal{N}$ the number of particles.

The system is at equilibrium: define the Gibbs Measure as

$$d\mathbb{P}_\varepsilon(\boldsymbol{Z}_\mathcal{N}) := \frac{1}{\mathcal{Z}_\varepsilon} \sum_{n\geq 0} \frac{\mu^n}{(2\pi)^{dn/2} n!} \mathbb{1}_{\mathbb{D}^n} e^{-\beta \mathcal{H}_n(\boldsymbol{Z}_n)} d\boldsymbol{Z}_n.$$

Note that $\mathbb{E}_\varepsilon[\mathcal{N}] \sim \mu$.

We fix the Boltzmann Grad scaling $\mu\varepsilon^{d-1} = 1$.

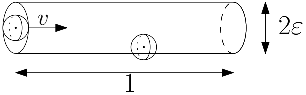


Figure: The density of particle is of order μ, the speed of order 1

Law of large number and central limit theorem

Define the empirical measure: for $g \in \mathcal{C}_c^\infty(\mathbb{D})$ a test function

$$\pi_\varepsilon^t(g) := \frac{1}{\mu} \sum_{i=1}^{\mathcal{N}} g(\mathbf{z}_i(t)).$$

Theorem (Law of large number)

In the Boltzmann-Grad scaling, for $M(v) := e^{-\frac{\|v\|^2}{2}}/(2\pi)^{d/2}$, $\forall g \in \mathcal{C}_c^\infty(\mathbb{D})$

$$\mathbb{E}_\varepsilon \left[\pi_\varepsilon^t(g) - \int g(z) M(v) dv \right] \to 0$$

Define the fluctuation: for $g \in \mathcal{C}_c^\infty(\mathbb{D})$ a test function

$$\zeta_\varepsilon^t(g) := \sqrt{\mu} \left(\pi_\varepsilon^t(g) - \mathbb{E}_\varepsilon \left[\pi_\varepsilon^t(g) \right] \right).$$

We want to prove that the ζ_ε^t converge in Boltzmann-Grad scaling to a Gaussian field (Bodineau, Gallagher, Saint-Raymond, Simonella 2022)
The first step: prove the convergence of the covariance for g, h two test functions

$$\mathbb{E}_\varepsilon \left[\zeta_\varepsilon^t(h) \zeta_\varepsilon^0(g) \right]$$

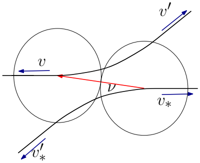
Theorem

In the Boltzmann-Grad scaling, for $\forall g, h \in \mathcal{C}_c^\infty(\mathbb{D})$, $T > 0$

$$\sup_{t \in [0, (\log|\log \varepsilon|)^{\frac{1}{4}} T]} \left| \mathbb{E}_\varepsilon \left[\zeta_\varepsilon^t(h) \zeta_\varepsilon^0(g) \right] - \int h(z) \mathbf{g}(t,z) M(v) dz \right| \underset{\varepsilon \to 0}{\longrightarrow} 0$$

with $\mathbf{g}(t)$ *solution of the linearized Boltzmann equation*

$$\partial_t \mathbf{g}(t) + v \cdot \nabla_x \mathbf{g}(t) = \mathcal{L}\mathbf{g}(t)$$
$$\mathbf{g}(t=0) = g$$
$$\mathcal{L}h(v) = \int (h(v') + h(v'_*) - h(v) - h(v_*)) M(v_*)((v - v_*) \cdot \nu)_+ d\nu dv_*$$



- For hard hard spheres: (Bodineau, Gallagher, Saint-Raymond, Simonella 2021, L. 2022).
- For other interaction potentials: you have to adapt the proof.

Pseudotrajectory devellopement

To simplify the presentation, take

$$\int h(z)M(v)dz = \int g(z)M(v)dz = 0.$$

Then

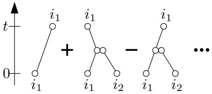
$$\mathbb{E}_\varepsilon\left[\zeta^t_\varepsilon(h)\zeta^0_\varepsilon(g)\right] = \mathbb{E}_\varepsilon\left[\frac{1}{\mu}\sum_{i=1}^{\mathcal{N}} h(\mathbf{z}_i(t))\sum_{j=1}^{\mathcal{N}} g(\mathbf{z}_j(0))\right]$$

We want to construct a family of functionals $\Phi^t_{1,n} : L^\infty(\mathbb{D}) \to L^\infty(\mathbb{D}^n)$ such that

$$\mathbb{E}_\varepsilon\left[\zeta^t_\varepsilon(h)\zeta^0_\varepsilon(g)\right] = \sum_{n\geq 1}\mathbb{E}_\varepsilon\left[\frac{1}{\mu}\sum_{\underline{i}_n}\Phi^t_{1,n}[h](\mathbf{Z}_{\underline{i}_n}(0))\sum_{j=1}^{\mathcal{N}} g(\mathbf{z}_j(0))\right]$$

where $\underline{i}_n := (i_1, \cdots, i_n)$, $1 \leq i_k \leq \mathcal{N}$, $i_k \neq i_l$.

We want to know the final position of particle i_1:

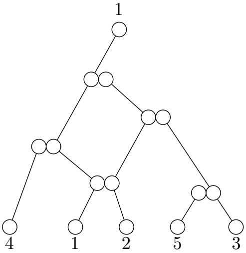


We can now develop the functionals $\Phi^t_{\varepsilon,n}$

$$\Phi^t_{1,n}[h](Z_n) := \frac{1}{n!} \sum_{\text{history}} h(z_k(t)) \mathbb{1}_{\text{history}} \sigma_{\text{history}}$$

There is two kind of collision:

- *annihilation*: we remove one of the particles,
- *recollision*: the two particles survive.

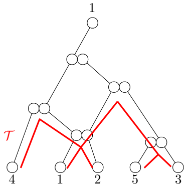


Bound of the $\Phi^t_{1,n}[h]$

We do not know how to use the cancellations due to σ_{history}.

$$\left\|\Phi^t_{1,n}[h]\right\|_{L^1(M^{\otimes n}dZ_n)} \le \frac{\|h\|_{L^\infty}}{n!}\sum_{\text{history}}\left\|\mathbb{1}_{\text{history}}\right\|_{L^1(M^{\otimes n}dZ_n)}$$

Construct the clustering tree $\mathcal{T}$:



Construct the clustering tree $\mathcal{T}$:

$$\left\|\Phi^t_{1,n}[h]\right\|_{L^1(M^{\otimes n}dZ_n)} \le \frac{\|h\|_{L^\infty}}{n!}\sum_{\text{history}}\sum_{\mathcal{T}}\underbrace{\left\|\mathbb{1}_{\substack{\text{history}\\ \mathcal{T}}}\right\|_{L^1}}_{\le C\left(\frac{Ct}{\mu}\right)^{n-1}} \le C\|h\|_{L^\infty}\left(\frac{Ct}{\mu}\right)^{n-1}\frac{|\{\mathcal{T}\}|}{n!}|\{\text{history}\}|$$

$$\le C\|h\|_{L^\infty}\left(\frac{Ct}{\mu}\right)^{n-1}|\{\text{history}\}|$$

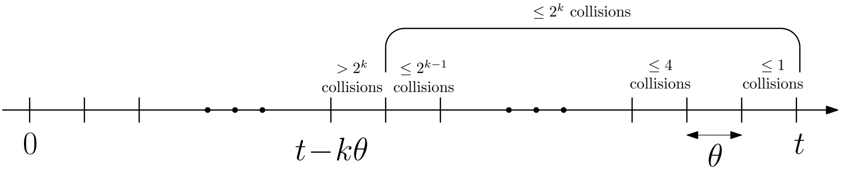
If there are a recollision, $\exists\alpha > 0$

$$\left\|\Phi^{t,\text{reco}}_{1,n}[h]\right\|_{L^1(M^{\otimes n}dZ_n)} \le C\|h\|_{L^\infty}\left(\frac{Ct}{\mu}\right)^{n-1}|\{\text{history}\}|\varepsilon^\alpha.$$

Bound of the number of annihilation

We stop the expansion when the number of history begin to explode.

For the annihilation, we consider a first sampling of size $\theta \sim |\log \varepsilon|^{-1}$:
Consider the case with no recollision



Stop the expension at time $t - k\theta$.
We have $|\{\text{history}\}| \leq C^{2^k}$ and

$$\left\|\Phi^{k\theta,\text{samp}}_{1,2^k+1}[h]\right\|_{L^1} \leq C\|h\|_{L^\infty} C^{2^k}\left(\frac{Ct}{\mu}\right)^{2^k}\left(\frac{C\theta}{\mu}\right)^{2^k} \leq \frac{C\|h\|_{L^\infty}}{\mu^{2^k-1}}\left(C't\theta\right)^{2^k}.$$

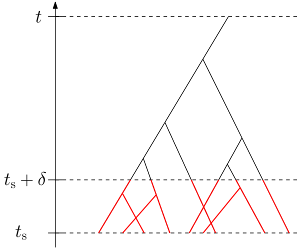
Useful to have long time convergence.

Bound of the number of recollision

We consider a second (smaller) sampling of size $\delta = \varepsilon^{\beta}$, $\beta \in (0,1)$.

Define $t_s = t - k\delta$ such that there is

- No recollision after time $t_s + \delta$,
- at least one in $(t_s, t_s + \delta)$



We need two conditioning on the initial data $\boldsymbol{Z}_{\mathcal{N}} \in \mathcal{D}$ to estimates the number of recollisions:

- one to provide that only $\gamma \in \mathbb{N}$ (fixed) particles can form a dynamical cluster on $(t_s, t_s + \delta)$, (a symmetric conditioning),
- in (Bodineau, Gallagher, Saint-Raymond, Simonella 2021) use a billard theory theorem (Burago, Ferleger, Konenko, 1998),
- a second to provide that the collision graph on $(t_s, t_s + \delta)$ has no cycle (an asymmetric conditioning)

Derviation of linearized Landau equation

We add an other scaling parameters α

$$\mathcal{H}_N(Z_N) := \frac{1}{2}\|V_N\|^2 + \alpha \sum_{1\leq i<j\leq N} \mathcal{V}\left(\frac{\|x_i - x_j\|}{\varepsilon}\right)$$

Take $\mathcal{V}(r) = \frac{1-r}{r}\mathbb{1}_{r\leq 1}$.

Conjecture

We take the Boltzmann-Grad scaling $\mu\varepsilon^2 = \alpha^{-2}|\log\alpha|^{-1}$ *and the grazing collision limit* $\alpha = (\log\log|\log\varepsilon|)^{-1}$.
Then for $f, g \in \mathcal{C}_c^\infty(\mathbb{D})$ *we have* $\forall T > 0$

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\left|\mathbb{E}_\varepsilon\left[\zeta_\varepsilon^t(h)\zeta_\varepsilon^0(g)\right] - \int h(z)\mathbf{g}_L(t,z)M(v)dz\right| = 0.$$

where $\mathbf{g}_L(t)$ *is solution of the linearized Boltzmann equation*

$$\partial_t\mathbf{g}_L(t) + v\cdot\nabla_x\mathbf{g}_L(t) = \mathcal{K}\mathbf{g}_L(t)$$
$$\mathbf{g}_L(t=0) = g,$$

$$\mathcal{K}h(v) := \frac{2\pi}{M(v)}\nabla_v\cdot\left(\int\frac{P^\perp_{v-v_*}}{|v-v_*|}(\nabla h(v) - \nabla h(v_*))M(v)M(v_*)dv_*\right)$$

with $P^\perp_{v-v_*}$ *the projector on the plan orthogonal to* $v - v_*$.

Thank you