# Continuous Box-Ball systems

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# Box-Ball system

by Takahashi-Satsuma (1990) [29]

Box at each $z \in \mathbb{Z}$. Ball configuration $\eta \in \{0,1\}^{\mathbb{Z}}$

$$\eta(z) = 0 \quad \text{empty box,} \qquad \eta(z) = 1 \quad \text{ball at } z$$

Carrier visits boxes from left to right.

Carrier picks balls from occupied boxes.

Carrier deposits one ball, if carried, at empty boxes.

0 0 1 0 1 1 0 0 0 1 1 1 0 1 0 0 0 0 0 $\eta$
0 0 1 0 1 2 1 0 0 1 2 3 2 3 2 1 0 0 0 carrier load
0 0 0 1 0 0 1 1 0 0 0 0 1 0 1 1 1 0 0 $T\eta$

$T\eta$ : configuration after the carrier visited all boxes.

# Motivation: Korteweg & de Vries equation

$$\dot{u} = u''' + u\,u'$$

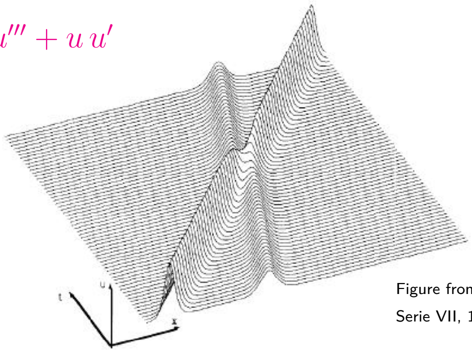



Figure from Rendiconti di Matematica, Serie VII, 11, p.351-376, 1991

Interacting soliton solutions

Soliton: a solitary wave that propagates with little loss of energy and retains its shape and speed after colliding with another such wave.

# Related mathematical works

Kato, Tsujimoto and Zuk [19], spectral decomposition of BBS.

Levine, Lyu and Pike [23] big solitons in a semi-infinite interval.

Kuniba and Lyu [21] one-sided multicolor solitons.

Croydon and Sasada: Invariant measures [7], Pitman transformation [11], Hydrodynamics [10],

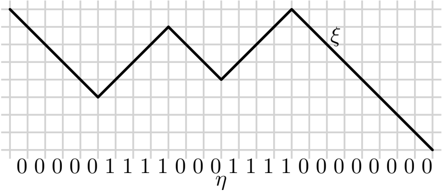
Linearizations: Kirillov-Sakamoto [20], Kuniba, Okado, Sakamoto,Takagi, Yamada [22] F, Nguyen, Rolla, Wang [17]

Mucciconi, Sasada, Sasamoto, Suda [24] relation between linearizations

F. Gabrielli [15, 16]: Big family of excursion based invariant measures.

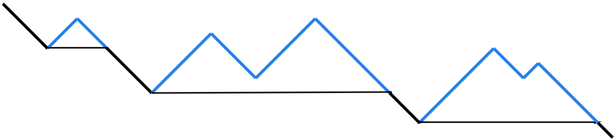
# Soliton Identification

# Walk representation




$$\xi(z) - \xi(z-1) = 2\eta(z) - 1, \qquad \xi(0) = 0$$

Records: $\{z : \xi(y) > \xi(z) \text{ for all } y < z\}$ (non-local function of $\eta$).

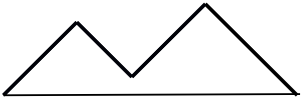


Excursion: configuration between two successive records

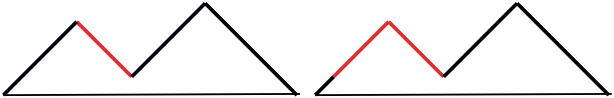
# Soliton identification for excursions [16, 29]

Let $t_i, s_i$ the localization of maxima and minima of the excursion.

*runs:* segments induced by broken lines of the walk: $[s_{i-1}, t_i]$ or $[t_i, s_i]$



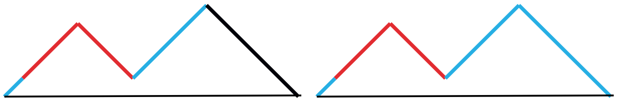
(1) Look for the shortest run $I$. Let $L$ be its length, and $s, t$ its extremes.



Identify the interval of length $2L$ centered at $t$ as an $L$-soliton. Call it $\gamma$.

(2) Ignore previously identified solitons $\tilde{\gamma}$ by gluing $\inf \tilde{\gamma}$ with $\sup \tilde{\gamma}$.

Iterate.

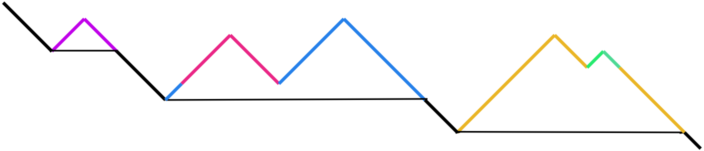


Solitons may be broken to give place to smaller solitons.

# Soliton identification for infinite configurations

Assume that the path $\xi$ has all records.

Apply the identification algorithm to each *excursion*.



Records are black, solitons are colored.

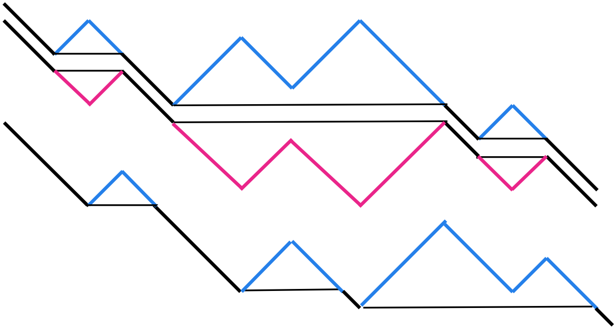
Any site is either a record or belongs to a soliton.

Call $s(\gamma)$ and $t(\gamma)$ the positions of the path min and max contained in $\gamma$.

# Dynamics: BBS and Pitman transformation

# Pitman transformation [6, 9, 12, 18, 26]

Reflect each excursion with respect to the current minimum of the walk:



The dynamics is well defined if all excursions are finite.

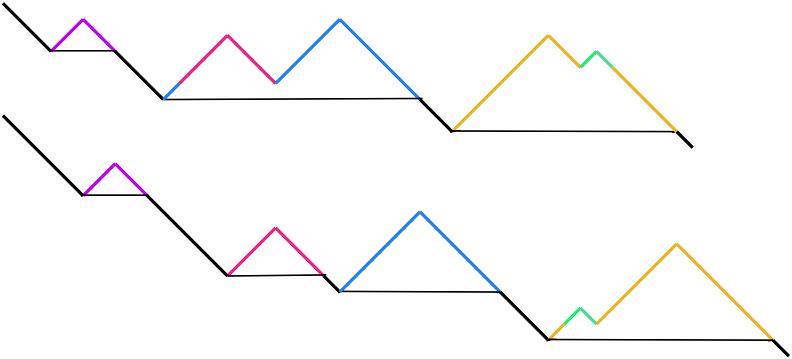
Call $T\xi$ the path obtained after one iteration of $\xi$.

Pitman transformation yields BBS for the derivative: $\eta(T\xi) = T\eta(\xi)$.

# BBS conserves solitons

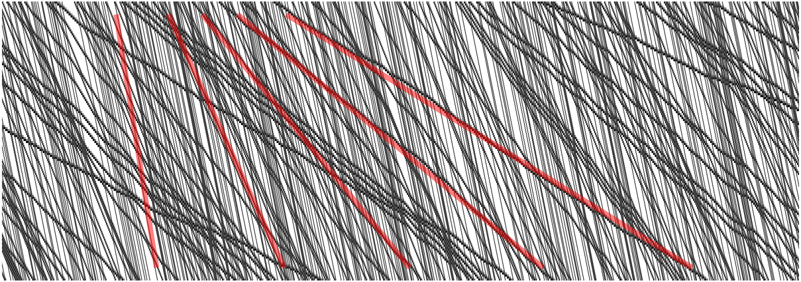
Proposition [29],[17] *For any path $\xi$:*

*$\xi$ has $L$-soliton $\gamma$ with maximum $t(\gamma)$*
*if and only if*
*$T\xi$ has $L$-soliton $\gamma'$ with minimum $s(\gamma') = t(\gamma)$.*



Corollary: We can follow solitons along time.

# Generalized hydrodynamics



ball density $= 0.25$

# Effective soliton speeds in the stationary regime

Theorem (F-Nguyen-Rolla-Wang [17])
*Let $\mu$ be a shift-ergodic $T$-invariant measure on the ball configuration space.*

$x(\gamma^t) :=$ *position of leftmost box of $k$-soliton $\gamma$ at time $t$.*

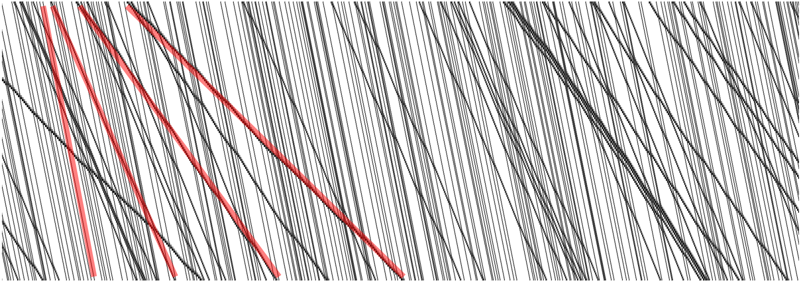
*There exists deterministic effective velocities $\underline{v} = (v_k)_{k\geq 1}$ such that*

$$\lim_{t\to\infty} \frac{x(\gamma^t)}{t} = v_k, \quad \mu\text{-a.s.}$$

*And $\underline{v}$ satisfies*

$$v_k = k + \sum_{m\neq k} 2(m \wedge k)\, \bar\rho_m\, (v_k - v_m)$$

$\bar\rho_k :=$ *mean number of $k$-solitons per site.*



Ball density $= 0.15$

# Hydrodynamics

Sequence of initial ball configurations $\eta^\epsilon$. Empirical density measures:

$$\mathtt{K}_t^\epsilon \varphi := \epsilon \sum_{x\in\mathbb{Z}} T_{t/\varepsilon}\eta^\epsilon(x)\,\varphi(\epsilon x)$$

$$\mathtt{K}_{k,t}^\epsilon \varphi := \epsilon \sum_{x\in\mathbb{Z}} S_{k,t/\varepsilon}\eta^\epsilon(x)\,\varphi(\epsilon x)$$

$\varphi$ test function; $S_{k,t}\eta(x) := 1\{x$ is leftmost box of a $k$-soliton in $T_t\eta\}$.

Theorem (Croydon and Sasada [10]) *Assume uniformly bounded solitons and no ball at negative sites. Let $g_{k,0}$ be densities such that*

$$\lim_{\epsilon\to 0} \mathtt{K}_{k,t}^\epsilon \varphi = \int g_{k,t}(x)\,\varphi(x)\,dx, \qquad k \ge 1, \tag{1}$$

*holds for $t = 0$. Then* (1) *holds for $t > 0$, where $g_{k,t}$ satisfy*

$$\frac{\partial}{\partial t} g_{k,t}(x) = -\frac{\partial}{\partial x} v_{k,t}(x)\,g_{k,t}(x),$$

*and the effective instantaneous velocities* $v_{k,t}$ *satisfy*

$$v_{k,t}(x) = k - \textstyle\sum_{j\geq 1} 2(k \wedge j)\, g_{j,t}(x)\, (v_{j,t}(x) - v_{k,t}(x)),$$

*Furthermore, the ball empirical measure converges:*

$$\lim_{\epsilon \to 0} \mathtt{K}_t^{\epsilon} \varphi = \sum_{k\geq 1} k \textstyle\int g_{k,t}(x)\, \varphi(x)$$

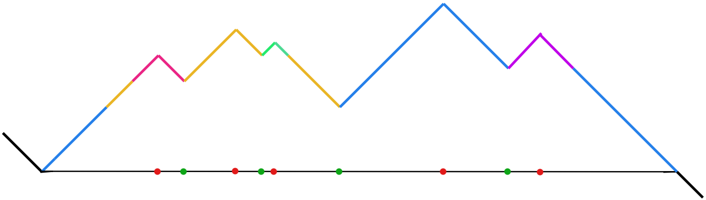
## Universality of speed equations

Toda lattice: Cao, Bulchadani, Spohn [5].

Hard-rods: Boldrighini, Dobrushin, Suhov [2, 3], F, Franceschini, Grevino, Spohn [14], F, Olla [25]

Surveys: Bulchandani-Cao-Moore [4], Doyon [13], Spohn [27, 28],. . .

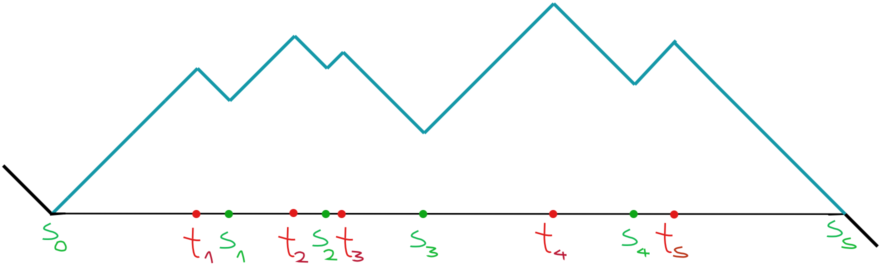
# Continuous BBS



# Space of continuous excursions

$\varepsilon : x \mapsto \varepsilon(x)$, $|\varepsilon'(x)| = 1$ at all $x$ but at maxima/minima located at

$$\underline{t}_n = (t_1, \ldots, t_n) \qquad \underline{s}_{n-1} = (s_1, \ldots, s_{n-1}):$$




$\mathcal{E}(n) :=$ Set of $(\underline{t}_n, \underline{s}_{n-1})$ representing excursions with $n$ maxima.

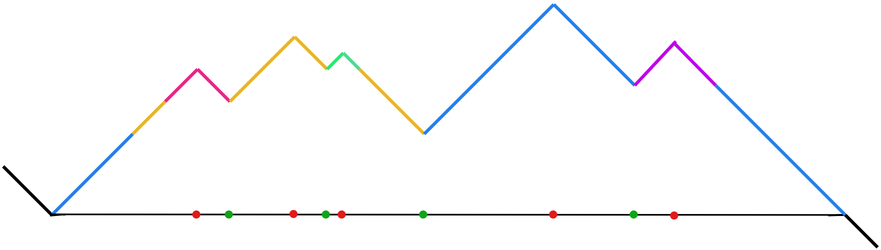
# Soliton identification

Same as in the discrete setup.

Look for shortest run $I$, denote $L$, $s$, $t$ its length and extremes.

Identify $\gamma =$ interval centered at $t$ with length $2L$.

Glue extremes of previous identified solitons and iterate



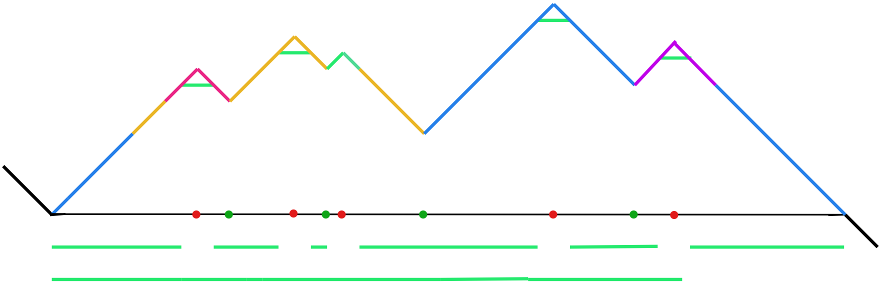
# Slot set

Denote $L_1 > \cdots > L_n > 0$ the maxima of $\varepsilon \in \mathcal{E}(n)$.

Define $L$-slot set of $\varepsilon$:

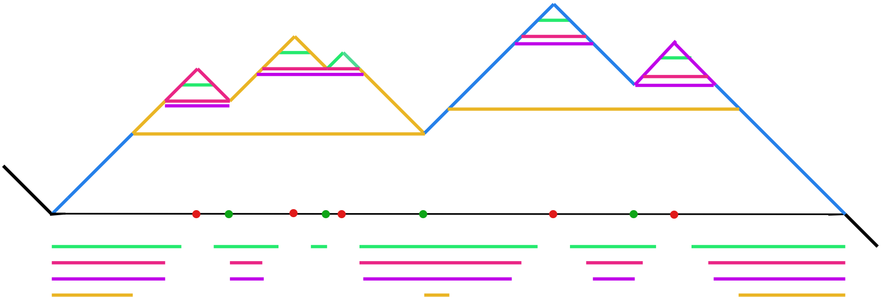
$$S(L|\varepsilon) := [0, s_n] \setminus \cup_{i=1}^{n} \big(t_i - (L \wedge L_i), t_i + (L \wedge L_i)\big),$$

where $s_n = 2(L_1 + \cdots + L_n)$ is the excursion size.



$L$-slot set is the set of points where an $L$-soliton may be inserted.

# Slot set

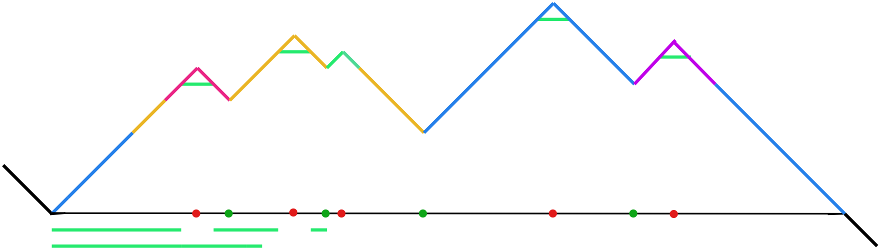


Slot sets of $L_5, \ldots, L_2$. There is only one slot point for $L_1$ (blue).

# Attachement slot

$$u_i := \left|S(L_i|\varepsilon_n) \cap [0, \inf \gamma_i]\right|$$

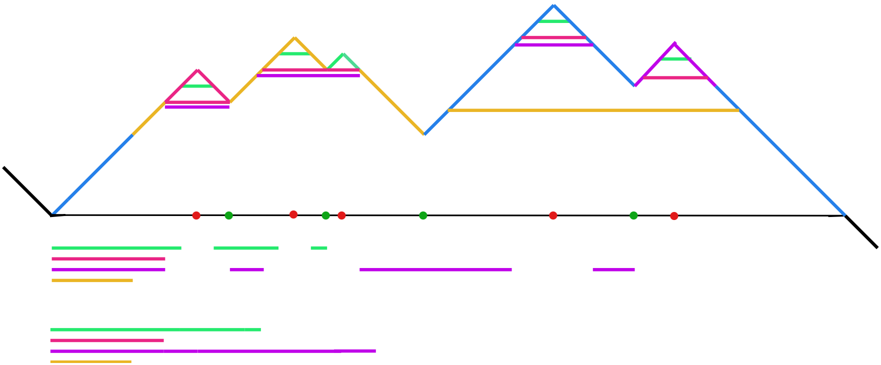
This is the $L_i$-slot length of $[0, \inf \gamma_i]$.



This is $u_5$

# Attachement slot

$u_i := L_i$-slot length in $[0, \inf \gamma_i]$.

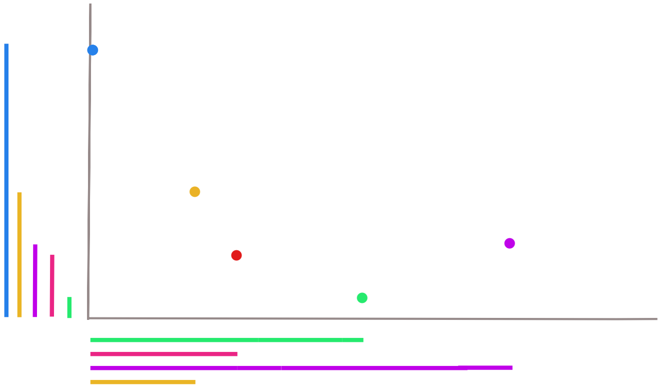


These are $u_5, \ldots, u_2$, $u_1 = 0$ is the origin of the excursion.

# Slot diagram

The slot diagram of the excursion $\varepsilon$ is the set of points

$(u_1, L_1), \ldots, (u_n, L_n)$:



Horizontal: attachement slot $u_i$,

Vertical: soliton size $L_i$.

## Space of slot diagrams with $n$ points

$$\mathcal{S}(n) := \Big\{(\underline{u}_n, \underline{L}_n) : L_1 > \cdots > L_n > 0,$$
$$u_1 = 0;\ 0 < u_i \leq 2\sum_{j=1}^{n}\big[L_j - (L_j \wedge L_i)\big],\ i = 2, \ldots, n\Big\}$$

## Bijection between excursions and slot diagrams

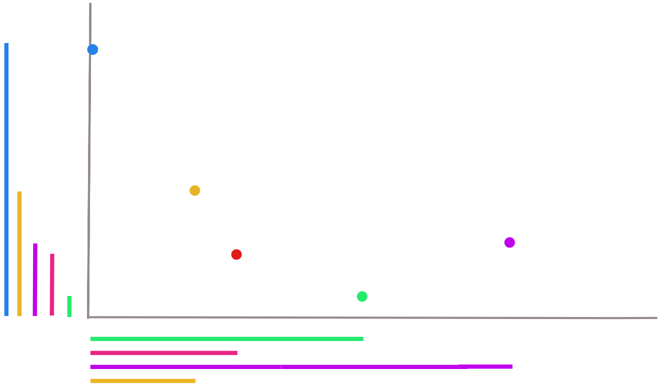
*The function*

$$\Phi : (\underline{t}_n, \underline{s}_{n-1}) \mapsto (\underline{u}_n, \underline{L}_n)$$

*is a bijection between $\mathcal{E}(n)$ and $\mathcal{S}(n)$.*

# Construction of $\Phi^{-1}$

We start with the slot diagram $(u_1, L_1), \ldots, (u_n, L_n)$:

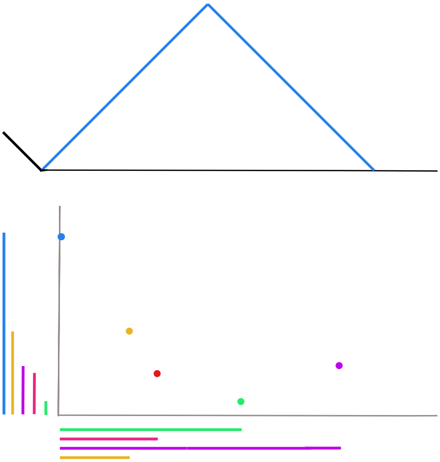


Horizontal: attachement slot $u_i$,

Vertical: soliton size $L_i$.

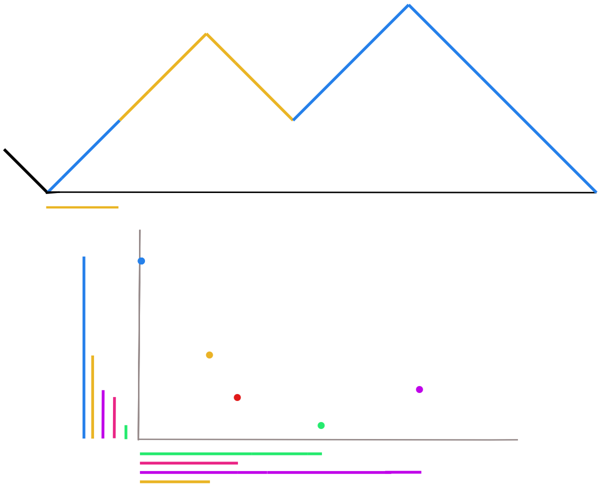
# Construction of $\Phi^{-1}$

(1) Insert biggest soliton $L_1$ at the origin.



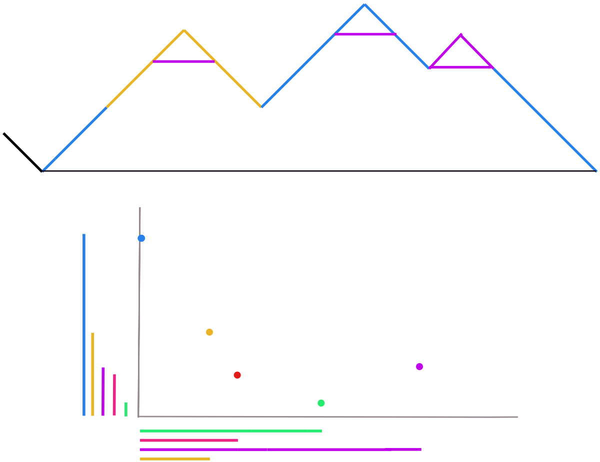
# Construction of $\Phi^{-1}$

(2) Insert second biggest soliton $L_2$ at $L_2$ slot distance $u_2$:



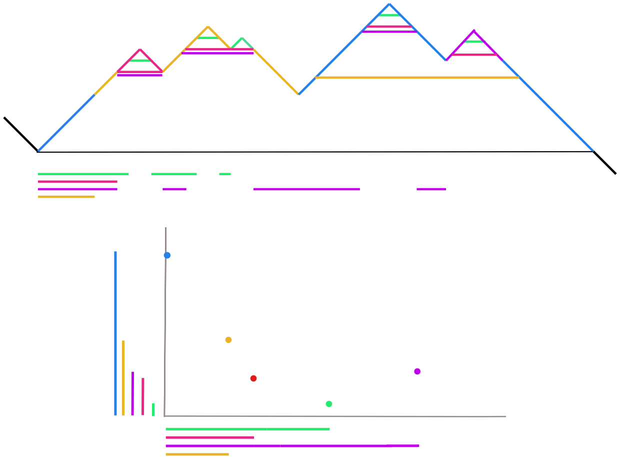
# Construction of $\Phi^{-1}$

(3) Insert third soliton $L_3$ at $L_3$-slot distance $u_3$:



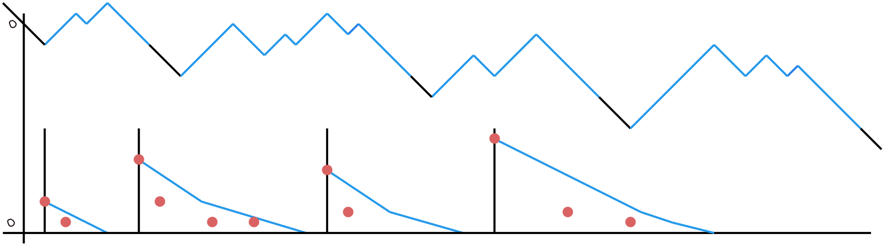
# Construction of $\Phi^{-1}$

(4-5) Insert remainig solitons $L_4$ and $L_5$ at slot distance $u_4$ and $u_5$:



Move slots sets together with the solitons containing them.

# Slot decomposition of infinite paths

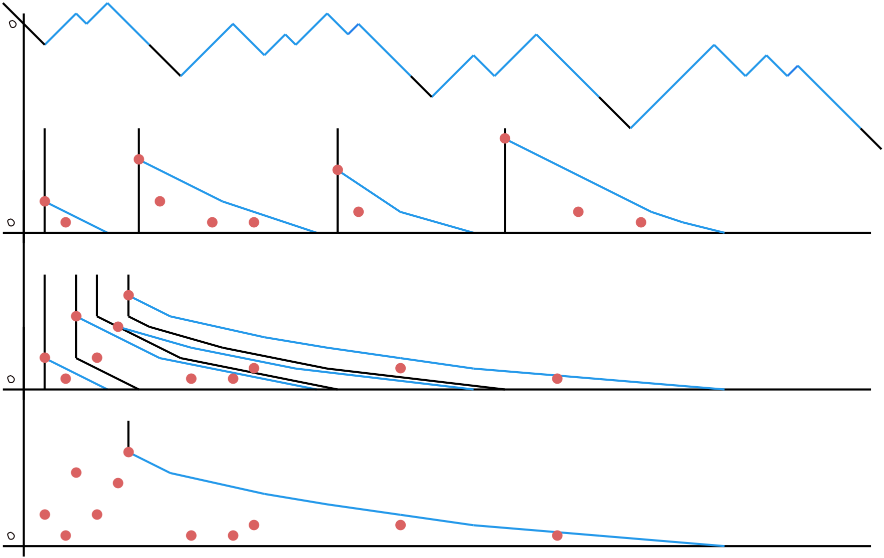


Concatenate the slot diagrams of the excursions

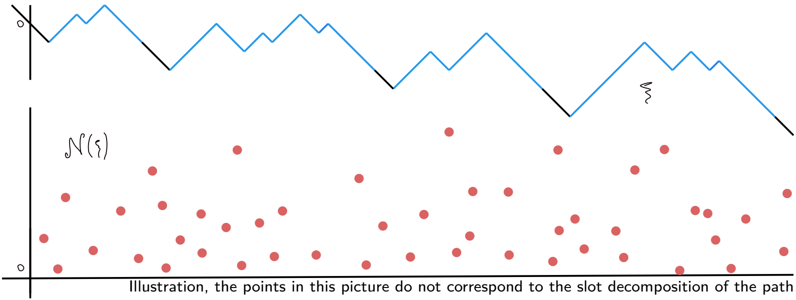
Keeping the same separation distances $D_i$.

Translate points associated to $\varepsilon_i$ to the left so that the left boundary of each diagram is at distance $D_i$ for all $y$.

# Slot decomposition of infinite paths



# Slot decomposition of an infinite path is a discrete set of points



Illustration, the points in this picture do not correspond to the slot decomposition of the path

Similar construction gives a point configuration in the left semiplane.

Analogous to discrete BBS FNRW [17] and FG [16].

# Random excursions

# Soliton based measures $\mu_\alpha$

Probability measures on $\mathcal{E} = \cup_n \mathcal{E}(n)$,

$\alpha : \mathbb{R}^+ \to [0,1)$ is a parameter function.

$$\mu_\alpha(\varepsilon) := \frac{1}{\mathcal{Z}(\alpha)} \prod_{i=1}^{n} \alpha(L_i)\, d\underline{t}_n\, d\underline{s}_{n-1}\,, \qquad \varepsilon = (\underline{t}_n, \underline{s}_{n-1}) \in \mathcal{E}(n)\,,$$

$$\mathcal{Z}(\alpha) := \sum_{n=1}^{+\infty} \int_{\mathcal{E}(n)} d\underline{t}_n\, d\underline{s}_{n-1} \prod_{i=1}^{n} \alpha(L_i)\,,$$

$L_1, \ldots L_n$ are the heigths of the solitons of $\varepsilon$.

$d\underline{t}_n := dt_1 \ldots dt_n$.

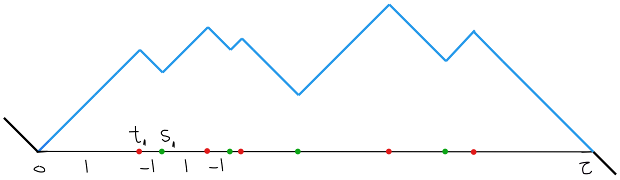
# Example: Zig-zag excursion

$X_t$ continuous time Markov process on $\{-1,+1\}$ [1,8]

Rates $c(-1,1) = \lambda_-$ and $c(1,-1) = \lambda_+$. $\lambda_- < \lambda_+$.

$$\varepsilon(t) := \int_0^t X_s\,ds, \qquad X_0 := 1$$

$$\varepsilon := \big(\varepsilon(t)\big)_{t\in[0,\tau]} \qquad \tau := \text{ hitting time of } 0.$$




Excursion $\varepsilon$ has law $\mu_\alpha$, with

$$\alpha(L) = e^{-(\lambda_+ + \lambda_-)L}\lambda_-\lambda_+, \qquad \mathcal{Z}(\alpha) = \lambda_-.$$

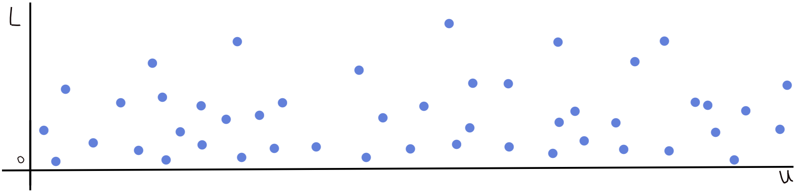
# Poisson$(q)$ process

Let $q : \mathbb{R}_+ \to \mathbb{R}_+$ be integrable:

$$\int_0^{+\infty} q(L)dL < +\infty$$

Let $\mathcal{N} \subset \mathbb{R} \times \mathbb{R}_+$ be a Poisson process with intensity

$$du\, q(L)\, dL.$$



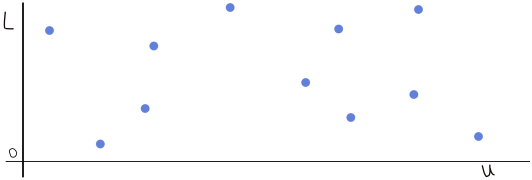
# Slot diagram based measures

$(u_1, L_1) :=$ leftmost positive point in $\mathcal{N}$

Grow region at rate twice the number of points in it

$g(\ell) = 2\sum_{(u,L)\in A(\ell)}(L-\ell)^+ \qquad G(\ell) := \int_0^\ell g(\ell')d\ell'.$

$A(\ell) = \{(u', \ell') : u_1 < u' < G(\ell');\, \ell \le \ell' \le L_1\}.$



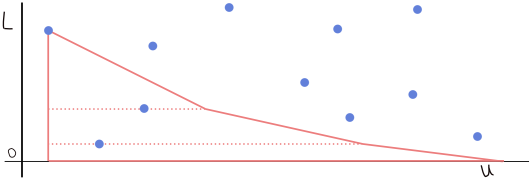
# Random slot diagram

$(u_1, L_1) :=$ leftmost positive point in $\mathcal{N}$

Grow region at rate twice the number of points in it

$g(\ell) = 2\sum_{(u,L)\in A(\ell)} (L-\ell)^+ \qquad G(\ell) := \int_0^\ell g(\ell')d\ell'.$

$A(\ell) = \{(u', \ell') : u_1 < u' < G(\ell'); \ell \le \ell' \le L_1\}.$




Sort $A(0) \cap \mathcal{N}$ to get $\{(u_1, L_1), \ldots, (u_n, L_n)\}$. ($n$ depends on $\mathcal{N}$)

Shift by $u_1$:

$$\varepsilon(\mathcal{N}) := \Phi^{-1}\big((0, L_1), (u_2 - u_1, L_2), \ldots, (u_n - u_1, L_n)\big) \in \mathcal{S}(n)$$

Lemma *The distribution of* $\varepsilon(\mathcal{N})$ *is given by*

$$\nu_q(d\underline{u}, d\underline{L}) := \frac{1}{\int_0^{+\infty} q(z)dz} \prod_{i=1}^{n} e^{-2\int_0^{L_i}(L_i-y)q(y)dy}\, q(L_i)\, dL_i \prod_{i=2}^{n} du_i$$

*Proof.* Explore the points of the Poisson process according to the construction of $\Phi^{-1}$. □

## Equivalence of measures

Proposition *If* $\alpha(L) = q(L)e^{-2\int_0^L (L-y)q(y)dy}$*, then*

$$\mu_\alpha = \nu_q,$$

*and* $\mathcal{Z}(\alpha) = \int_0^{+\infty} q(L)dL$*.*

*Proof.* The Jacobian is 1: $d\underline{y}_n\, d\underline{s}_{n-1} = d\underline{u}_n\, d\underline{L}_n$.

The identity is immediate. □

# Relation between $\alpha$ and $q$

Call $Q(L) = \int_0^L q(z)dz$. Integrating $q(L)e^{-2\int_0^L (L-y)q(y)dy}$ by parts gives

$$q(L) = \alpha(L)e^{2\int_0^L Q(z)dz} .$$

Calling $M(L) = \int_0^L Q(z)dz$, this is equivalent to

$$\begin{cases} M''(L) = \alpha(L)e^{2M(L)} , \\ M(0) = M'(0) = 0 . \end{cases}$$

# Explicit $q$ for the Zig-zag excursion

Proposition *If* $\alpha(L) = e^{-(\lambda_+ + \lambda_-)L}\,\lambda_-\lambda_+$*, then the solution* $q$ *of*

$$\alpha(L) = q(L)\,e^{-2\int_0^L (L-y)q(y)dy}$$

*is given by*

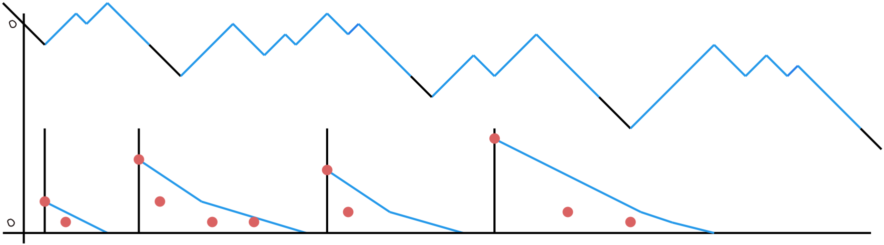
$$q(L) = E\,\left[\sinh\left(\sqrt{E}(L+K)\right)\right]^{-2},$$

$$E := \lambda_+^2 + \lambda_-^2 + \lambda_-\lambda_+$$

$$K := \frac{1}{\sqrt{E}}\coth^{-1}\left(\sqrt{1+\frac{\lambda_+\lambda_-}{E}}\right).$$

Normalized $q$ is the distribution of the maximum of an excursion with law $\mu_\alpha$.

# Slot decomposition of infinite paths

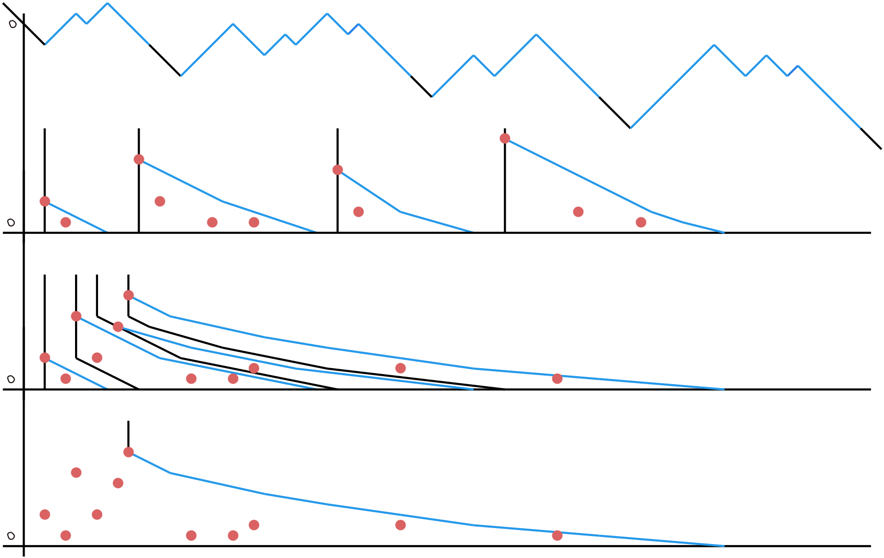



Concatenate the slot diagrams of the excursions.

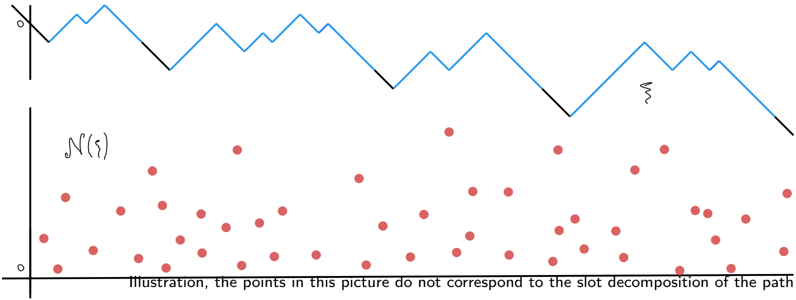
Keep the same separation distances $D_i$.

Translate points associated to $\varepsilon_i$ to the left so that the left boundary of each diagram is at distance $D_i$ for all $y$.

# Slot decomposition of infinite paths



# Slot decomposition of an infinite path is a point process




Illustration, the points in this picture do not correspond to the slot decomposition of the path

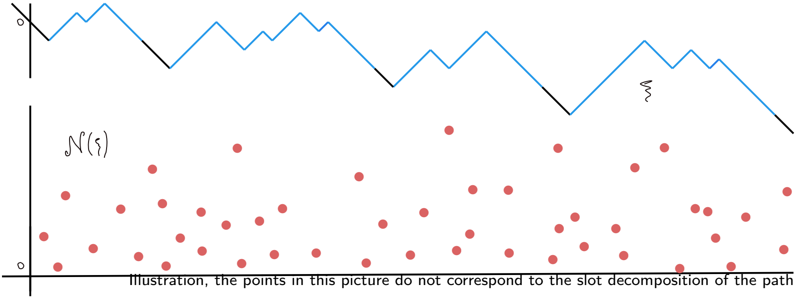
Similar construction gives a point configuration in the left semiplane.

Analogous to discrete BBS FNRW [17] and FG [16].

# $\mu_\alpha$ excursions generate Poisson$(q)$ process

Construct random path $\xi$:

(1) iid excursions with law $\mu_\alpha = \nu_q$
(2) iid random variables $D_i \sim$ Exponential$(\int q(x)dx)$ separate excursions



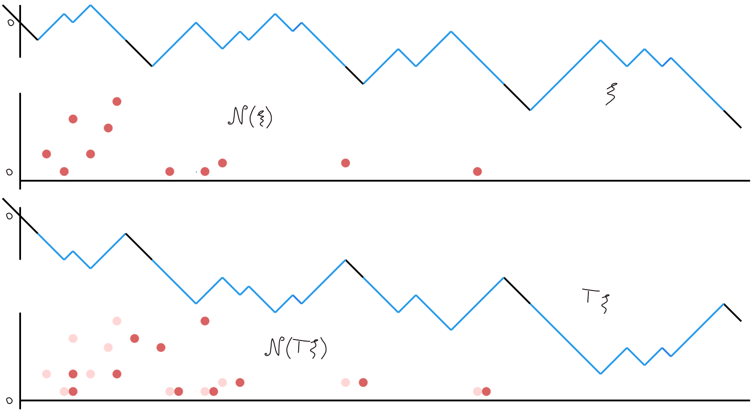

Proposition *Slot decomposition $\mathcal{N}(\xi)$ is a Poisson$(q)$ process*

We have constructed a random path $\xi$ with a record at the origin.

Its law $\hat{\mu}$ is invariant by translation to another record.

# Dinamics

# BBS operator commutes with slot decomposition




Theorem [17] Take $\eta$ with no ball at negative boxes and the origin and $\xi = \xi(\eta)$. Then $\mathcal{N}(T\xi) = \mathcal{T}\mathcal{N}(\xi)$.

$\mathcal{T}\mathcal{N} := \{(u + L, L) : (u, L) \in \mathcal{N}\}$, $\qquad$ $\mathcal{N}$ contained in positive quadrant.

$L$-soliton jumps $L$ to the right

# Doubly infinite case

$\hat{T}\xi :=$ "$T\xi$ as seen from record at level 0".

Theorem [17] *Take initial $\xi$ with a record at level 0 at the origin. Then*

$$\mathcal{N}(\hat{T}\xi) = \mathcal{T}\mathcal{N}(\xi)$$

$$\mathcal{T}\mathcal{N} := \Big\{\big(u + L - o(L,\mathcal{N}), L\big) : (u,L) \in \mathcal{N}\Big\}$$

$o(L,\mathcal{N}) :=$ *flow of $L$-slots through record 0.*

$L$-soliton jumps $L$ to the right, but $L$-slot crossing of the origin must be considered.

# Invariant measures for continuous BBS

Take $\alpha$ such that $\int L\,\alpha(L)\,dL < \infty$,

and $q$ compatible with $\alpha$.

This is equivalent to $\int y\,q(y)\,dy < \infty$ (finite maximum mean).

Consider iid excursions $\varepsilon_j \sim \mu_\alpha$, separated by iid $A_j \sim$ Exponential$(\int q)$.

We obtain a random path $\xi$ with a record at the origin, and invariant by translation to any record.

Call $\hat{\mu}$ the distribution of $\xi$.

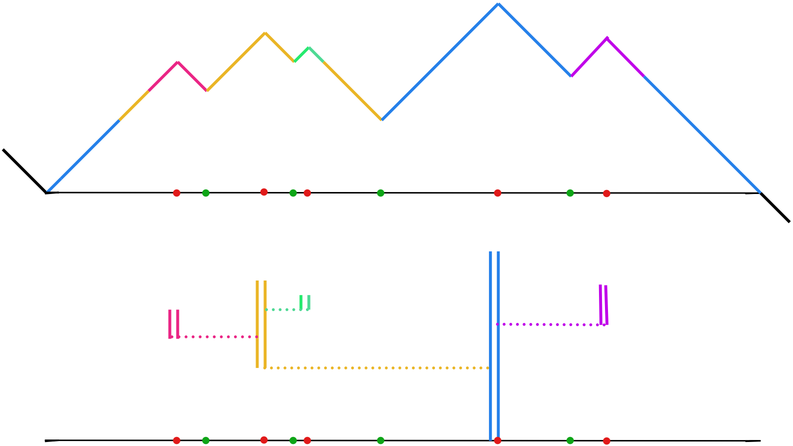
Call $\mu$ the anti-Palm distribution of $\hat{\mu}$.

$\mu$ is translation invariant.

Theorem [17] *The measure $\mu$ is invariant for $T$.*

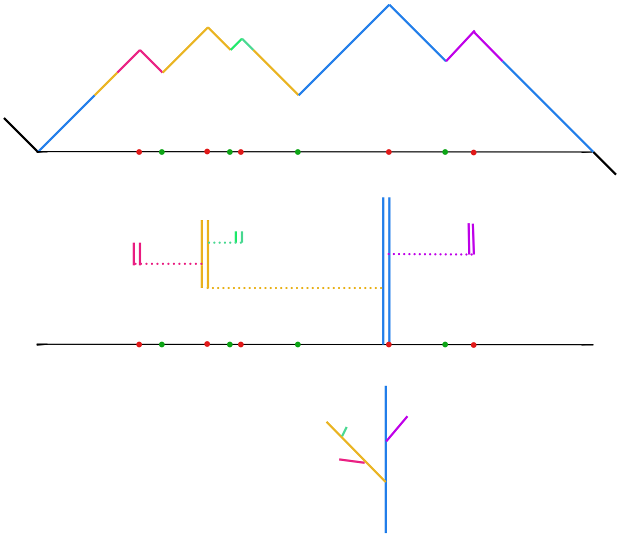
Corollary *The stationary zigzag process with $\lambda_+ > \lambda_-$ is invariant for $T$.*

# From excursions to trees



Vertical slot space

# From excursions to trees



Vertical slot space

To be continued

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