Hydrodynamic and Hydrostatic limit for a generalized contact process with slow reservoirs

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GSSI, Scaling limits and generalized hydrodynamics

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Introduction

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- Sterile males are introduced in a healthy population at a constant rate.
- The reproduction rate is therefore decreased.
- With a high enough rate of injection of sterile individuals, the healthy population becomes extinct.

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- An exchange dynamics with parameter D models the displacements of the mosquitoes.
- "Slow" reservoirs at the boundary to model migrations/immigrations.

Introduction

First issue of interest: Hydrodynamic limit (law of large numbers)

Microscopic probabilistic dynamics $\Rightarrow$ Macroscopic deterministic dynamics

Sequence of empirical measures $\{\pi_t^N, \ t \in [0,1]\}_{N \geq 1}$ defines a measure Q^N on $D([0,T], \mathcal{M}^+(V))$ where V is the underlying continuous space.

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Sequence of empirical measures $\{\pi_t^N,\ t \in [0,1]\}_{N\geq 1}$ defines a measure Q^N on $D([0,T],\mathcal{M}^+(V))$ where V is the underlying continuous space. Under some appropriate space-time renormalization, Q^N converges to Q, a measure on $D([0,T],\mathcal{M}^+(V))$ concentrated on $\{\rho(t,x)dx,\ t \in [0,T]\}$, where ρ is the solution of a PDE (hydrodynamic equation).

Second issue of interest: Hydrostatic limit

Microscopic equilibrium (invariant measure)

$\Rightarrow$ Macroscopic equilibrium (stationary solution of the PDE)

Invariant measure for the microscopic dynamics $\{\pi_t^N,\ t \in [0,1]\}_{N\geq 1}$ is *associated* (under same space time renormalisation) to the unique stationary solution of the hydrodynamic equation.

Interacting particle system at play

$d \geq 1$, $N \in \mathbb{N}$, finite volume $V_N = \{-N, ..., N\} \times (\mathbb{Z}/N\mathbb{Z})^{d-1}$.
State space of configurations:

$$E_N = \{0, 1, 2, 3\}^{V_N}$$

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Markovian jump process $(\eta_t^N)_{t\geq 0}$ parameterized by N on the state space E_N. For $t \geq 0$, $N \in \mathbb{N}$ and $x \in V_N$, $\eta_t^N(x)$ satisfies:

$$\eta_t^N(x) = \begin{cases} 0 & \text{if } x \text{ is empty,} \\ 1 & \text{if there are only healthy mosquitoes in } x, \\ 2 & \text{if there are only sterile mosquitoes in } x, \\ 3 & \text{if there are healthy and sterile mosquitoes in } x. \end{cases}$$

Interacting particle system at play

The global dynamics is a superposition of three dynamics: a generalized contact dynamics, an exchange and a boundary dynamics.

- Rates for the generalized contact dynamics: $0 < \lambda_2 < \lambda_1$, and $r > 0$. Denote $n_i(x, \eta)$ the number of neighbors of x in state i in η.
  - **Birth rates**:

$$0 \to 1: \text{ at rate } \lambda_1 n_1(x,\eta) + \lambda_2 n_3(x,\eta), \quad 0 \to 2: \text{ at rate } r$$

$$2 \to 3: \text{ at rate } \lambda_1 n_1(x,\eta) + \lambda_2 n_3(x,\eta), \quad 1 \to 3: \text{ at rate } r$$

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  - **Death rates:**

$$1 \to 0,\ 2 \to 0,\ 3 \to 1,\ 3 \to 2: \text{ at rate } 1$$

Interacting particle system at play

- Rates for the exchange dynamics: two neighbouring sites exchange states at rate DN^2 with $D > 0$, i.e. for $x, y \in V_N$ two neighbours,

$$(\eta(x), \eta(y)) \to (\eta(y), \eta(x)) : \text{ at rate } DN^2.$$

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- For the boundary dynamics: fix a function $\widehat{b} = (b_1, b_2, b_3) : \Gamma \to \mathbb{R}^3$ and two slow-down parameters $\theta_\ell, \theta_r \geq 0$. For $x \in \Gamma_N^+$ and $i \in \{0, 1, 2, 3\}$,

$$\eta(x) \to i : \text{ at rate } N^{2-\theta_r} b_i(x/N)$$

with $b_0 = 1 - b_1 - b_2 - b_3$ and $\eta_0 = 1 - \eta_1 - \eta_2 - \eta_3$. For $x \in \Gamma_N^-$, replace θ_r by θ_ℓ in the above rates.

Hydrodynamic limit

Denote $V = [-1,1] \times \mathbb{T}^{d-1}$ the continuous macroscopic counterpart of V_N. For $t \geq 0$, denote $\widehat{\pi}_t^N \in \big(\mathcal{M}_+(V)\big)^3$ the vector of empirical measures associated to the configuration η_t^N:

$$\widehat{\pi}_t^N = \Big(\frac{1}{N^d} \sum_{x \in V_N} \eta_{t,1}^N(x) \delta_{x/N}, \frac{1}{N^d} \sum_{x \in V_N} \eta_{t,2}^N(x) \delta_{x/N}, \frac{1}{N^d} \sum_{x \in V_N} \eta_{t,3}^N(x) \delta_{x/N}\Big)$$

where $\delta_{x/N}$ is the Dirac measure on x/N and $\eta_{t,i}^N(x) = \mathbf{1}_{\eta_t^N(x)=i}$.

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Fix a time horizon $T > 0$.

$$(\eta_t^N)_{t\in[0,T]} \in D\Big([0,T], E_N\Big) \Rightarrow (\widehat{\pi}_t^N)_{t\geq 0} \in D\Big([0,T], \big(\mathcal{M}_+(V)\big)^3\Big).$$

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Question: Asymptotic behavior of

$$(\widehat{\pi}_t^N)_{t \geq 0} \text{ when } N \to \infty?$$

Hydrodynamic limit

Fix $\widehat{\theta} = (\theta_\ell, \theta_r)$. Denote $(Q_N^{\widehat{\theta}})_{N\geq 1}$ the law of $(\widehat{\pi}_t^N)_{t\in[0,T]}$ in the Skorohod space of measure trajectories $D\Big([0,T], \big(\mathcal{M}_+(V)\big)^3\Big)$. The hydrodynamic limit is given by the following system of coupled non linear equations with mixed boundary conditions depending on the values of θ_ℓ and θ_r:

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- **Bulk equation:** $\widehat{\rho} = (\rho_1, \rho_2, \rho_3) : [0,T] \times V \to [0,1]^3$

$$\partial_t \widehat{\rho} = D\Delta\widehat{\rho} + \widehat{F}(\widehat{\rho}) \quad \text{in} \quad V \times (0,T)$$

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where $\widehat{F} = (F_1, F_2, F_3) : [0,1]^3 \to \mathbb{R}^3$ will be given by

$$\begin{cases} F_1(\rho_1,\rho_2,\rho_3) = 2d(\lambda_1\rho_1 + \lambda_2\rho_3)\rho_0 + \rho_3 - (r+1)\rho_1 \\ F_2(\rho_1,\rho_2,\rho_3) = r\rho_0 + \rho_3 - 2d(\lambda_1\rho_1 + \lambda_2\rho_3)\rho_2 - \rho_2 \\ F_3(\rho_1,\rho_2,\rho_3) = 2d(\lambda_1\rho_1 + \lambda_2\rho_3)\rho_2 + r\rho_1 - 2\rho_3 \end{cases}$$

with $\rho_0 = 1 - \rho_1 - \rho_2 - \rho_3$.

Hydrodynamic limit

- **Boundary conditions:**
  - Dirichlet regime: for $\theta_r \in [0,1)$, resp. $\theta_\ell \in [0,1)$, for $0 < t \leq T$,

$$\widehat{\rho}(t,.)_{|\Gamma^+} = \widehat{b}_{|\Gamma^+}, \quad \text{resp.} \quad \widehat{\rho}(t,.)_{|\Gamma^-} = \widehat{b}_{|\Gamma^-}.$$

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  - Neumann regime: for $\theta_r > 1$, resp. $\theta_\ell > 1$, for $0 < t \le T$,

$$\partial_{e_1}\widehat{\rho}(t,.)_{|\Gamma^+} = 0, \quad \text{resp.} \quad \partial_{e_1}\widehat{\rho}(t,.)_{|\Gamma^-} = 0.$$

Theorem

The sequence of measures $(Q_N^{\widehat{\theta}})_{N\ge 1}$ is relatively compact and any limit $Q^{\widehat{\theta}}$ is a Dirac measure concentrated on trajectories which are absolutely continuous with respect to the Lebesgue measure. The density trajectory must satisfy one of the above PDEs with mixed boundary conditions depending on $\widehat{\theta}$, as well as some regularity assumptions.

Hydrodynamic limit

Uniqueness of the limit: If for any $\widehat{G} = (G_1, G_2, G_3) : V \to \mathbb{R}^3$,

$$\limsup_{N\to\infty} \mathbb{E}_{\mu_N}\Big[\Big| < \widehat{\pi}^N, \widehat{G} > - \int_V \widehat{\gamma}(x).\widehat{G}(x)dx \Big|\Big] = 0,$$

Then, $(Q_N^{\widehat{\theta}})_{N\geq 1}$ converges to the Dirac measure concentrated on $\{\widehat{\rho}(t,x)dx,\ t \in [0,T]\}$ where $\widehat{\rho}$ is the **unique** solution of the PDE corresponding to $\widehat{\theta}$ with initial condition $\widehat{\gamma}$.

Proof:

- Martingale problem, entropy method
- Analysis of coupled equations in dimensions $d \geq 2$.

Hydrostatic limit

Macroscopic equilibrium:
Under conditions (1), for a given $\widehat{\theta}$, there exists a unique stationary solution $\overline{\rho}$ of the hydrodynamic equation associated to $\widehat{\theta}$ and we refer to it as a macroscopic equilibrium.

$$\begin{cases} D \geq 1 \\ r + 1 > 2d(\lambda_1 - \lambda_2) \\ 1 > 2d\lambda_2. \end{cases} \tag{1}$$

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The invariant measure and the stationary profile are related by the Hydrostatic limit which states as follows:

Hydrostatic limit

Theorem (Mourragui, Saada, V.)

Suppose (1) *holds and consider a* $\widehat{\theta} \in (\mathbb{R}^+)^2$. *For any continuous* $\widehat{G} = (G_1, G_2, G_3) : [-1,1] \times \mathbb{T}^{d-1} \to \mathbb{R}^3$,

$$\limsup_{N\to\infty} \mathbb{E}_{\mu_{ss}^N(\widehat{\theta})}\Big[\Big| < \widehat{\pi}^N, \widehat{G} > - < \overline{\rho}, \widehat{G} > \Big|\Big] = 0$$

where $\overline{\rho}$ *is the unique stationary solution of the hydrodynamic equation associated to* $\widehat{\theta}$.

$\Longrightarrow$ The sequence of probability measures $(\mu_{ss}^N(\widehat{\theta}))_{N\geq 1}$ is associated to $\overline{\rho}$.

Sketch of the proof of the Hydrostatic limit

Consider $\mu_{ss}^{N_k}(\widehat{\theta})$ a subconverging sequence of $\mu_{ss}^{N}(\widehat{\theta})$. By stationarity of $\mu_{N}^{ss}(\widehat{\theta})$, for any continuous $\widehat{G}$ and any $T \geq 0$,

$$\mathbb{E}_{\mu_{ss}^{N_k}(\widehat{\theta})}\Big(\Big| < \widehat{\pi}^{N}, \widehat{G} > - < \overline{\rho}, \widehat{G} > \Big|\Big) = \mathbb{E}_{\mu_{ss}^{N_k}(\widehat{\theta})}\Big(\Big| < \widehat{\pi}_{T}^{N}, \widehat{G} > - < \overline{\rho}, \widehat{G} > \Big|\Big).$$

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$$\begin{aligned} &\lim_{k\to\infty} \mathbb{E}_{\mu_{ss}^{N_k}(\theta)}\Big(\Big| < \widehat{\pi}^N_T, \widehat{G} > - < \overline{\rho}, \widehat{G} > \Big|\Big) \\ &\leq \sum_{i=1}^{3} \|G_i\|_\infty \mathbb{E}_{\mu_{ss}^*(\theta)}\Big(\sum_{i=1}^{3} \|\rho_i(T,.) - \overline{\rho}_i(.)\|_1\Big) \end{aligned}$$

where we used the Hydrodynamic limit to replace $\widehat{\pi}^N_T$ by its associated density limit $\widehat{\rho}(T,.) = (\rho_1(T,.), \rho_2(T,.), \rho_3(T,.))$ and the triangular inequality.

Sketch of the proof of the Hydrostatic principle

The last term is bounded by

$$\sup_{\widehat{\rho}} \sum_{i=1}^{3} \|G_i\|_\infty \Big(\sum_{i=1}^{3} \|\rho_i(T,.) - \overline{\rho}_i(.)\|_1 \Big),$$

where the supremum is carried over all solutions of the hydrodynamic equation.

Hydrostatic limit

Theorem

Suppose conditions (1) hold. For any $\widehat{\theta} = (\theta_\ell, \theta_r) \in (\mathbb{R}^+)^2$, the hydrodynamic equation with mixed boundary conditions associated to $\widehat{\theta}$ has a unique stationary solution $\overline{\rho}$. Furthermore, given $\widehat{\rho} = (\rho_1, \rho_2, \rho_3)$ a solution to that equation with arbitrary initial condition,

$$\lim_{t\to\infty} \sum_{i=1}^{3} \|\rho_i(t,.) - \overline{\rho}_i(.)\|_1 = 0.$$

$$(1) : \begin{cases} D \geq 1 \\ r + 1 > 2d(\lambda_1 - \lambda_2) \\ 1 > 2d\lambda_2. \end{cases}$$

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Difficulty: No comparison principle for (ρ_1, ρ_2, ρ_3). Instead:

- Work under the change of coordinates

$$(\rho_1, \rho_2, \rho_3) \leftrightarrow (\rho_1, \rho_1 + \rho_3, 1 - (\rho_2 + \rho_3)) =: (\beta_1, \beta_2, \beta_3)$$

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- Prove a comparison principle to get that $\widehat{\beta}^1$ the solution starting from $(\beta_1, \beta_2, \beta_3) = (1, 1, 1)$ is decreasing, $\widehat{\beta}^0$ the solution starting from $(\beta_1, \beta_2, \beta_3) = (0, 0, 0)$ is non decreasing.

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- Prove that

$$\lim_{t \to \infty} \sum_{i=1}^{3} \|\beta_i^1 - \beta_i^0\|_1 = 0.$$

Hydrostatic limit

Necessity to have conditions on the parameters?

$$F_1\left(0, \frac{r}{r+1}, 0\right) = F_2\left(0, \frac{r}{r+1}, 0\right) = F_3\left(0, \frac{r}{r+1}, 0\right) = 0.$$

So the constant function $\widehat{\rho} = \left(0, \frac{r}{r+1}, 0\right)$ is a stationary solution of

$$\partial_t \widehat{\rho} = D\Delta\widehat{\rho} + \widehat{F}(\widehat{\rho})$$

with Neumann boundary conditions. Simulations show that when conditions (1) are not satisfied, solutions to the PDE do not necessarily converge to $\left(0, \frac{r}{r+1}, 0\right)$ hence non uniqueness.

Simulations in one dimension for Neumann boundary conditions

On the left, we started from the initial profile $(\rho_1, \rho_2, \rho_3) = (1, 0, 0)$ and on the right, with $(\rho_1, \rho_2, \rho_3) = (0, 1, 0)$.

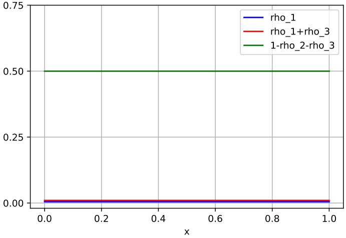
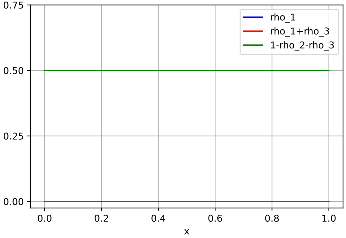


Figure: $\lambda_1 = 0.75$, $\lambda_2 = 0.25$, $\widehat{\rho}(0, .) = (1, 0, 0)$.



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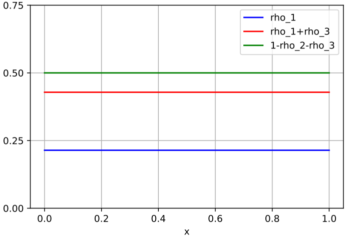

Figure: $\lambda_1 = 1$, $\lambda_2 = 0.75$, $\widehat{\rho}(0,.) = (1, 0, 0)$.

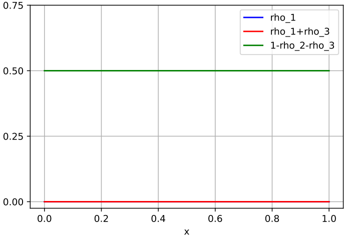

Figure: $\lambda_1 = 1$, $\lambda_2 = 0.75$, $\widehat{\rho}(0,.) = (0, 1, 0)$.

Some perspectives

- Infinite volume case.

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- Large deviations and fluctuations.

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- Large deviations and fluctuations.
- Microscopic dynamics: information about the invariant measure for the superposed dynamics with reservoirs.

Some references

- Mustapha Mourragui, Ellen Saada and Sonia Velasco. Hydrodynamic and Hydrostatic limit for a generalized contact process with slow reservoirs. arXiv:2212.07762 *(preprint)*, December 2022.
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Thank you for listening!

Microscopic equilibrium in infinite volume for the birth and death process

Theorem

There exists a value $\lambda_c(d) = \lambda_c$ *such that*

(i) *For* $\lambda_2 < \lambda_1 \leq \lambda_c$*, for any* $r \geq 0$*, the birth and process dies out.*

(ii) *For* $\lambda_c < \lambda_2 < \lambda_1$*, for every* $r \geq 0$*, the birth and death process survives.*

(iii) *If* $\lambda_2 < \lambda_c < \lambda_1$ *are fixed,*

- *There exists* $r_0 \in (0, \infty)$ *such that if* $r < r_0$*, the process survives.*
- *There exists* $r_1 \in (0, \infty)$ *such that if* $r > r_1$*, the process dies out.*

K. Kevin. Phase transition for a contact process with random slowdowns. *Markov Processes And Related Fields*, 22(1):53–86, 2016.