Renormalized spectral zeta function of Rabi model

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The Rabi model describes a two-level atom coupled to a single mode photon by the dipole interaction term. The single photon is represented by the 1D harmonic oscillator. This model is introduced by I.I. Rabi at 1937.

Let $\sigma_x, \sigma_y, \sigma_z$ be the 2×2 Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Suppose that the eigenvalues of the two level atom is $\{-\triangle, \triangle\}$. Here $\triangle > 0$. Then the Hamiltonian of the two level atom is represented by $\triangle\sigma_z$. On the other hand a and $a^\dagger$ be the annihilation operator and the creation operator in $L^2(\mathbb{R})$. It is given by

$$a = \frac{1}{\sqrt{2}}\left(\frac{\mathrm{d}}{\mathrm{d}x} + x\right), \quad a^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\mathrm{d}}{\mathrm{d}x} + x\right).$$

They satisfy the canonical commutation relation $[a, a^\dagger] = 1$. The harmonic oscillator is given by $a^\dagger a$, i.e.,

$$a^\dagger a = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2 - \frac{1}{2}.$$

The harmonic oscillator $a^\dagger a$ is self-adjoint on $D(\frac{\mathrm{d}^2}{\mathrm{d}x^2}) \cap D(x^2)$ and the spectrum of $a^\dagger a$ is $\sigma(a^\dagger a) = \{n\}_{n=0}^\infty$. The Rabi Hamiltonian is defined as a self-adjoint operator on the tensor product Hilbert space $\mathbb{C}^2 \otimes L^2(\mathbb{R})$ by

$$H_{Rabi} = \triangle\sigma_z \otimes \mathbb{1} + \mathbb{1} \otimes a^\dagger a + g\sigma_x \otimes (a + a^\dagger).$$

Here

$$\sigma_x \otimes (a + a^\dagger) = \sigma_x \otimes \sqrt{2}x$$

is the interaction term and $g \in \mathbb{R}$ stands for a coupling constant. It can be seen that H_{Rabi} has the parity symmetry:

$$[H_{Rabi}, \sigma_z \otimes (-1)^{a^\dagger a}] = 0.$$

We set $\sigma(H_{Rabi}) = \{E_n(g)\}_{n=0}^\infty$, where $E_n(g) \le E_{n+1}(g)$ for $n \ge 0$. The spectral zeta function of the Rabi model is defined by

$$\zeta_g(s; \tau) = \sum_{n=1}^\infty \frac{1}{(E_n(g) + \tau)^s}, \quad \tau > 0.$$

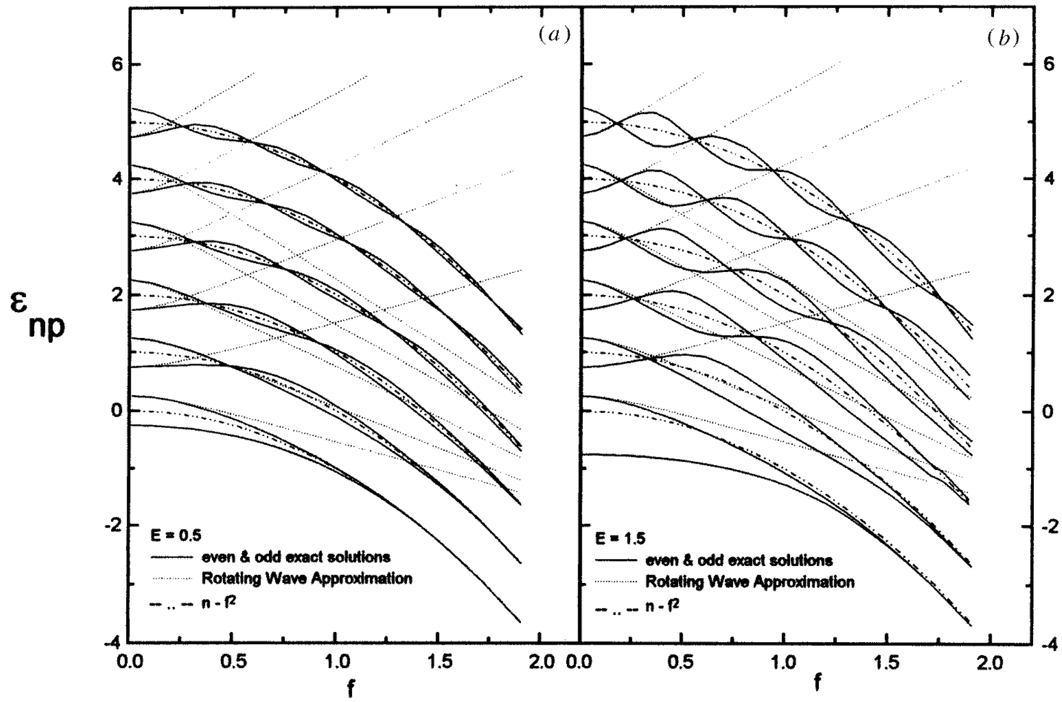

Figure 1: $E_n(g) \to -\infty$ as $|g| \to \infty$ I. D. Feranchuk, L. I. Komarov and A. P. Ulyanenkov, J. Phys. A: Math. Gen. 29 (1996) 4035-4047.

Since $H_{Rabi} + g^2 \geq -\triangle$, instead of $\zeta_g(s;\tau)$, we are interested in investigating the asymptotic behaviour of

$$\zeta_g^{\rm ren}(s;\tau) = \sum_{n=1}^{\infty} \frac{1}{(E_n(g) + \tau + g^2)^s}, \quad \tau > 0$$

as $g \to \infty$. $\zeta_g^{\rm ren}(s;\tau)$ is called the renormalized spectral zeta function. Let $\tau > 0$ and

$$\zeta(s;\tau) = \sum_{n=0}^{\infty} \frac{1}{(n+\tau)^s}, \quad s > 1$$

be the Hurwiz zeta function.

Theorem 0.1 *It is shown that*

$$\lim_{g\to\infty} \zeta_g^{\rm ren}(s;\tau) = 2\zeta(s;\tau).$$

Moreover it can be shown that

$$\lim_{g\to\infty} E_{2n}(g) + g^2 = n,$$
$$\lim_{g\to\infty} E_{2n+1}(g) + g^2 = n.$$