

Convergence of a dilute gas to the fluctuating Boltzmann equation

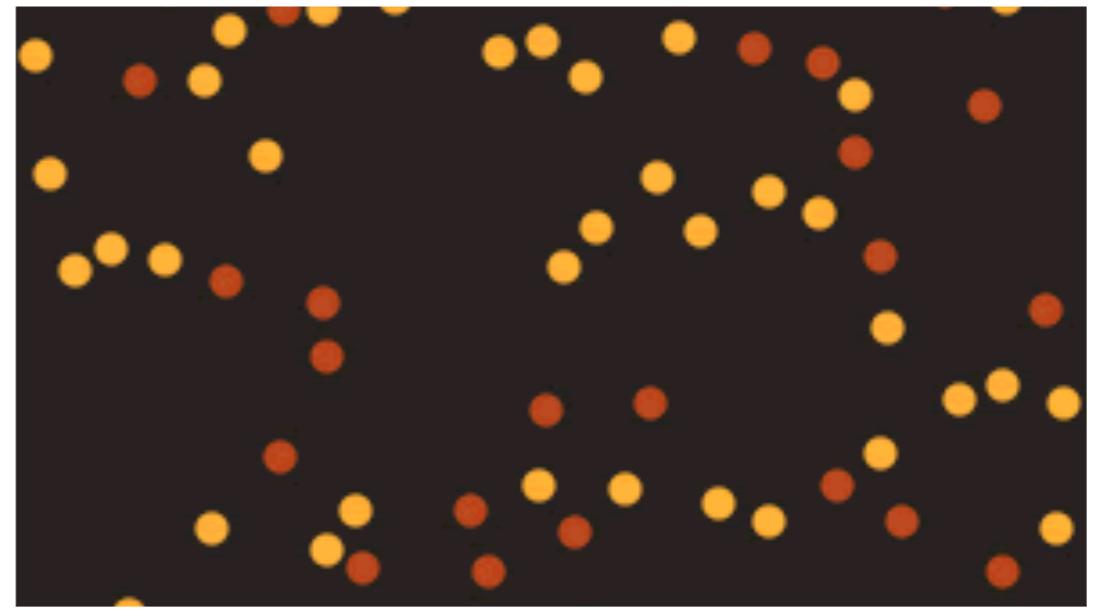
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Joint work with

I. Gallagher, L. Saint-Raymond, S. Simonella

Outline.

- Hard sphere dynamics
- Fluctuating Boltzmann equation
 - out of equilibrium (short time)
 - at equilibrium (long time)
- Long time derivation : sketch of the proof



Dilute gas of hard spheres

Gas of N hard spheres with deterministic Newtonian dynamics (elastic collisions).

Dimension : $d \geq 2$

Periodic domain: $\mathbf{T}^d = [0,1]^d$

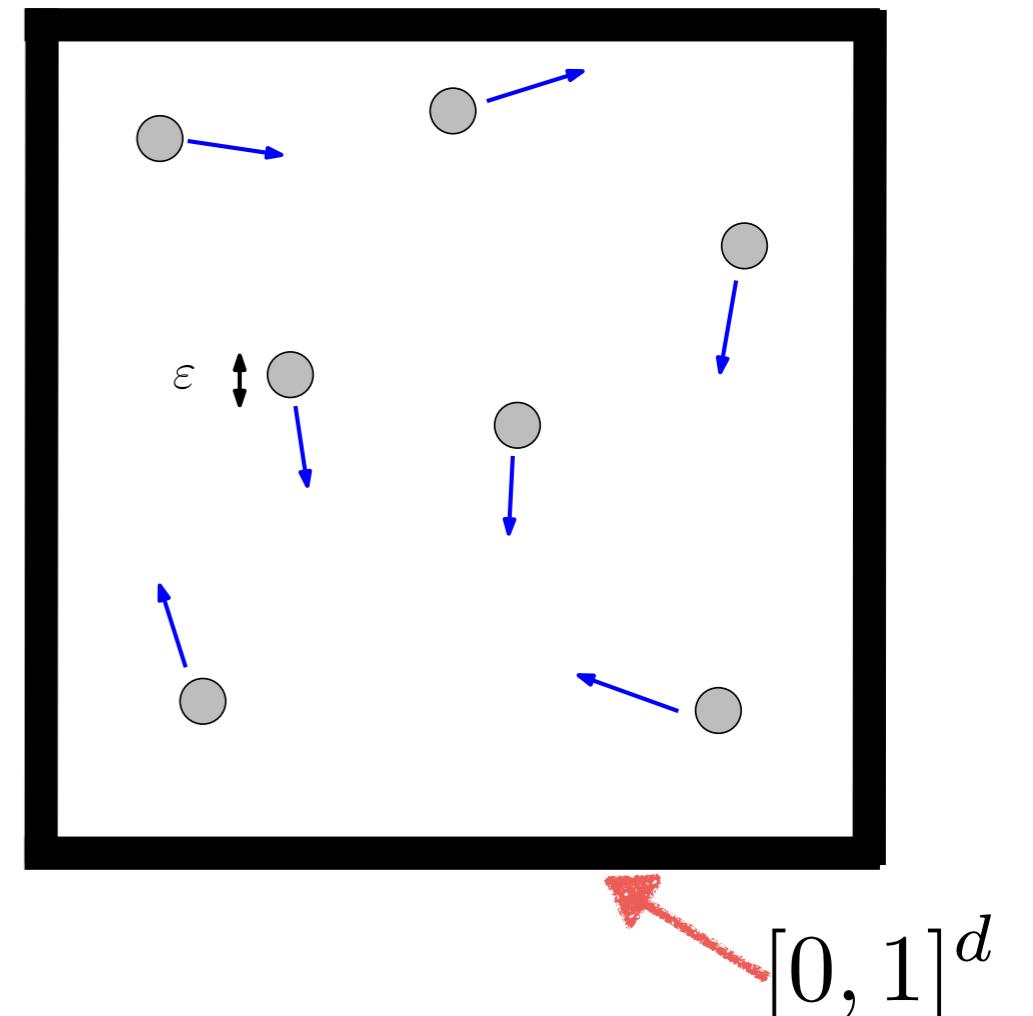
Sphere radius = ε

Strongly unstable dynamics

Boltzmann-Grad scaling $N\varepsilon^{d-1} = 1$

Microscopic scale :

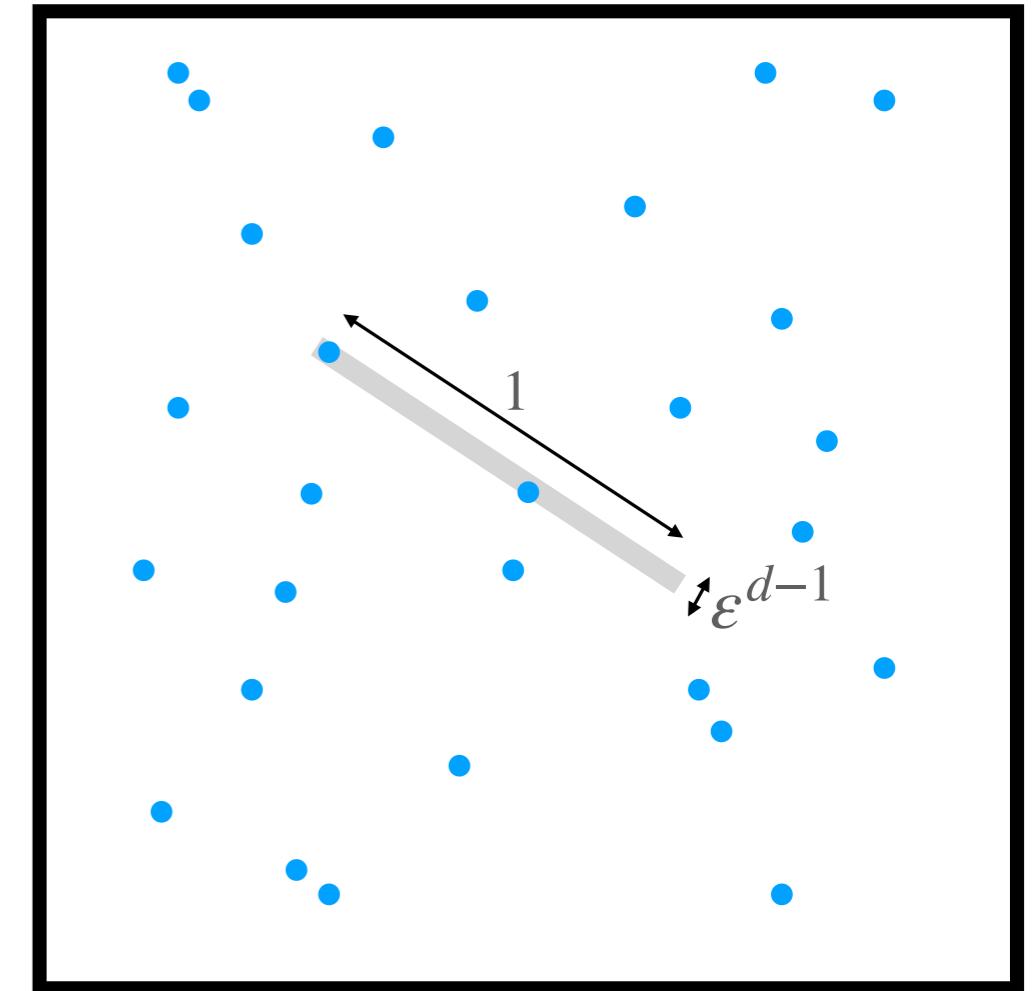
$$Z_N(t) = (x_i(t), v_i(t))_{i \leq N}$$



Boltzmann-Grad scaling

Mean free path $\simeq 1$

$$\Rightarrow N\varepsilon^{d-1} = 1$$



- Typical volume covered by a particle $= \varepsilon^{d-1}$
- N particles per unit volume

Limit $\varepsilon \rightarrow 0$: dilute gas

Hard sphere dynamics

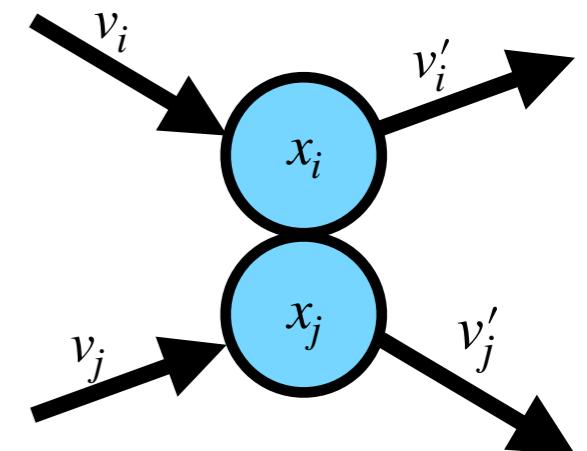
Gas of hard spheres

$$Z_N = \{(x_i(t), v_i(t))\}_{i \leq N}$$

$$\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = 0 \quad \text{as long as} \quad |x_i(t) - x_j(t)| > \varepsilon$$

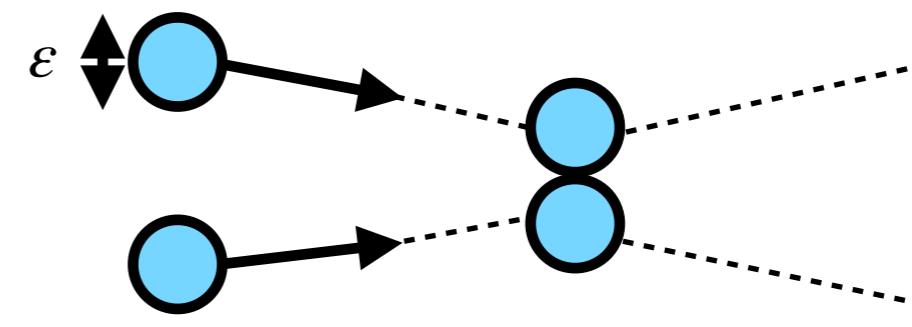
and elastic collisions if $|x_i(t) - x_j(t)| = \varepsilon$

$$\begin{cases} v'_i + v'_j = v_i + v_j \\ |v'_i|^2 + |v'_j|^2 = |v_i|^2 + |v_j|^2 \end{cases}$$



Deterministic dynamics :

- Reversible
- Unstable



ε no collision

Hard sphere dynamics

Gas of hard spheres

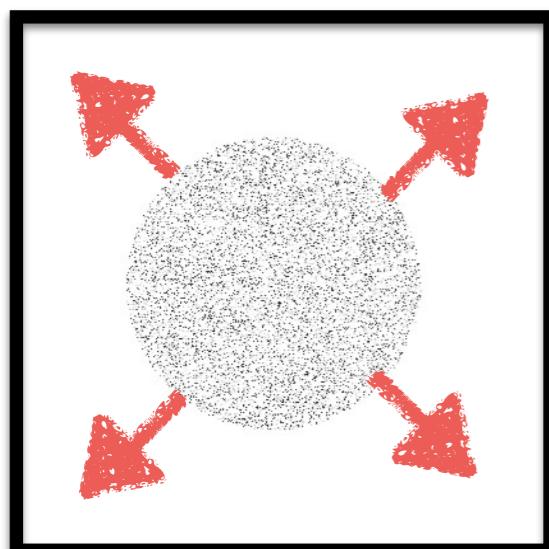
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$$\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = 0 \quad \text{as long as} \quad |x_i(t) - x_j(t)| > \varepsilon$$

and elastic collisions if

$$|x_i(t) - x_j(t)| = \varepsilon$$

Statistical description



$$W_N(0, Z_N)$$

Liouville equation

$$\partial_t W_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} W_N = 0$$

with specular reflection



$$W_N(t, Z_N)$$

Initial density.

$$W_N(0, Z_N) = \prod_{i=1}^N f^0(z_i) \times \frac{1}{\mathcal{Z}_N} \prod_{i \neq j} 1_{|x_i - x_j| > \varepsilon}$$

Dilute gas :
almost product
measure

Grand canonical formalism :

$$N \text{ random : } \mu_\varepsilon = \mathbb{E}(N) \quad \text{with} \quad \mu_\varepsilon \varepsilon^{d-1} = 1$$

Boltzmann-Grad
scaling

Typical density of a particle at time t : $F_1^\varepsilon(t, z_1)$

Typical density of k particles at time t : $F_k^\varepsilon(t, Z_k)$

Theorem (Convergence to the Boltzmann equation)

Density f^0 smooth and bounded $f^0(x, v) \leq C_0 \exp(-v^2)$

There exists $T > 0$ such that

$$t \leq T, \quad \|F_1^\varepsilon(t) - f(t)\|_\infty = \gamma(\varepsilon) \xrightarrow[\varepsilon \rightarrow 0]{} 0$$

Solution of the Boltzmann
equation starting from f^0

$$\mu_\varepsilon \varepsilon^{d-1} = 1$$

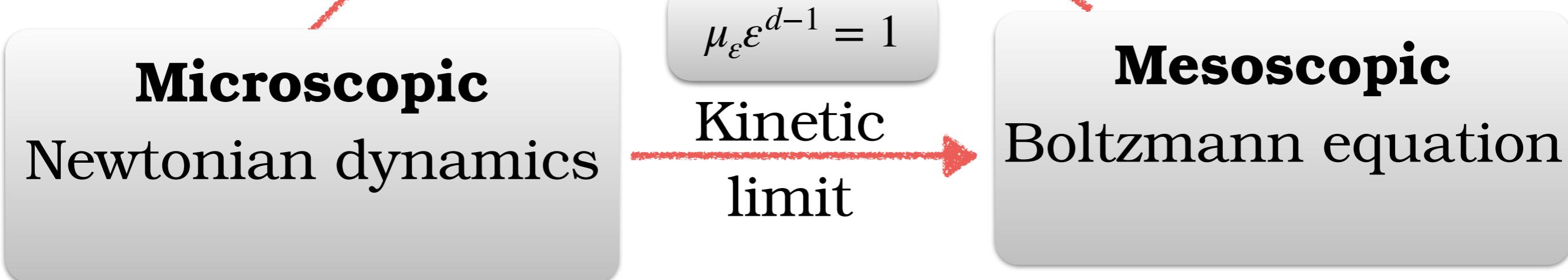
$$\partial_t f + v \cdot \nabla_x f = \iint [f(x, v') f(x, v'_2) - f(x, v) f(x, v_2)] ((v - v_2) \cdot \nu)_+ dv_2 d\nu$$

Theorem (Convergence to the Boltzmann equation)

Density f^0 smooth and bounded $f^0(x, v) \leq C_0 \exp(-v^2)$

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Deterministic description

- Hamiltonian dynamics
- Reversible

Reduced description

- Dissipative equation
- Irreversible

Theorem (Convergence to the Boltzmann equation)

Density f^0 smooth and bounded $f^0(x, v) \leq C_0 \exp(-v^2)$

There exists $T > 0$ such that

$$t \leq T, \quad \|F_1^\varepsilon(t) - f(t)\|_\infty = \gamma(\varepsilon) \xrightarrow[\varepsilon \rightarrow 0]{} 0$$

$$\mu_\varepsilon \varepsilon^{d-1} = 1$$



Lanford; King; Alexander; van Beijeren, Lanford,
Lebowitz, Spohn; Uchiyama; Cercignani, Illner,
Pulvirenti; V.Gerasimenko, D. Petrina; Simonella ...

Quantitative convergence :

Gallagher, Saint-Raymond, Texier; Pulvirenti, Saffirio,
Simonella; Denlinger; Pulvirenti, Simonella

Convergence as a law of large numbers

Test function $h(z) = h(x, v)$

$$\mathbb{E} \left[\left(\frac{1}{N} \sum_{i=1}^N h(z_i(t)) - \int f(t, z) h(z) dz \right)^2 \right] \xrightarrow{\varepsilon \rightarrow 0} 0$$

Assuming
 N constant

$$= \frac{N}{N^2} \mathbb{E} \left[\left(h(z_1(t)) - \int f(t) h dz \right)^2 \right] \xrightarrow{\varepsilon \rightarrow 0} 0$$

$$+ \frac{2N(N-1)}{N^2} \mathbb{E} \left[\left(h(z_1(t)) - \int f(t) h dz \right) \left(h(z_2(t)) - \int f(t) h dz \right) \right]$$

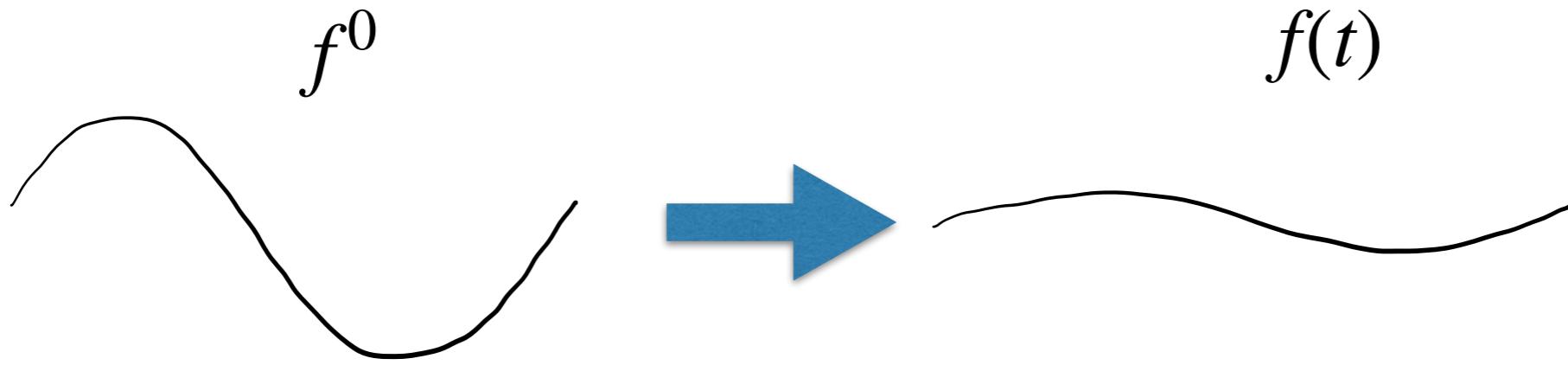
correlations

$$\simeq \int F_N^{(2)}(t, z_1, z_2) h(z_1) h(z_2) - \left(\int f(t) h dz \right)^2 \xrightarrow{\varepsilon \rightarrow 0} 0$$

Asymptotic independence : chaos property

Fluctuating Boltzmann equation

Law of large numbers : Boltzmann equation

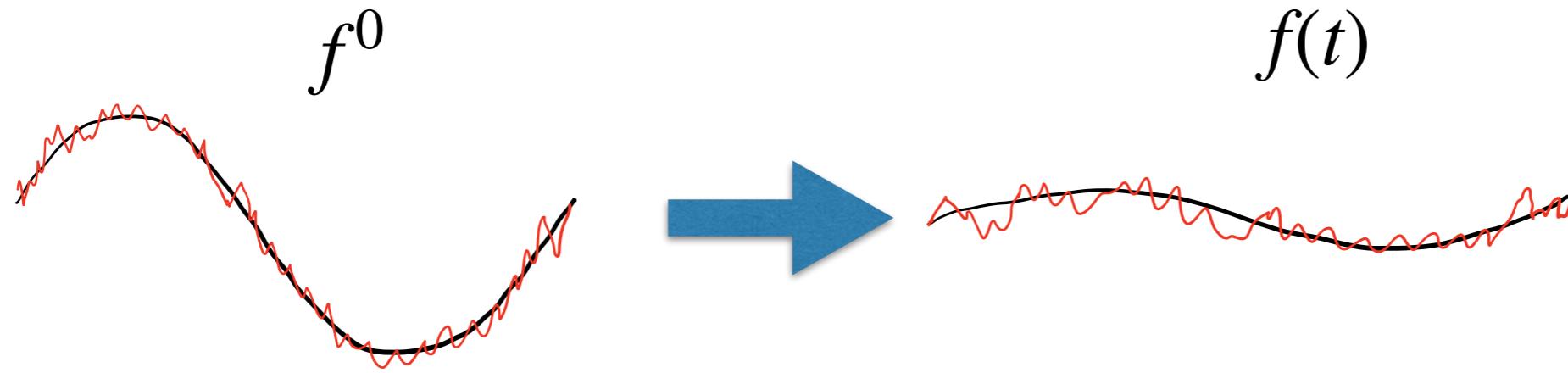


Test function $h(z) = h(x, v)$

$$\mathbb{E} \left[\left(\frac{1}{N} \sum_{i=1}^N h(z_i(t)) - \int f(t, z) h(z) dz \right)^2 \right] \xrightarrow[\varepsilon \rightarrow 0]{} 0$$

Fluctuating Boltzmann equation

Law of large numbers : Boltzmann equation



Corrections to the mean behavior : CLT

Fluctuation field : choose a test function h

$$\zeta_t^\varepsilon(h) = \frac{1}{\sqrt{N}} \sum_{i=1}^N \left(h(z_i(t)) - \int dz h(z) f(t, z) \right)$$

Question: Time evolution of the fluctuations ?

Central limit theorem

Test function $h(z) = h(x, v)$

$$\mathbb{E} \left[\left(\frac{1}{\sqrt{N}} \sum_{i=1}^N \left(h(z_i(t)) - \int f(t, z) h(z) dz \right) \right)^2 \right]$$

Assuming
 N constant

$$= \frac{N}{N} \mathbb{E} \left[\left(h(z_1(t)) - \int f(t) h dz \right)^2 \right]$$

$$+ \frac{2N(N-1)}{N} \underbrace{\mathbb{E} \left[\left(h(z_1(t)) - \int f(t) h dz \right) \left(h(z_2(t)) - \int f(t) h dz \right) \right]}$$

$\rightarrow +\infty$

the decay needs to be quantified [Spohn]

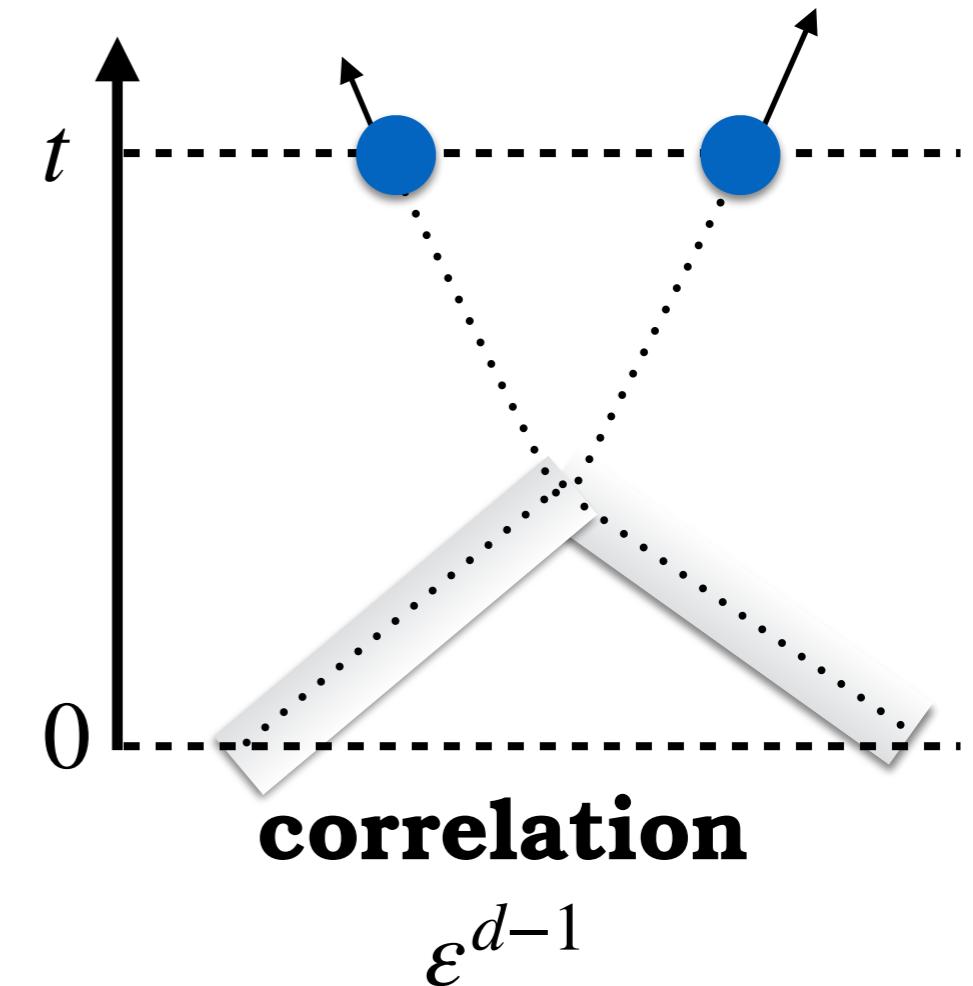
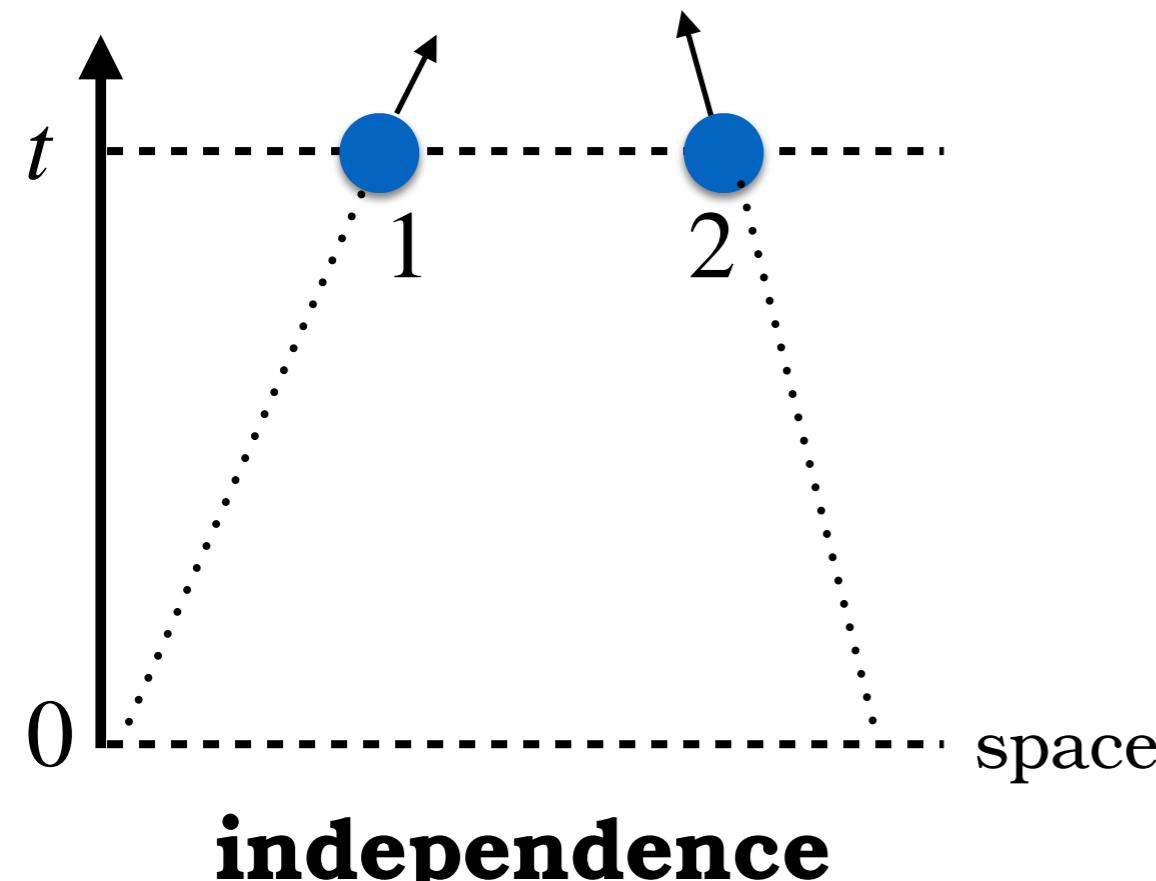
For fluctuations the correlations matter

Central limit theorem

$$N\varepsilon^{d-1} = 1$$

$$N \left[\left(h(z_1(t)) - \int f(t) h dz \right) \left(h(z_2(t)) - \int f(t) h dz \right) \right] = O(1) \quad \text{↗}$$

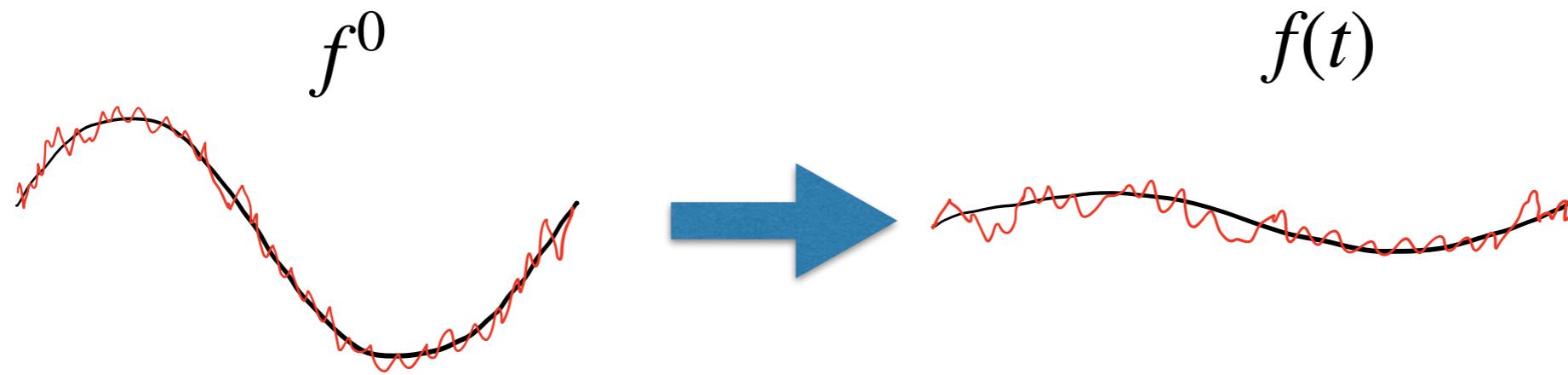
Time



Recollision operator [Spohn]

Fluctuating Boltzmann equation

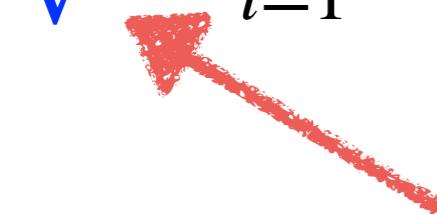
Law of large numbers : Boltzmann equation



Corrections to the mean behavior : CLT

Fluctuation field : choose a test function h

$$\zeta_t^\varepsilon(h) = \frac{1}{\sqrt{\mu_\varepsilon}} \sum_{i=1}^N \left(h(z_i(t)) - \int dz h(z) f(t, z) \right)$$



Grand canonical formalism :

$$\mu_\varepsilon = \mathbb{E}(N) \quad \text{with} \quad \mu_\varepsilon \varepsilon^{d-1} = 1$$

Fluctuating Boltzmann equation

Fluctuation field.

$$\zeta_t^\varepsilon(h) = \frac{1}{\sqrt{\mu_\varepsilon}} \sum_{i=1}^N \left(h(z_i(t)) - \int dz h(z) f(t, z) \right)$$

Theorem. [B., Gallagher, Saint-Raymond, Simonella]

Grand canonical distribution $W_N(0, Z_N) = \prod_{i=1}^N f^0(z_i) \times \text{exclusion}$

Convergence to the fluctuating Boltzmann equation in the time interval $[0, T^*]$

$$\zeta_t^\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{\text{(law)}} \zeta_t \quad \text{with} \quad d\zeta_t = \mathcal{L}_t \zeta_t + d\eta_t \quad \text{SPDE}$$

[Spohn] : conjecture & computation of the covariance

Same structure

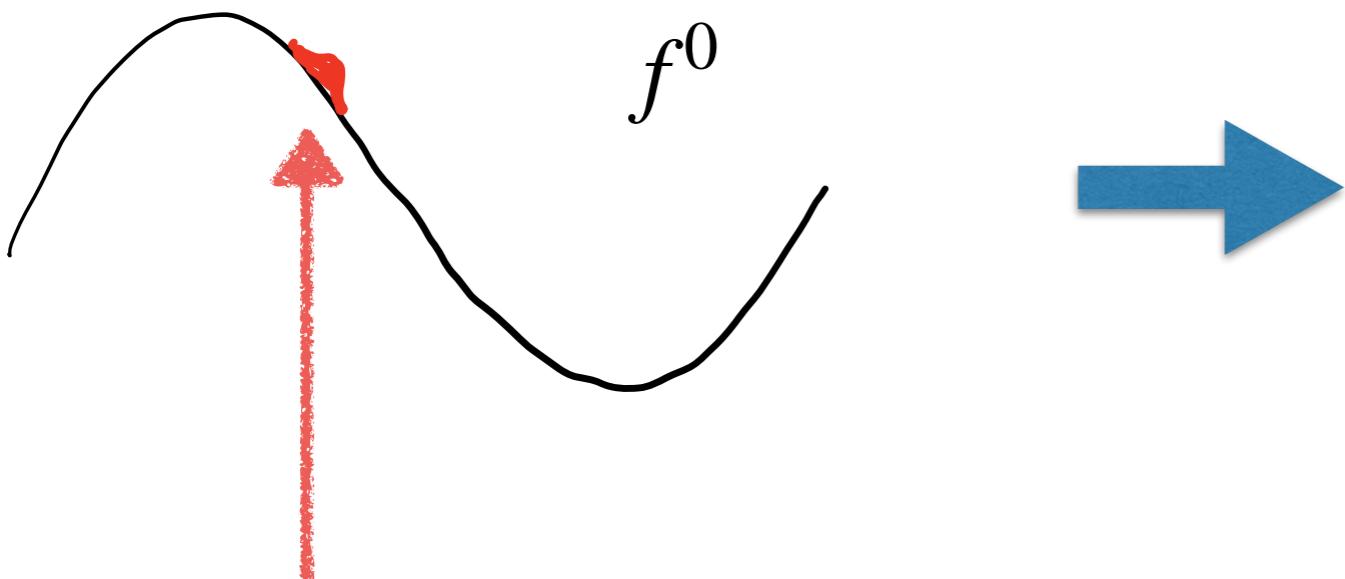
Kac model : [Kac, Logan; Méléard]

Model with stochastic collisions : [Rezakhanlou]

$$d\zeta_t = \mathcal{L}_t \zeta_t + d\eta_t$$

Linearised Boltzmann
operator around $f(t)$

- Dissipation



initial perturbation

$$d\zeta_t = \mathcal{L}_t \zeta_t + d\eta_t$$

Linearised Boltzmann
operator around $f(t)$

- Dissipation

Noise

- Creates entropy
- Covariance : dynamical correlations [Spohn]

- **Deterministic microscopic dynamics**
- **Randomness only in the initial conditions**
- **Dynamical instabilities \Rightarrow Space/time white noise**

$$\mathbb{E}_\varepsilon(\eta_t(\varphi)\eta_s(\varphi)) = \frac{1}{2}\delta_{t=s} \int dz_1 dz_2 d\nu ((v_1 - v_2) \cdot \nu)_+ f(t, z_1) f(t, z_2)$$

$\delta_{x_1=x_2} \times [\varphi(z'_1) + \varphi(z'_2) - \varphi(z_1) - \varphi(z_2)]^2$

Same noise structure as for Kac's model

No stochastic process at the microscopic level

Strategy of the proof.

1/ *Convergence in law of the process :*

- Control the characteristic function by cluster expansion

$$\lambda \mapsto \mathbb{E}_\varepsilon \left(\exp \left(\frac{i\lambda}{\sqrt{\mu_\varepsilon}} \sum_{i=1}^N \left(h(z_i(t)) - \int dz h(z) f(t, z) \right) \right) \right)$$

- Estimates on the size of the cumulants imply that only two-point correlations are relevant
- Identification of the covariance [Spohn]

2/ *Tightness of the process.*

Cluster expansion \Rightarrow better controls
 \Rightarrow large deviations

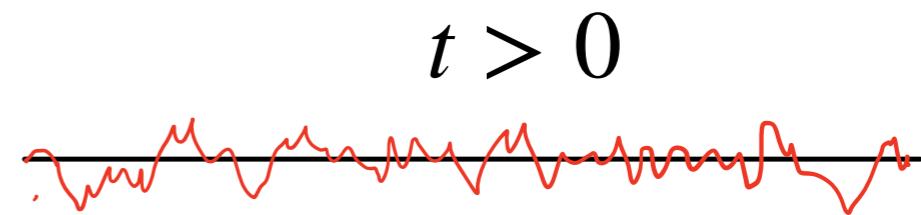
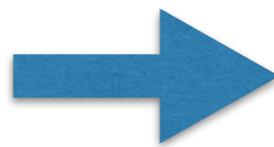
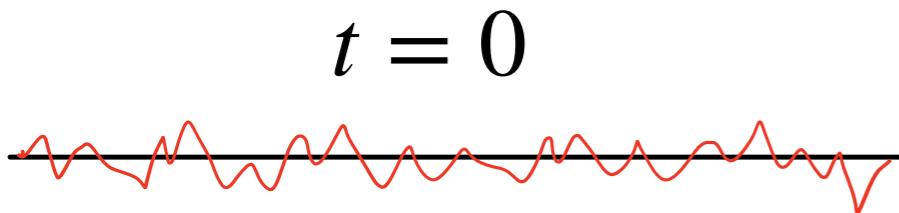
Equilibrium initial data :

$$W_N(0, Z_N) = \prod_{i=1}^N M(v_i) \times \frac{1}{\mathcal{Z}_N} \prod_{i \neq j} 1_{|x_i - x_j| > \varepsilon}$$

$$M(v) = \frac{1}{c_d} \exp\left(-\frac{v^2}{2}\right)$$

invariant under the dynamics

Fluctuation field : $\zeta_t^\varepsilon(h) = \frac{1}{\sqrt{\mu_\varepsilon}} \sum_{i=1}^N \left(h(z_i(t)) - \int dz h(z) M(v) \right)$



Remark.

The convergence time to the Boltzmann equation cannot be improved close to equilibrium by applying directly Lanford's strategy.

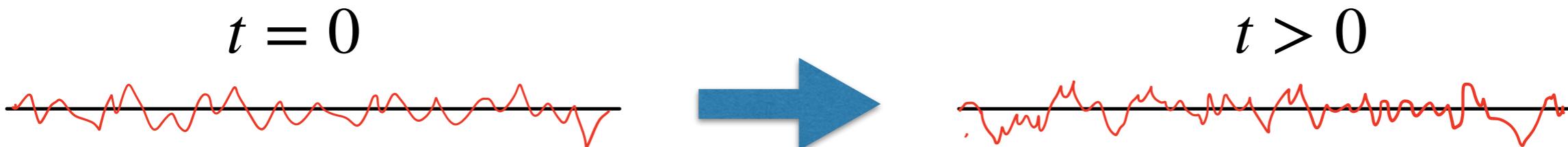
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Theorem. [B., Gallagher, Saint-Raymond, Simonella]

Convergence to the fluctuating Boltzmann equation

$$\forall t > 0, \quad \zeta_t^\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{(law)} \zeta_t \quad \text{with} \quad d\zeta_t = \mathcal{L} \zeta_t + d\eta_t$$

Consequences of $d\zeta_t = \mathcal{L} \zeta_t + d\eta_t$:

- Fluctuation/dissipation :

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}\left(\zeta_0^\varepsilon(h)\zeta_0^\varepsilon(g)\right) = \int dz h(z)g(z)M(v) \quad \text{initial correlations}$$

The diagram illustrates the evolution of initial correlations over time. At $t = 0$, two points on a red wavy line are highlighted by green and orange circles. A large blue arrow points to the right, indicating the transition to $t > 0$. At $t > 0$, the same two points are shown, but the orange circle has expanded significantly, indicating that the correlation between them has decayed to zero. A green arrow points from the initial correlation expression to the final state at $t > 0$.

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}\left(\zeta_0^\varepsilon(h)\zeta_t^\varepsilon(g)\right) = \mathbb{E}\left(\zeta_0(h)\zeta_t(g)\right) \xrightarrow{t \rightarrow \infty} 0$$

- Time correlations decay : dissipation
- Noise preserves the invariant measure

Hydrodynamic scalings

Convergence quantitative for large times:

New scaling: $N\varepsilon^{d-1} = \alpha$ with $\alpha = \log \log |\log \varepsilon|$

then $\forall t \ll \alpha$, $\zeta_t^\varepsilon \simeq \xi_t^{\alpha}$ with $d\xi_t^{\alpha} = \mathcal{L}^\alpha \xi_t^{\alpha} + d\eta_t$

overall scaling α^2

Rescaling time by α , we test :

- the energy

$$\Theta_\tau^\varepsilon = \zeta_{\tau\alpha}^\varepsilon \left(\psi \left(\frac{\nu^2}{d+2} - 1 \right) \right)$$

$$\psi(x) : [0,1]^d \mapsto \mathbb{R}^d$$

- the momentum

$$U_\tau^\varepsilon = \zeta_{\tau\alpha}^\varepsilon (\varphi \cdot \nu)$$

$$\varphi(x) : [0,1]^d \mapsto \mathbb{R}^d, \quad \nabla \varphi = 0$$

Hydrodynamic scalings

New scaling: $N\varepsilon^{d-1} = \alpha$ with $\alpha = \log \log |\log \varepsilon|$

then $\forall t \ll \alpha$, $\zeta_t^\varepsilon \simeq \xi_t^\alpha$ with $d\xi_t^\alpha = \mathcal{L}^\alpha \xi_t^\alpha + d\eta_t$

Theorem. [B., Gallagher, Saint-Raymond, Simonella]

Convergence of $(U_\tau^\varepsilon, \Theta_\tau^\varepsilon)_{\tau \leq T}$ to the fluctuating Fourier-

Stokes equations as $\varepsilon \rightarrow +\infty$:

$$\begin{cases} \partial_\tau \Theta = \kappa \Delta_x \Theta + \sqrt{\frac{4\kappa}{d+2}} \nabla \cdot \dot{W}_t \\ \partial_\tau \mathcal{U} = \nu \Delta_x \mathcal{U} + \sqrt{2\nu} P \nabla \cdot \dot{W}_t \end{cases}$$



projection on divergent free fields

Back to the fluctuating Boltzmann equation

$$W_N(0, Z_N) = \prod_{i=1}^N M(v_i) \times \frac{1}{\mathcal{Z}_N} \prod_{i \neq j} 1_{|x_i - x_j| > \varepsilon}$$
$$M(v) = \frac{1}{c_d} \exp\left(-\frac{v^2}{2}\right)$$

invariant under the dynamics

Fluctuation field : $\zeta_t^\varepsilon(h) = \frac{1}{\sqrt{\mu_\varepsilon}} \sum_{i=1}^N \left(h(z_i(t)) - \int dz h(z) M(v) \right)$

Theorem. [B., Gallagher, Saint-Raymond, Simonella]

Convergence to the fluctuating Boltzmann equation

$$\forall t > 0, \quad \zeta_t^\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{(law)} \zeta_t \quad \text{with} \quad d\zeta_t = \mathcal{L} \zeta_t + d\eta_t$$

[Rezakhanlou]

Strategy of the proof.

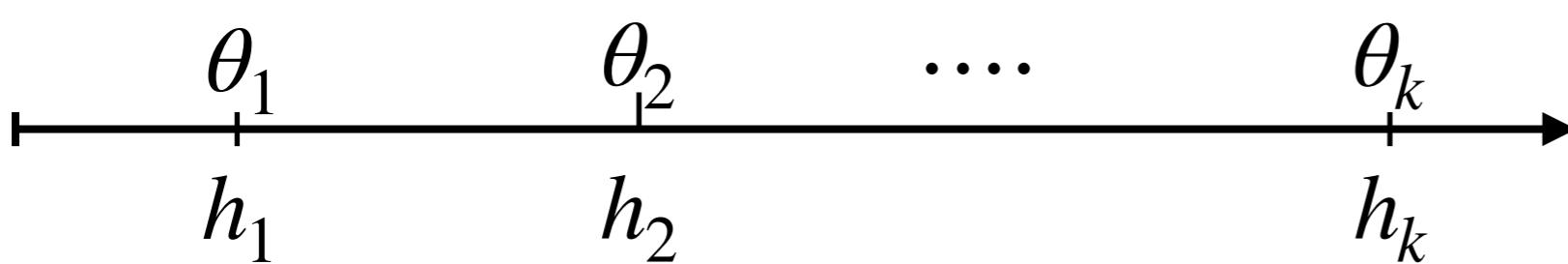
$$d\zeta_t = \mathcal{L} \zeta_t + d\eta_t$$

- *Tightness argument :*

$$\zeta_t^\varepsilon \xrightarrow[\mu_\varepsilon \rightarrow \infty]{(law)} \zeta_t$$

**Limiting process
is Gaussian**

- *Characterizing the limit : Proof of the Wick rule*



*k times
test functions*

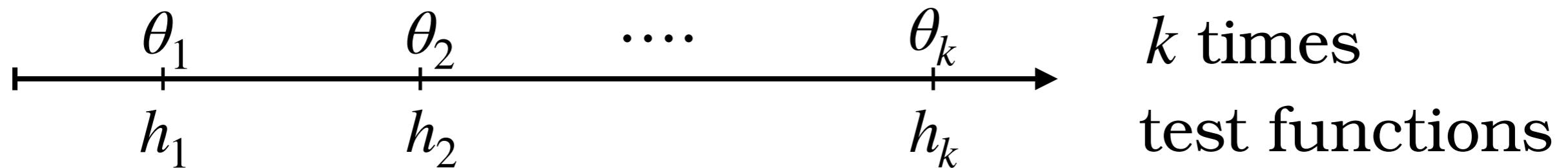
$$\mathbb{E}_\varepsilon \left(\prod_{i=1}^k \zeta_{\theta_i}^\varepsilon(h_i) \right) = \sum_{\substack{\eta \\ pairings}} \prod_{(i,j) \in \eta} \mathbb{E}_\varepsilon \left(\zeta_{\theta_i}^\varepsilon(h_i) \zeta_{\theta_j}^\varepsilon(h_j) \right) + \text{quantified error}$$

Wick rule : $\mathbb{E} \left(\prod_{i=1}^k \zeta_{\theta_i}(h_i) \right) = \sum_{\substack{\eta \\ pairings}} \prod_{(i,j) \in \eta} \mathbb{E} \left(\zeta_{\theta_i}(h_i) \zeta_{\theta_j}(h_j) \right)$

Strategy of the proof.

$$d\zeta_t = \mathcal{L} \zeta_t + d\eta_t$$

Proof of the *Wick rule* :



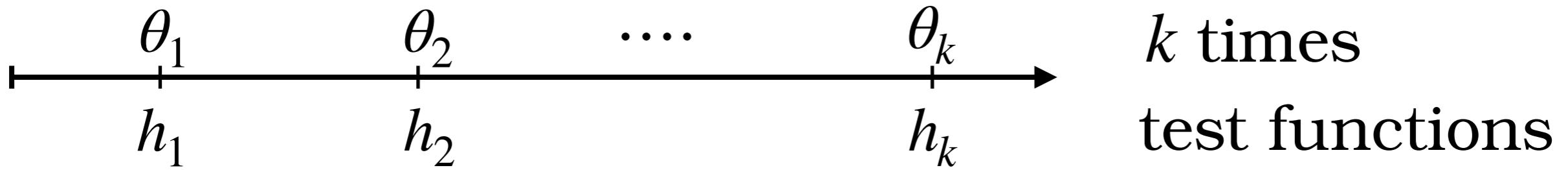
$$\mathbb{E}_\varepsilon \left(\prod_{i=1}^k \zeta_{\theta_i}^\varepsilon(h_i) \right) = \sum_{\substack{\eta \\ pairings}} \prod_{(i,j) \in \eta} \mathbb{E}_\varepsilon \left(\zeta_{\theta_i}^\varepsilon(h_i) \zeta_{\theta_j}^\varepsilon(h_j) \right) + \text{quantified error}$$

and identification of the covariance

$$\mathbb{E}_\varepsilon \left(\zeta_0^\varepsilon(h_0) \zeta_\theta^\varepsilon(h_1) \right) = \int dz M(v) h_0(z) \exp(-\theta \mathcal{L}) h_1(z) + \text{quantified error}$$

[van Beijeren, Lanford, Lebowitz, Spohn] (short time)

[B., Gallagher, Saint-Raymond] (d=2)



Summary.

The limiting Gaussian process $d\zeta_t = \mathcal{L} \zeta_t + d\eta_t$ is fully characterized by

$$\text{Wick rule : } \mathbb{E}\left(\prod_{i=1}^k \zeta_{\theta_i}(h_i)\right) = \sum_{\substack{\eta \\ pairings}} \prod_{(i,j) \in \eta} \mathbb{E}\left(\zeta_{\theta_i}(h_i)\zeta_{\theta_j}(h_j)\right)$$

$$\text{Covariance : } \mathbb{E}\left(\zeta_0(h_0)\zeta_\theta(h_1)\right) = \int dz M(v) h_0(z) \exp(-\theta \mathcal{L}) h_1(z)$$

Identification of the covariance

Mean zero test functions :

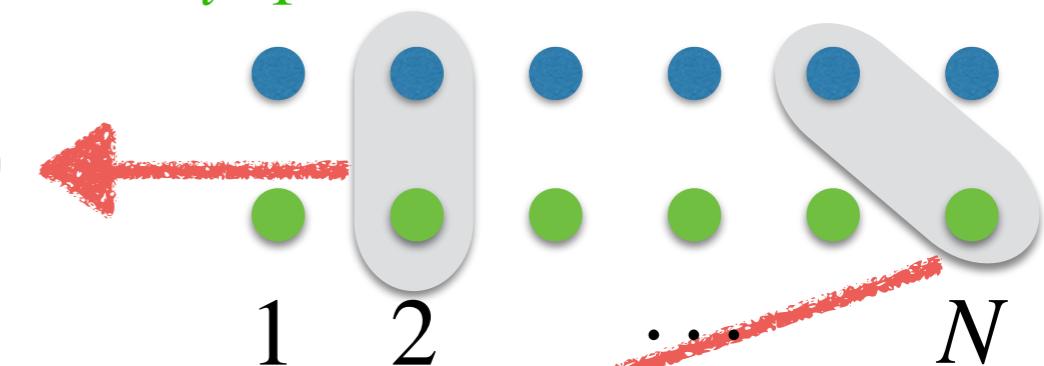
$$\zeta_t^\varepsilon(h) = \frac{1}{\sqrt{\mu_\varepsilon}} \sum_{i=1}^N h(z_i(t))$$

Correlations at time 0 :

$$\mathbb{E}_\varepsilon(\zeta_0^\varepsilon(h_0)\zeta_0^\varepsilon(h_1)) = \frac{1}{\mu_\varepsilon} \mathbb{E}_\varepsilon \left(\left(\sum_{i=1}^N h_0(z_i(0)) \right) \times \left(\sum_{i=1}^N h_1(z_i(0)) \right) \right)$$

$$= \frac{1}{\mu_\varepsilon} \mathbb{E}_\varepsilon \left(\sum_{i=1}^N h_0(z_i(0)) h_1(z_i(0)) \right)$$

$$+ \frac{1}{\mu_\varepsilon} \mathbb{E}_\varepsilon \underbrace{\left(\sum_{i \neq j} h_0(z_i(0)) h_1(z_j(0)) \right)}_{\simeq O(\varepsilon^d)}$$



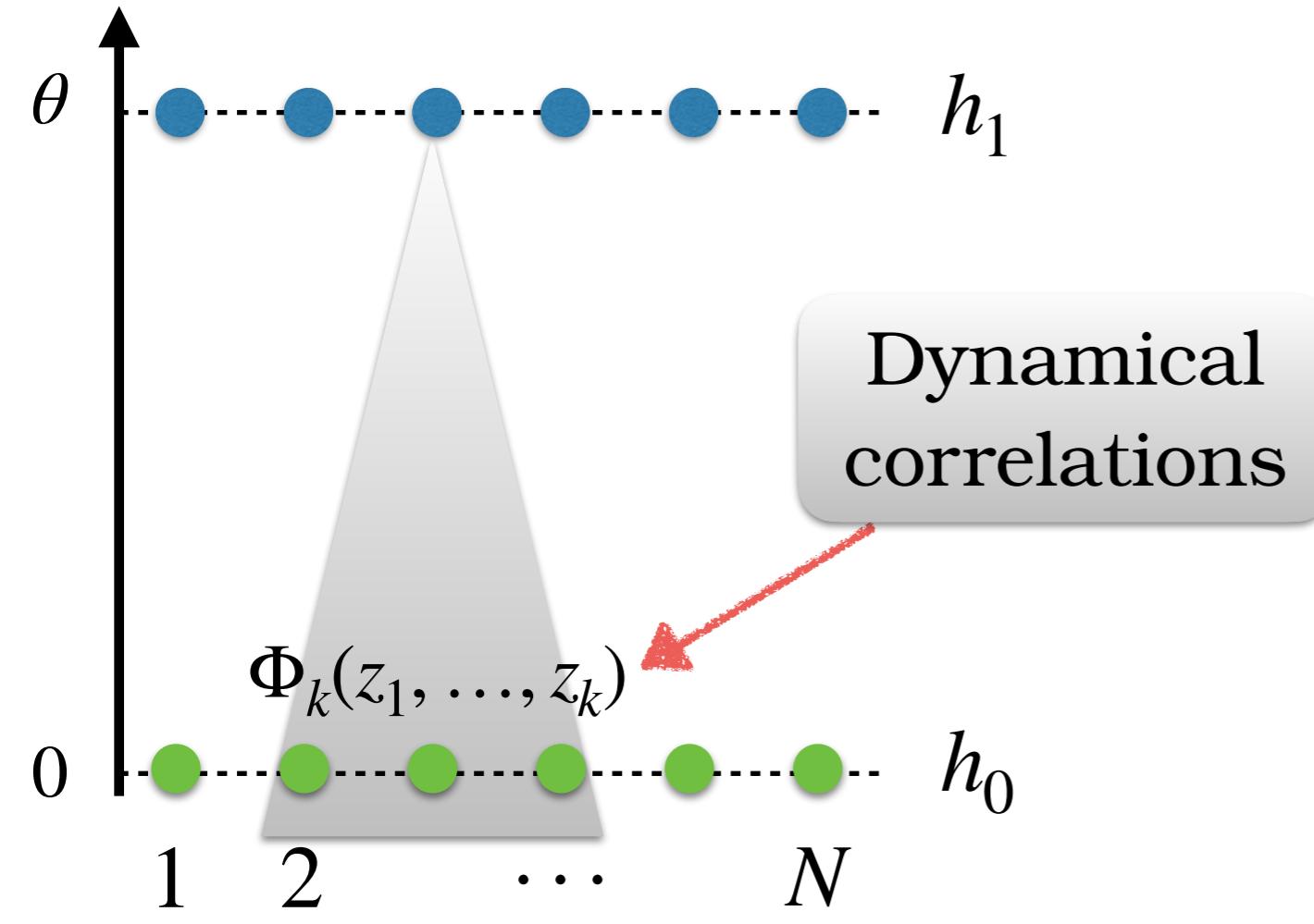
initial measure
 \simeq product

$$\mathbb{E}_\varepsilon(\zeta_0^\varepsilon(h_0)\zeta_0^\varepsilon(h_1)) = \int dz M(v) h_0(z) h_1(z) + o(\varepsilon)$$

pairing

Identification of the covariance : correlations at time θ

$$\mathbb{E}_\varepsilon \left(\zeta_0^\varepsilon(h_0) \zeta_\theta^\varepsilon(h_1) \right) = \frac{1}{\mu_\varepsilon} \mathbb{E}_\varepsilon \left(\left(\sum_{i=1}^N h_0(z_i(0)) \right) \times \left(\sum_{i=1}^N h_1(z_i(\theta)) \right) \right)$$



h_1 is represented by a function $\Phi_k(z_1, \dots, z_k)$ at time 0 coding the linearized evolution
(k is random)

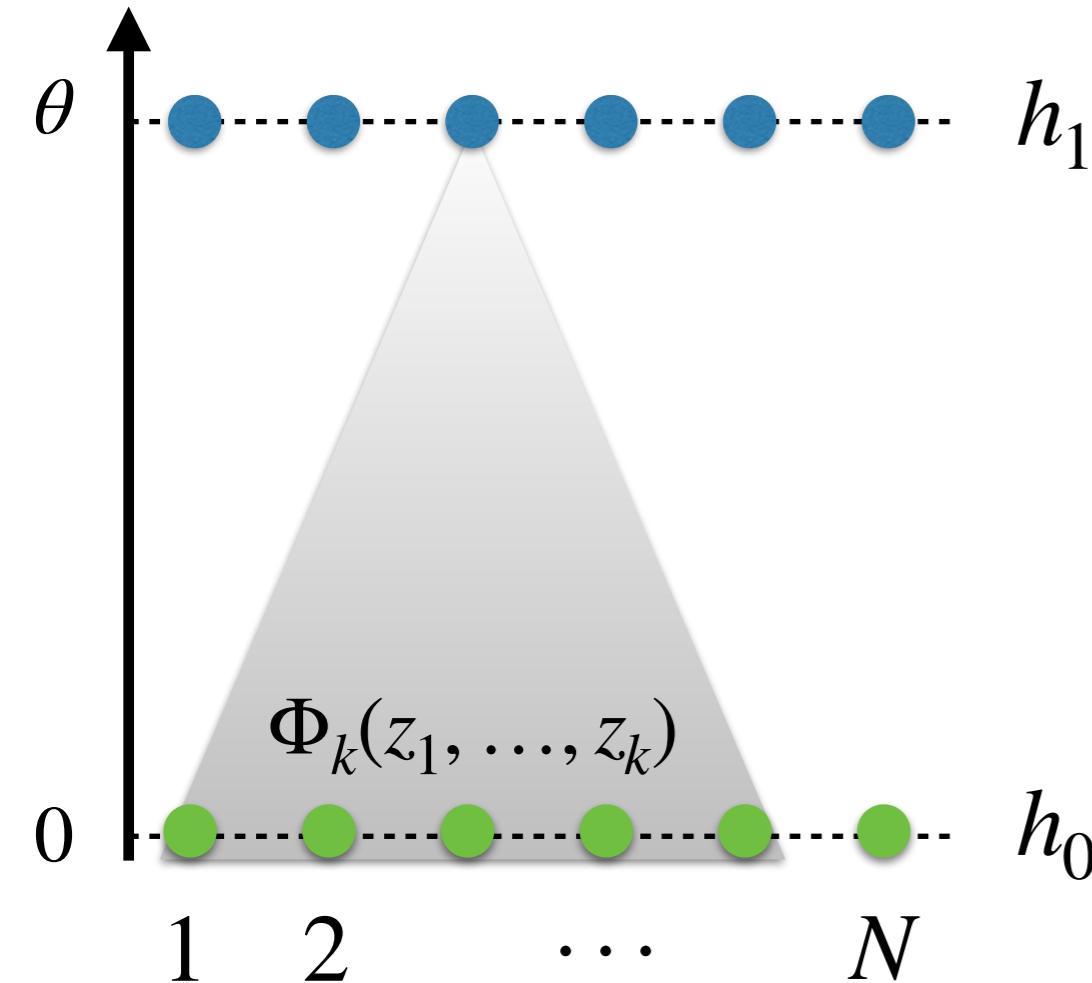
$$\hat{\zeta}_0^\varepsilon(\Phi_k) = \frac{1}{\sqrt{\mu_\varepsilon}} \sum_{i_1, \dots, i_k} \Phi_k(z_{i_1}, \dots, z_{i_k})$$

Correlations at time 0 : $\mathbb{E}_\varepsilon \left(\zeta_0^\varepsilon(h_0) \zeta_\theta^\varepsilon(h_1) \right) = \sum_k \mathbb{E}_\varepsilon \left(\zeta_0^\varepsilon(h_0) \hat{\zeta}_0^\varepsilon(\Phi_k) \right)$

Controlling correlations for large θ

Reducing to time 0 :

$$\hat{\zeta}_0^\varepsilon(\Phi_k) = \frac{1}{\sqrt{\mu_\varepsilon}} \sum_{i_1, \dots, i_k} \Phi_k(z_{i_1}(0), \dots, z_{i_k}(0))$$



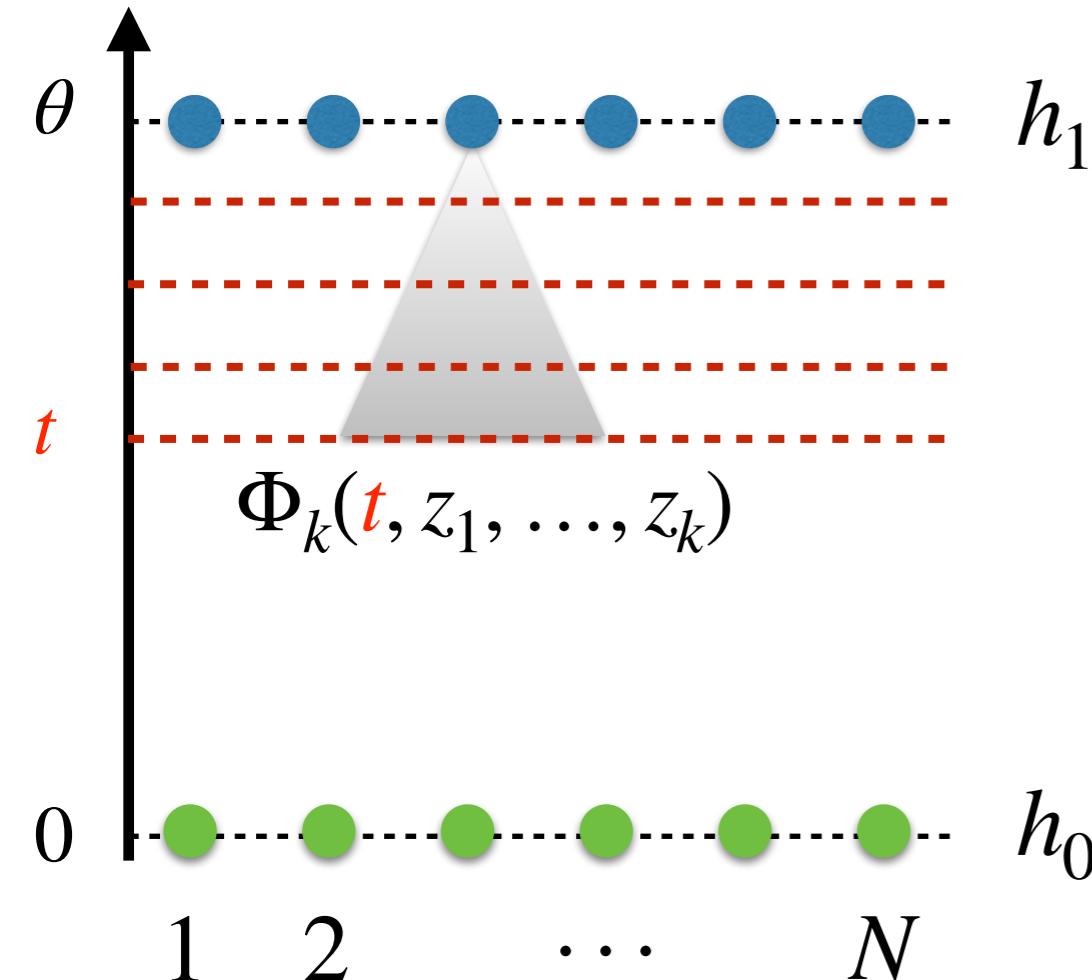
$$\mathbb{E}_\varepsilon \left(\zeta_0^\varepsilon(h_0) \zeta_{\theta}^\varepsilon(h_1) \right) = \sum_k \mathbb{E}_\varepsilon \left(\zeta_0^\varepsilon(h_0) \hat{\zeta}_{\textcolor{blue}{0}}^\varepsilon(\Phi_k) \right)$$

k has to be controlled for θ large

Controlling correlations for large θ

Reducing to time t :

$$\widehat{\zeta}_t^\varepsilon(\Phi_k) = \frac{1}{\sqrt{\mu_\varepsilon}} \sum_{i_1, \dots, i_k} \Phi_k(t, z_{i_1}(t), \dots, z_{i_k}(t))$$



Controlled backward propagation

$$\mathbb{E}_\varepsilon \left(\zeta_0^\varepsilon(h_0) \zeta_{\theta}^\varepsilon(h_1) \right) = \sum_k \mathbb{E}_\varepsilon \left(\zeta_0^\varepsilon(h_0) \widehat{\zeta}_{\textcolor{blue}{t}}^\varepsilon(\Phi_k(t)) \right)$$

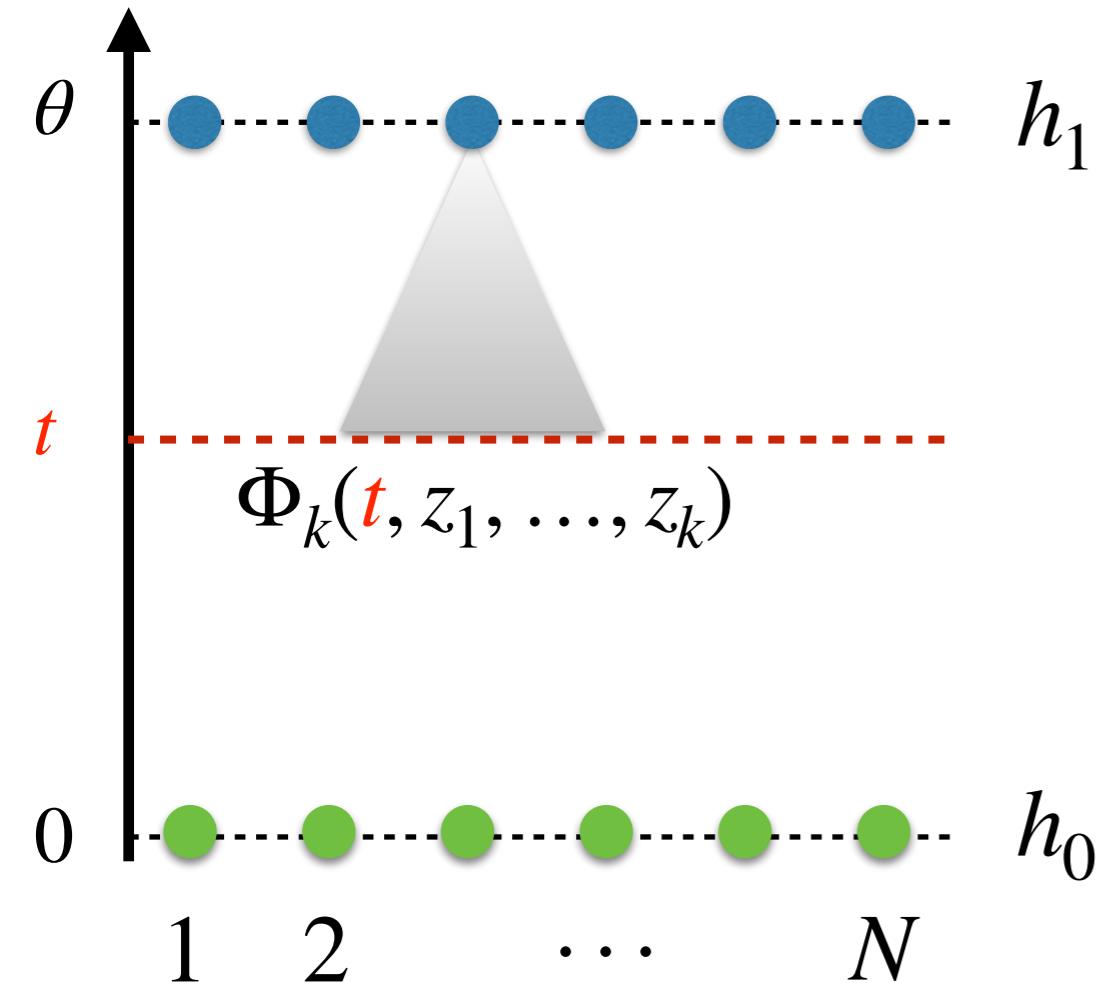
Stop at time $t > 0$ to remove :

- large k
- recollisions

Controlling correlations for large θ

Reducing to time t :

$$\widehat{\zeta}_t^\varepsilon(\Phi_k) = \frac{1}{\sqrt{\mu_\varepsilon}} \sum_{i_1, \dots, i_k} \Phi_k(t, z_{i_1}(t), \dots, z_{i_k}(t))$$



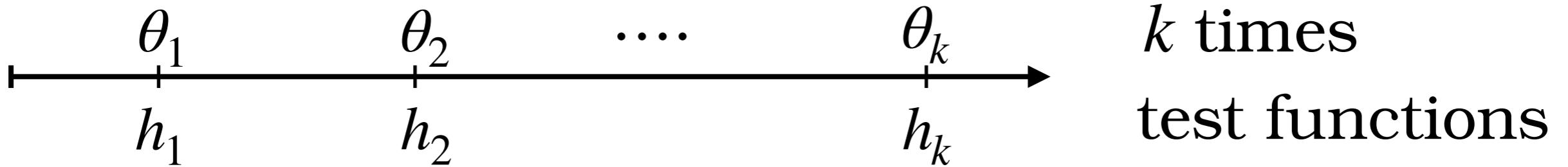
Large k can be neglected

$$\left| \mathbb{E}_\varepsilon \left(\zeta_0^\varepsilon(h_0) \widehat{\zeta}_{\textcolor{blue}{t}}^\varepsilon(\Phi_k(t)) \right) \right| \leq \mathbb{E}_\varepsilon \left(\zeta_0^\varepsilon(h_0)^2 \right)^{1/2} \mathbb{E}_\varepsilon \left(\widehat{\zeta}_{\textcolor{blue}{t}}^\varepsilon(\Phi_k(t))^2 \right)^{1/2}$$

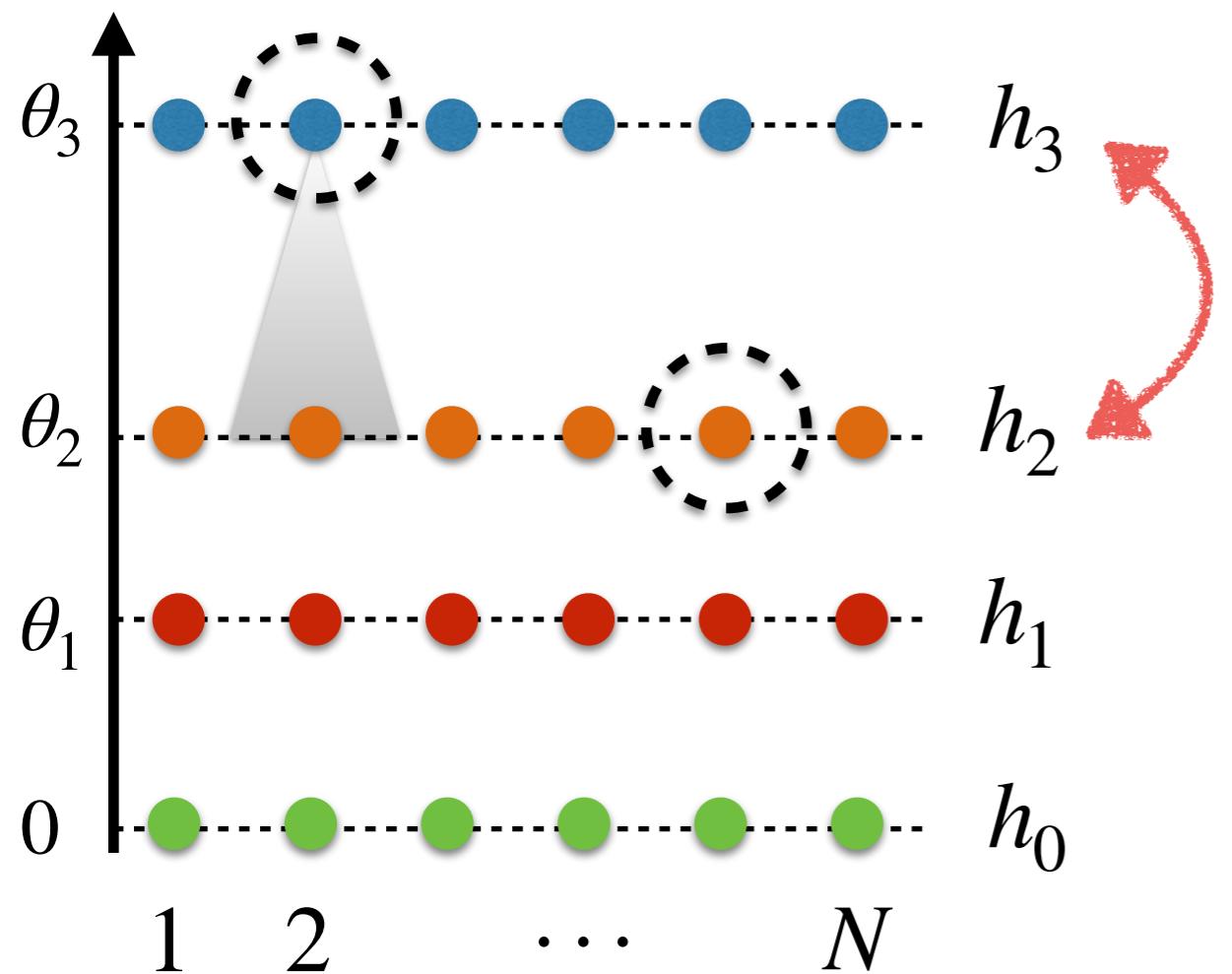
Time decoupling :

- (local) equilibrium
- symmetrisation \Rightarrow averaging
(better than just BBGKY)
- geometric estimates on trajectories

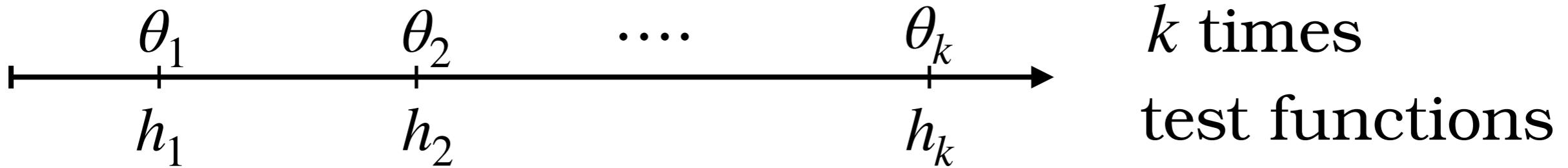
Wick rule for several times.



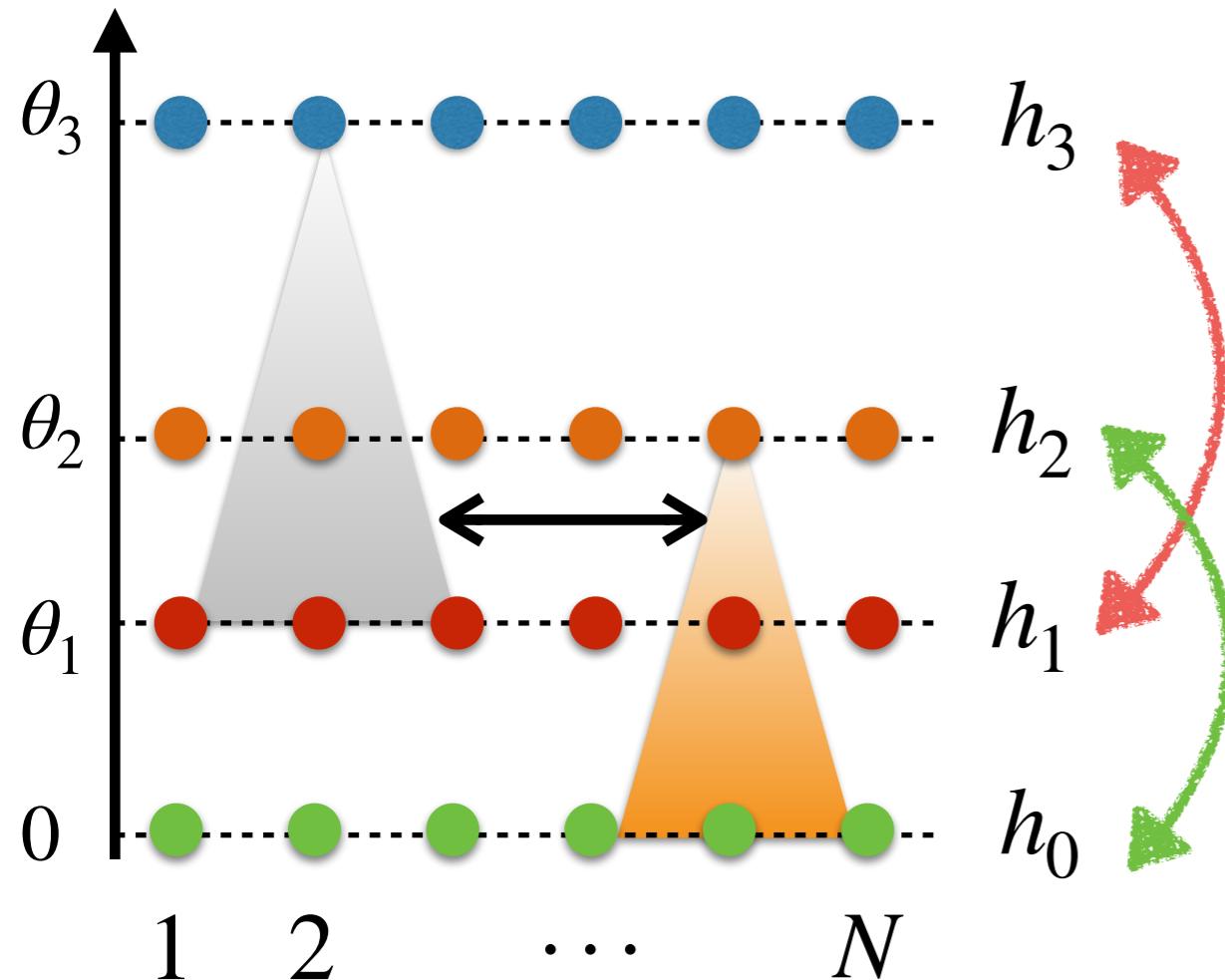
$$\mathbb{E}_\varepsilon \left(\prod_{i=1}^k \zeta_{\theta_i}^\varepsilon(h_i) \right) = \sum_{\substack{\eta \\ pairings}} \prod_{(i,j) \in \eta} \mathbb{E}_\varepsilon \left(\zeta_{\theta_i}^\varepsilon(h_i) \zeta_{\theta_j}^\varepsilon(h_j) \right) + \text{quantified error}$$



Wick rule for several times.



$$\mathbb{E}_\varepsilon \left(\prod_{i=1}^k \zeta_{\theta_i}^\varepsilon(h_i) \right) = \sum_{\substack{\eta \\ pairings}} \prod_{(i,j) \in \eta} \mathbb{E}_\varepsilon \left(\zeta_{\theta_i}^\varepsilon(h_i) \zeta_{\theta_j}^\varepsilon(h_j) \right) + \text{quantified error}$$



Difficulty :

- intertwined pairings
- dynamical interactions

Tool :

- dynamical cluster expansion

Remark.

Equilibrium measure : $W_N(0, Z_N) = \prod_{i=1}^N M(v_i) \times \frac{1}{\mathcal{Z}_N} \prod_{i \neq j} 1_{|x_i - x_j| > \varepsilon}$

Perturbation size in L^∞ of the equilibrium measure :

- Tagged particle : $O(1) \Rightarrow$ Brownian motion

- Boltzmann equation : $\prod_{i=1}^N f^0(z_i) = O(\exp(\mu_\varepsilon))$

- Fluctuating equation : $\prod_{i=1}^k \zeta_{\theta_i}^\varepsilon(h_i) = O(\mu_\varepsilon^{k/2})$

Averaging is key
to go beyond L^∞

Weak convergence method :

no estimate of the correlation functions

Conclusion.

Deterministic dynamics of a dilute gas of hard spheres

- Fluctuating Boltzmann equation :
 - stochastic corrections to Boltzmann equation
 - same structure as for the *Kac model*
 - fluctuation/dissipation relation
- Long time fluctuations at equilibrium
- Hydrodynamic scalings
 - fluctuating Stokes equation (momentum)
 - fluctuating Fourier equation (energy)