Crossover scaling functions in the asymmetric avalanche process

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Motivation

Stochastic integrable systems of interacting particles is a tool to study such phenomena as random growth of interfaces:

- borders of bacterial colonies
- shapes of crystals
- combustion and wetting fronts
- polymers in random media
- traffic flows

The common point is **the universal behaviour at large scales**.

Universality classes

- Kardar-Parisi-Zhang (KPZ)
- Edwards-Wilkinson (EW)

To describe random many-particle systems we need **exactly solvable models** with stochastic dynamics. Universal scaling exponents and crossover scaling functions can be obtained from exact results in the scaling limit.

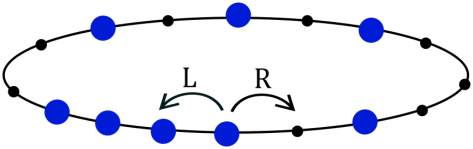
Outline

- **Definition:** Asymmetric avalanche model and its particle current
- **Methods:** stationary state analysis, Bethe anzats approach and Baxter's TQ-equation
- **Results:** Present exact formulas for mean particle current and diffusion coefficient. Obtain scaling exponents and crossover scaling functions in the thermodynamic limit

Asymmetric Avalanche Process

is a 1-dimensional stochastic process on a ring evolving in continuous time

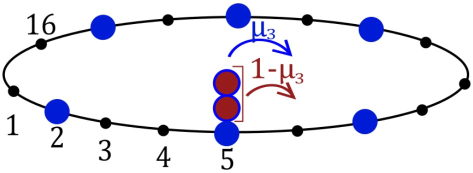
- States $\eta(t) : \mathbb{R}_{\geq 0} \to \{0, 1\}^{\mathbb{Z}/N\mathbb{Z}}$ with exactly p particles and no more than one particle per site;
- Evolution:
  - all particles occupy different sites: jump randomly and independently having waited for $\mathbb{P}(t(\eta_k) < T) = 1 - e^{-T}$ either left or right, $R + L = 1$
  - particle comes to already occupied site the avalanche dynamics starts



(Priezzhev, Ivashkevich, Povolotsky, Hu, 2001)

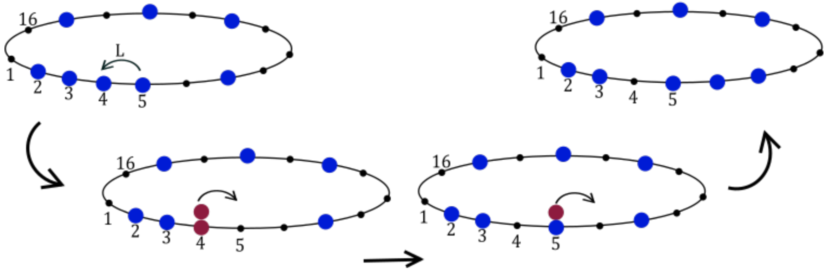
Avalanche dynamics

- with probability μ_n, n particles go to the next site;
- with probability $1 - \mu_n$, $n - 1$ particles go to the next site and one particle stays at the current site.
- **occurs instantly**



transition rate $u(\eta' \to \eta) = R$

transition rate $u(\eta' \to \eta) = L\mu_2(1 - \mu_2)$



Master equation

$P_t(\boldsymbol{\eta}) := \mathbb{P}(\eta(t) = \eta)$ - probability to be at state $\boldsymbol{\eta}$ at time t.

Given an initial distribution $P_0(\boldsymbol{\eta})$, $P_t(\boldsymbol{\eta})$ satisfies forward Kolmogorov equation

$$\partial_t P_t(\boldsymbol{\eta}) = \mathcal{L} P_t(\boldsymbol{\eta}),$$

$$\mathcal{L} P_t(\boldsymbol{\eta}) = \sum_{\boldsymbol{\eta}'} \left(u(\boldsymbol{\eta}' \to \boldsymbol{\eta}) P_t(\boldsymbol{\eta}') - u(\boldsymbol{\eta} \to \boldsymbol{\eta}') P_t(\boldsymbol{\eta}) \right)$$

Bethe ansatz integrability condition + positivity of rates (Priezzhev, Ivashkevich, Povolotsky, Hu, 2001)

$$\mu_n = 1 - [n]_q = 1 - \frac{1 - q^n}{1 - q}, \quad -1 < q < 0$$

Why this process is interesting ?

- Unstable states may appear randomly
- Specific transition into a totally unstable state, when $\rho \to \rho_c$ and an avalanche never stops in the thermodynamic limit $(p, N \to +\infty, \rho = const)$
- Unusual universal scaling behaviour, for ex. for the average particle current per site j_N

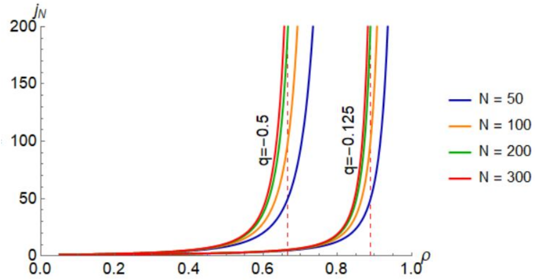


Figure: j_N for $q = -0.5$, $\rho_c = 2/3$ and $q = -0.125$, $\rho_c = 8/9$.

Stationary probability measure

is extremely simple
$$\boldsymbol{P}_{st}(\boldsymbol{x}) = \frac{1}{C_N^p}.$$

Analysis of discretized AAP stationary measure reveals the structure of avalanches resulting in

$$j_N = \frac{(1-q)}{C_N^p} \sum_{m=0}^{p-1} (m+1) \frac{(-1)^m C_N^{p-m-1}}{1-q^{m+1}} (Rq^m - L)$$

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$$= \frac{(1-q)}{C_N^p} \oint \frac{(1+z)^N}{z^p} \Big[Rg'(zq) - Lg'(z) \Big] \frac{dz}{2\pi i}.$$

in terms of

$$g(z) = -\sum_{k=0}^{\infty} \frac{(-z)^{k+1}}{1-q^{k+1}} = \sum_{k=0}^{\infty} \frac{q^i z}{1+q^i z}.$$

(it has poles $z_i = -q^i, i \geq 0$)

Total distance Y_t

$Y_0 = 0$,
$Y_t : \Omega \times \mathbb{R} \to \mathbb{Z}_{\geq 0}$ - random variable of total number of jumps made by all particles

$$Y_t \to Y_t + \Delta Y_t,\ \ \Delta Y_t \in \{1, -1, n \leq p\}$$

The behaviour of moment generating function in the large time limit $t \to \infty$ is dominated by the largest eigenvalue $\lambda(\gamma)$ of the deformed model generator $\mathcal{L}(\gamma)$

$$\lambda(\gamma) = \lim_{t\to\infty} \frac{\ln \mathbb{E} e^{\gamma Y_t}}{t} = \sum_{n=1}^{\infty} c_n \frac{\gamma^n}{n!},$$

First and second scaled cumulants:

$$J := c_1 = \lim_{t\to\infty} \frac{\mathbb{E}(Y_t)}{t}, \quad \Delta := c_2 = \lim_{t\to\infty} \frac{\mathbb{E}(Y_t^2) - \mathbb{E}(Y_t)^2}{t},$$

Methods: Bethe anzatz, Baxter's TQ-equation
(Baxter, 1972, Prolhac, Mallick, 2008)

Mean particle current

Introducing *normalized differential*

$$D_{N,p}(t) := \frac{dz}{2\pi \mathrm{i}} \frac{1}{C_N^p} \frac{(1+t)^N}{t^{p+1}}.$$

we reproduce the stationary state result

$$j_N = R j_N^R - L j_N^L$$

$$j_N^R = (1-q) \oint D_{N,p}(z) z g'(zq), \qquad j_N^L = (1-q) \oint D_{N,p}(z) z g'(z).$$

The group diffusion coefficient is

$\Delta = R\Delta^R - L\Delta^L$, where both right and left parts are given by the formula

$$\Delta^I = \epsilon(I)pNj_N^I + 2N^2 \sum_{i=0}^{\infty} \oint \oint D_{N,p}(t) D_{N,p}(y) ty \frac{a^I(y)}{t - q^i y}$$
$$+2N^2 \sum_{i=1}^{\infty} \oint \oint D_{N,p}(t) D_{N,p}(y) ty \frac{q^i a^I(q^i y)}{t - q^i y}$$

for $I \in \{R, L\}$, where function $\epsilon(R) = 1, \epsilon(L) = -1$ stands for sign and functions

$$a^R(y) = (1-q)g'(qy) - \frac{j_N^R}{\rho(1+y)},$$
$$a^L(y) = (1-q)g'(y) - \frac{j_N^L}{\rho(1+y)}.$$

Asymptotic analysis in the thermodynamic limit $p, N \to \infty, p/N = \rho$

The critical density is a point of model phase transition $\rho_c = \frac{1}{1-q}$.

$$j_N(\rho) \simeq \begin{cases} \frac{\rho(1-\rho)(R\rho_c+(1-\rho_c)L)}{(\rho-\rho_c)^2} + j_\infty^{\mathrm{reg}}(\rho), & \rho < \rho_c, \\ N(R\rho_c + L(1-\rho_c)), & \rho = \rho_c, \\ N^{3/2} e^{Ns(\rho|\rho_c)} \frac{\sqrt{2\pi\rho(1-\rho)}}{\rho_c(1-\rho_c)}(\rho-\rho_c)(\rho_c R + (1-\rho_c)L), & \rho > \rho_c, \end{cases}$$

where

$$j_\infty^{\mathrm{reg}}(\rho) = \frac{\rho_c R + (1-\rho_c)L}{\rho_c(1-\rho_c)} \sum_{k=1}^{\infty} k \frac{\left[\frac{(\rho_c-1)^2}{\rho-1}\frac{\rho}{\rho_c^2}\right]^k}{1-\left[\frac{\rho_c-1}{\rho_c}\right]^k} - \frac{L\rho(1-\rho)}{\rho_c}$$

$$s(\rho|\rho_c) = (1-\rho)\ln\left(\frac{1-\rho}{1-\rho_c}\right) + \rho\ln\left(\frac{\rho}{\rho_c}\right)$$

Crossover function for j_N

Result 2: Under the scaling of

$$\beta = \frac{\sqrt{N}(\rho_c - \rho)}{\sqrt{\rho_c(1-\rho_c)}}$$

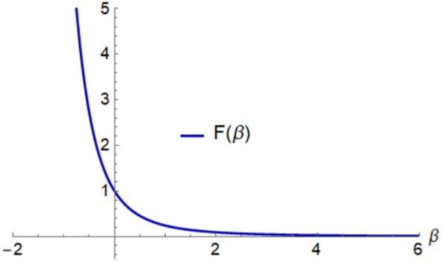
the particle current is described by

$$j_N(\rho) = N(R\rho_c + L(1-\rho_c))\mathcal{F}(\beta) + O(N^{\frac{1}{2}}),$$

where

$$\mathcal{F}(\beta) = 1 - \sqrt{\frac{\pi}{2}}\beta \, \mathrm{erfc}\left(\frac{\beta}{\sqrt{2}}\right) e^{\frac{\beta^2}{2}}.$$

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2} dt.$$



Asymptotic analysis in the thermodynamic limit $p, N \to \infty, p/N = \rho$

$$\Delta_N(\rho) \simeq \begin{cases} N^{3/2}\left(\frac{f(\rho)}{2(\rho-\rho_c)^4} + \Delta_\infty^{reg}(\rho)\right), & \rho < \rho_c \\ N^{7/2}(R\rho_c + L(1-\rho_c))\sqrt{\pi\rho_c(1-\rho_c)}, & \rho = \rho_c \\ N^4 e^{2Ns(\rho|\rho_c)} 4\pi(\rho-\rho_c)(R\rho_c + L(1-\rho_c))\frac{\rho(1-\rho)}{\rho_c(1-\rho_c)}, & \rho > \rho_c \end{cases}$$

where

$$f(\rho) = \sqrt{\pi}(R\rho_c + L(1-\rho_c))(\rho_c(1-\rho_c))^{3/2}(\rho_c^2 - 2\rho_c(1-\rho) - \rho)$$

$$\Delta_\infty^{\mathrm{reg}}(\rho) \simeq \frac{\sqrt{\pi}(R\rho_c + L(1-\rho_c))}{4\sqrt{\rho(1-\rho)}\rho_c(1-\rho_c)} \sum_{k=1}^{\infty} \frac{\left[\frac{(\rho_c-1)^2}{\rho-1}\frac{\rho}{\rho_c^2}\right]^k}{1-\left[\frac{\rho_c-1}{\rho_c}\right]^k}\left(k^2(1-2\rho) - k^3\right) - \frac{\sqrt{\pi}(\rho(1-\rho))^{3/2}}{4\rho_c}L.$$

Result 3: Under the scaling of

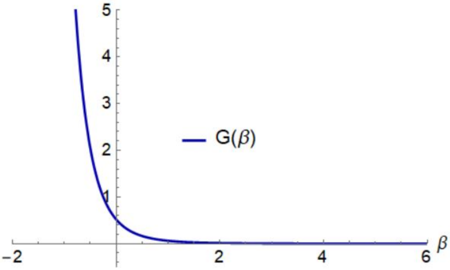
$$\beta = \frac{\sqrt{N}(\rho_c - \rho)}{\sqrt{\rho_c(1-\rho_c)}}$$

the group diffusion coefficient is

$$\Delta_N(\rho) = N^{\frac{7}{2}} \left(R\rho_c + L(1-\rho_c)\right) \sqrt{\rho_c(1-\rho_c)}\, \mathcal{G}(\beta) + O(N^3)$$

where the crossover function is

$$\mathcal{G}(\beta) = \sqrt{\pi}(2\mathcal{F}(\sqrt{2}\beta) - \mathcal{F}(\beta))$$



Thank you!