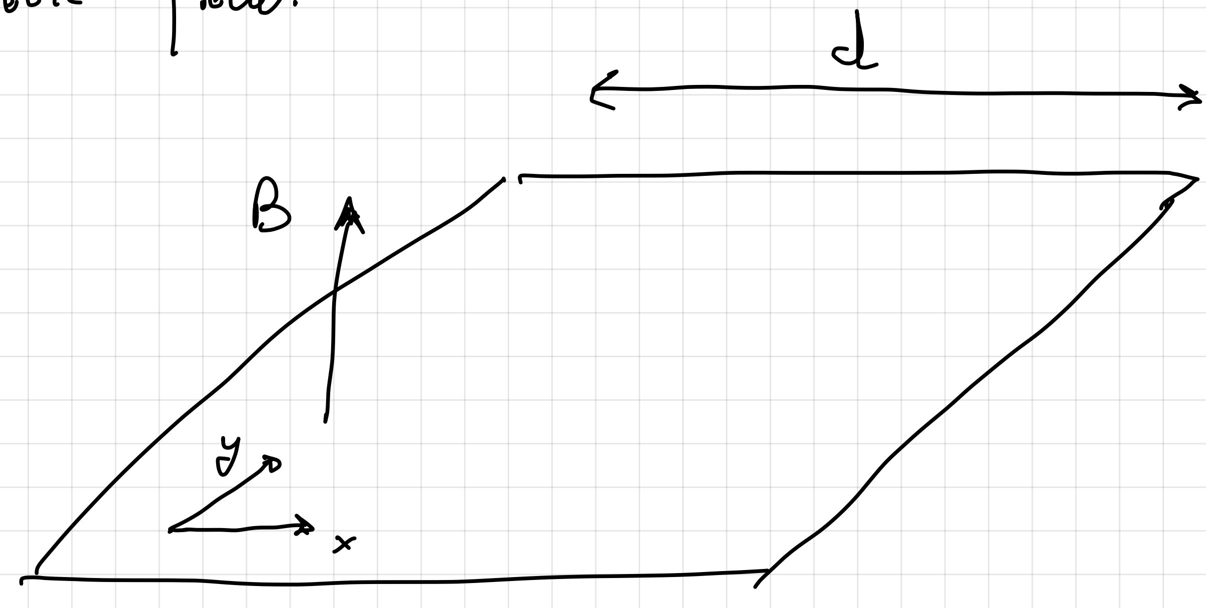
JAN 9 2023

An Overview of Math. Aspects of Top. Insulators

1. Entry point: IQHE

Take 2D GaAs (interface between two materials) at very low temp. and strong perp. magnetic field.



Apply electric field in $\hat{x}$ direction.
Due to magnetic field in $\hat{z}$ direction there will be a HALL CURRENT in the $\hat{y}$ direction.

Hall conductance: $\boxed{\sigma_{xy} := \frac{I_y}{V_x}}$.

Simplest model: classical EM

Lorentz force on e^-: ($q \equiv 1$ kept for clarity)

$$F = q\, v \wedge B$$

At equilibrium this should equal the force due to the electric field:

$$q\, v \wedge B = qE = q \frac{V_x}{d} \hat{e}_x$$

$$\Longrightarrow V_x = d \|v\| \, \|B\| .$$

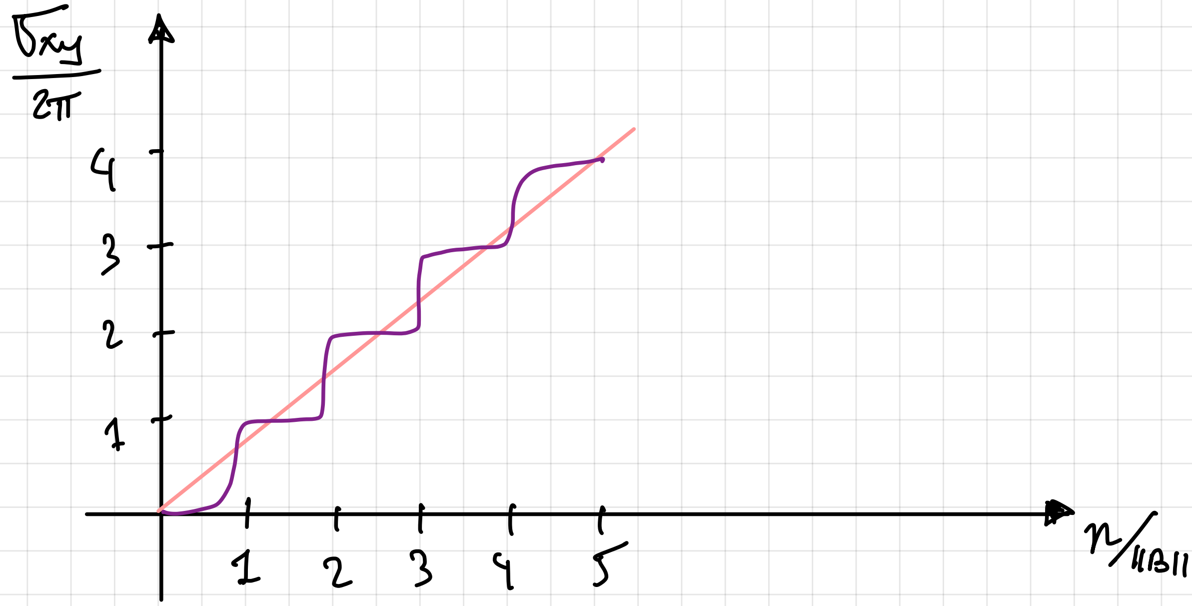
We also know $I_y \hat{e}_y = d\, j = d\, q n v$ ($n \equiv$ charge density)

current density

$$\Longrightarrow \sigma_{xy} = \frac{I_y}{V_x} = \frac{\cancel{d}\, q n \cancel{\|v\|}}{\cancel{d}\, \cancel{\|v\|}\, \|B\|} = \frac{q n}{\|B\|} \equiv \frac{n}{\|B\|} .$$

So expect that as we increase e^- density n or decrease $\|B\|$, we'll see a <u>linear</u> increase in σ_{xy}.

Experimental observation (von Klitzing, 1979):

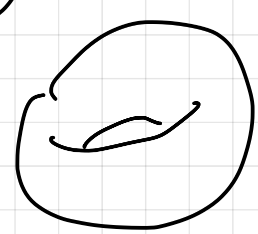


Expect pink plot from our calculation.

Observe purple quantized plot in experiment.

WHY?

Answer: "Algebraic topology of spaces of quantum mechanical Hamiltonians".

2. Some words on topology beyond  and .

⊛ A discrete space D has $\{s\} \in \mathrm{Open}(D)$ $\forall s \in D$.
Think of $D = \mathbb{Z}, \mathbb{Z}_2, \dots$

⊛ For any top. sp. S, it is called

"connected"

iff any continuous map $f: S \to \{0,1\}$ is

constant.

The conn. comp. are the collection of maximal conn. subsets.

For any discrete D, continuous $f: S \to D$, f is constant on connected components of S.

<u>IDEA</u>: Take S as topological space of QM Hamiltonians. If we succeed to find a continuous map $f: S \to D$ s.t. $|f(S)|$ is the # of conn. comp. of S then we arrive at a "complete top. classification" of S by $f: S \to D$.

$f \equiv$ top. invariant $\equiv$ physically msrbl. qty.

<u>Note on conn. vs. path conn.:</u>

⊛ Define an equiv. rel $\sim$ on S by:

$s \sim \tilde{s}$ iff $\exists$ cont. map $\gamma: [0,1] \to S$:

$\gamma(0) = s$, $\gamma(1) = \tilde{s}$ (path $\gamma: s \to \tilde{s}$).

$\pi_0(S) := \{ [s] \subseteq S \mid s \in S \}$ set of path conn. comp.

If $|\pi_0(S)| = 1$, S is path-connected.

Any $f: S \to D$ cont. is const. within PCCs.

[Indeed, assume otherwise. Then $f \circ \gamma : [0,1] \to D$ is not const., but $[0,1]$ is conn. & $f \circ \gamma$ is cont. $\boxed{\perp}$].

Claim: Path-conn. $\Rightarrow$ conn.

(Converse is false!)

Top. sine curve

$$\left\{ \left(x, \sin\left(\tfrac{1}{x}\right)\right) \in \mathbb{R}^2 \,\middle|\, x \in (0,1] \right\}$$

So two notions are distinct, but the distinction is exotic (sp. must be NOT locally path conn.). Since path-conn. is simpler, we concentrate on that.

ULTIMATE GOAL: Find a continuous $f: S \to D$ which lifts to a bijection $\hat{f}: \pi_0(S) \to D$.

BASIC PRINCIPLE:

EXPERIMENTAL QUANTIZATION & STABILITY

$\updownarrow$

CONTINUITY PROPERTY OF TOP. INVARIANT INTO DISCRETE SPACE.

"Identifying the ambient topology of space of QM Hamiltonians will predict what kind of <u>experimental</u> perturbations leave top. invariant constant..."

J. QM Models of Insulators

QM of non int. e^- in solids.

X configuration space $\big(\underset{\text{cont.}}{\mathbb{R}^d}, \underset{\text{TB}}{\mathbb{Z}^d}, \underset{\text{edge}}{\mathbb{Z}^{d-1}\times\mathbb{N}}, \dots\big)$

Hilbert sp. $\mathcal{H} = L^2(X) \otimes \mathbb{C}^N$

int. DoF / spin / ...

<u>Def.</u>: A "material" is a linear S.A. op.

$$H : \mathcal{H} \to \mathcal{H}$$

which is "local":

$$\underbrace{\| \langle \delta_x, H \delta_y \rangle_{\mathcal{H}} \|}_{N\times N \text{ matrix}}{}_{\mathbb{C}^N} \leq C e^{-\mu \|x-y\|_X} \qquad (x,y \in X)$$

<u>Claim</u>: The space of local S.A. bdd. op. on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$ is path-conn. w.r.t. s/sp. top. from op. norm. top.

Proof: Let H be a local S.A. op. and define

$$\gamma(t) := (1-t)H + t\mathbb{1}$$

Note:
- $\mathbb{1}$ is local
- sum of local is local
- $\gamma : [0,1] \to \mathcal{B}(\ell^2(\mathbb{Z}^d)\otimes\mathbb{C}^N)$ is cont. and hence restricted to s/sp. remains cont.

$\Rightarrow$ any local S.A. bdd. op. is path-conn. to $\mathbb{1}$.

$\Rightarrow$ Use "insulators" from the title:

Def.: A "material" H is "an insulator @ $\mu \in \mathbb{R}$" iff $\mu \notin \sigma(H)$, the spectrum of H.

Claim: If H is "an insulator @ $\mu \in \mathbb{R}$" then the zero temp. DC conduc. of H is zero.

Proof: Kubo formula.

⚠ Random op. exhibit a more general gap cond. called "mobility gap" for which H also has zero DC cond.
Many analytic hurdles.
Will not comment further.

Now it is clear that space of insulators contains <u>at least</u> 3 PCCs (for fixed $\mu=0$):

① Spec. all below 0.
② Spec. all above 0.
③ Spec. above <u>and</u> below 0.

First two classes not so interesting.

It turns out that the third class further breaks into $\mathbb{Z}$ PCCs IF $d \in 2\mathbb{N}$.
$d=2$ is the aforementioned IQHE, $\boxed{D=\mathbb{Z},\ f=\frac{\sigma_{xy}}{2\pi}}$.
CHERN #.

3.1 Concrete models for the IQHE

① <u>Landau Hamiltonian</u> (param. by $B\in\mathbb{R}$)

$$X=\mathbb{R}^2,\quad N=1$$

pos. op.

$$H=\left(P-\tfrac{1}{2}B\,e_3\wedge X\right)^2$$

$$\sigma(H)=B(2\mathbb{N}_{\geq 0}+1) \rightsquigarrow \text{many gaps.}$$

$$\frac{\sigma_{xy}(\mu)}{2\pi} \equiv \#\text{ of eigenvalues below } \mu.$$

② $\frac{1}{2}$ BHZ Hamiltonian (param. by $a \in \mathbb{R}$)

$$\mathcal{X} = \mathbb{Z}^2, \quad N = 2,$$

R_j right shift on j^{th} axis: $(R_j \psi)(x) \equiv \psi(x - e_j)$

$$H = \left[R_1 \otimes \tfrac{1}{2}(\sigma_3 - i\sigma_1) + R_2 \otimes \tfrac{1}{2}(\sigma_3 - i\sigma_2) + h.c.\right] + a\,\mathbb{1} \otimes \sigma_3$$

$$\sigma(H) = \bigcup_{k_1, k_2 \in [0, 2\pi]} \pm\sqrt{\sin(k_1)^2 + \sin(k_2)^2 + [a + \cos(k_1) + \cos(k_2)]^2}$$

If $a = 2$ $\nexists$ gap about zero.

$$\sigma_{xy}(0) = \begin{cases} \pm 1 & a \in \pm(0, 2) \\ 0 & |a| > 2 \end{cases}.$$

(Even though the space of all models has $\mathbb{Z}$ PCCs, this family of models only explores $\{0, \pm 1\}$ as a varies on $\mathbb{R}$).

3.2. Altland-Zirnbauer Symm. Classes

Around 2005 it was realized that restricting the space of insulators further by imposing symm. constraints leads to richer classification.

One popular scheme is due to Altland & Zirnbauer, originally from random matrices:

Let ① $\Theta : \mathcal{H} \to \mathcal{H}$ be anti-unitary: (TR)

$\Theta^{-1} = \Theta^*$ but Θ is anti-$\mathbb{C}$-lin:

$$\Theta(\alpha\psi) = \bar{\alpha}\,\Theta\psi \qquad (\alpha \in \mathbb{C}).$$

$$\langle \Theta\varphi, \Theta\psi \rangle \equiv \langle \psi, \varphi \rangle$$

$$\Theta^2 = \pm \mathbb{1}$$

H is TRI $\Leftrightarrow$ $[H, \Theta] = 0$.

② $\Xi : \mathcal{H} \to \mathcal{H}$ anti-unitary (particle-hole)

$$\Xi^2 = \pm \mathbb{1}$$

H is PHS $\Leftrightarrow$ $\{H, \Xi\} = 0$

Both Θ and Ξ commute w/ position op.

③ $\Pi := \Theta\,\Xi$ (chiral symm.)

H is chiral iff $\{H, \Pi\} = 0$.

(could be even if not TRI / PHS!)

All together, $\exists$

10 (AZ) symm. classes

$\updownarrow$

10 symm. spaces / Classical gps.

$\updownarrow$

10 Clifford alg. Morita classes

KITAEV TABLE

	Symmetry			d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

4. The Chiral 1D Example

Let $\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^{2N} \cong \underbrace{\left(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^N\right)}_{=: \mathcal{H}_+} \oplus \underbrace{\left(\ell^2(\mathbb{Z}) \otimes \mathbb{C}^N\right)}_{\mathcal{H}_-}$

Implement Π (chiral symm.) as

$$\Pi := \begin{bmatrix} \mathbb{1}_{\mathcal{H}_+} & 0 \\ 0 & -\mathbb{1}_{\mathcal{H}_-} \end{bmatrix}.$$

$\Rightarrow \{H, \Pi\} = 0 \iff H = \begin{bmatrix} 0 & S^* \\ S & 0 \end{bmatrix}$

$\exists\ S : \mathcal{H}_+ \to \mathcal{H}_-$ not necessarily S.A.

$\{H, \Pi\} = 0 \Rightarrow \sigma(H)$ symm. about zero,

So $\not\exists$ to set $\mu = 0$.

$\Rightarrow 0 \notin \sigma(H) \Leftrightarrow S$ is invertible.

Moreover, H is local iff S is local.

$\boxed{\text{Example: } S := R, \text{ right shift } (R\psi)(x) \equiv \psi(x-1) \quad (x\in\mathbb{Z})}$.

We find that the space of (spectrally gapped) chiral insulators is isomorphic to the space

$$\mathcal{A} := \left\{ S \in \mathcal{B}(\ell^2(\mathbb{Z})\otimes\mathbb{C}^N) \;\middle|\; S \text{ inv. and local} \right\}.$$

Let $\mathcal{A}$ have the s/sp. top. from the op. norm top. on $\mathcal{B}(\ell^2(\mathbb{Z})\otimes\mathbb{C}^N)$.

Define $\mathcal{N} : \mathcal{A} \to \mathbb{Z}$ via

$$\mathcal{N}(S) := \text{index}(\Lambda S) \in \mathbb{Z}$$

where: $\Lambda S \equiv \Lambda S \Lambda + \mathbb{1} - \Lambda$

$\Lambda \equiv$ proj. to RHS of space $((\Lambda\psi)(x) \equiv \chi_{\mathbb{N}}(x)\psi(x))$

index: $\mathcal{H} \to \mathbb{Z}$ is the Fredholm index of a Fredholm op.: $\text{index}(F) \equiv \dim(\ker F) - \dim(\ker F^*)$.

Claim: $\mathcal{N}$ is a well-defined continuous surj. map which obeys a log. law $\mathcal{N}(ST) = \mathcal{N}(S) + \mathcal{N}(T)$.

Thm. (CHUNG–S. '23): $\mathcal{N}$ ascends to a bijection

$\hat{\mathcal{N}} : \pi_0(\mathcal{A}) \xrightarrow{\sim} \mathbb{Z}$ so that we obtain a complete

Classification of $\mathcal{A}$.

<u>Remarks</u>: ① Assuming transl. invar. this reduces to the classical fact that the winding # classifies the space of continuous maps $\mathbb{S}^1 \to \mathbb{C} \setminus \{0\}$. (See e.g. Krein, Widom, ... ; explained in [Graf-S. 2018]).

② Assuming covariant disorder this reduces to the classical fact that $K_1(C^*(\mathbb{R})) \cong \mathbb{Z}$.
(See e.g. Prodan-Schulz-Baldes 2016 book on Bulk-Bdry. Corresp. for $\mathcal{P}$-classes).

③ In this level this statement is new AFAIK.

④ No assertion on mob. gap regime yet (but w/ obvious conj. ... working on it).

⑤ Cf. Atiyah-Jänich thm.: If X is cpt. Hausdorf then $[X \to \mathcal{F}(\mathcal{H})] \cong K_0(C(X))$. In particular, if $X = \{0\}$, $\pi_0(\mathcal{F}(\mathcal{H})) \cong K_0(C(\{0\})) \cong \mathbb{Z}$.

5. Small Detour into Fredholm Op.

$F \in \mathcal{B}(\mathcal{H})$ is Fredholm iff

① $\ker F$, $\ker F^*$ finite dim.

② $\text{range}(F)$ closed in $\mathcal{H}$ ($\Leftrightarrow \|F\psi\|^2 \geq c\|\psi\|^2$
$\exists c>0, \forall \psi \in \ker(F)^\perp$).

Whence $\text{index}(F) \equiv \dim\ker F - \dim\ker F^* \in \mathbb{Z}$
is well-defined.

The space $\mathcal{F} \subseteq \mathcal{B}(\mathcal{H})$ of such
operators obeys:

① $\text{index}(F+G) = \text{index}(F) \quad \forall G \in \text{Cpt},$

② $\text{index}(F+G) = \text{index}(F) \quad \forall \|G\|$ small.

and BTW, $\pi_0(\mathcal{F}) \cong \mathbb{Z}$ (cont. w.r.t. s/sp. top. from op. norm top.).

Examples: ① $\text{index}(\text{invertible}) = 0$

② 0 is NOT Fredholm!

③ $\text{index}(M_{m\times n}) = m-n$ by rank-nullity.

④ Let $\hat{R}$ be the right-shift on $\ell^2(\mathbb{N})$.

$$(\hat{R}\psi)(n) \equiv \psi(n+1) \qquad n \in \mathbb{N}.$$

$$\ker(\hat{R}) \cong \{0\}$$
$$\ker(\hat{R}^*) \cong \mathbb{C}\delta_1$$

$$\Rightarrow \operatorname{index}(\hat{R}) = -1.$$

BTW $\Lambda R \cong \hat{R}$ in the following sense:

$\Lambda : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ proj. onto states within resp. $\ell^2(\mathbb{N}) \subseteq \ell^2(\mathbb{Z})$.

Hence $\ker_{\ell^2(\mathbb{Z})}(\Lambda R \Lambda) \cong \ker_{\ell^2(\mathbb{N})}(\hat{R})$.

So $\mathcal{N}(S) = -1$.

(5) Avron-Seiler-Simon '94: (IQHE)

For Landau Hamil., let P proj. onto some L.L.

Note $U \equiv \exp(i \arg(X_1 + i X_2))$ is a right shift within angular momentum space:

If Θ is the angle op. in the plane,

$U = \exp(i\Theta)$

But $\exp(i\Theta)$ is a shift in ang. mom. just as $\exp(iX)$ is a shift in mom.

Within one L.L., the angular mom. sector $\cong \ell^2(\mathbb{N})$ since $\nexists$ states of negative ang. mom.

(Basic fact about sol-n of Landau H.)

$\Rightarrow \quad PUP + P^\perp \cong$ right shift on $\ell^2(\mathbb{N})$

$\Rightarrow \quad \operatorname{index}(PUP + P^\perp) = -1.$

It turns out that if $P = \chi_{(-\infty,\mu)}(H)$,

$\operatorname{index}(PUP + P^\perp)$ gives the Hall conductance

(Laughlin argument, Bellissard-van Elst-Schulz-Baldes)

6. Proof of Chiral 1D Thm.

6.1. Why is ΛS Fredholm when S is local

$$S \text{ is local} \iff \|S_{xy}\| \leq C e^{-\mu|x-y|} \quad (x,y\in\mathbb{Z})$$

Claim: S is local $\Rightarrow$ $[\Lambda, S]$ is cpt.

Proof: FACT: ① F is cpt. $\Leftarrow$ F is Schatten class.

② $\|F\|_1 \leq \sum_{x,y} \|F_{xy}\|$

$$[\Lambda, S]_{xy} \equiv (\Lambda S - S\Lambda)_{xy} = (\Lambda S)_{xy} - (S\Lambda)_{xy}$$

$$= \chi_{\mathbb{N}}(x) S_{xy} - S_{xy}\chi_{\mathbb{N}}(y)$$

$$= (\chi_{\mathbb{N}}(x) - \chi_{\mathbb{N}}(y)) S_{xy}$$

$$\Rightarrow \|[\Lambda, S]\|_1 \leq \sum_{x,y\in\mathbb{Z}} |\chi_{\mathbb{N}}(x) - \chi_{\mathbb{N}}(y)| \, \|S_{xy}\|$$

$$= \sum_{x\in\mathbb{N},\, y\in\mathbb{Z}\setminus\mathbb{N}} \|S_{xy}\| + \sum_{\substack{x\in\mathbb{Z}\setminus\mathbb{N} \\ y\in\mathbb{N}}} \|S_{xy}\|$$

By loc.,

$$\sum_{x\in\mathbb{N}, y\in\mathbb{Z}\setminus\mathbb{N}} \|S_{xy}\| \leq \sum_{x\in\mathbb{N}, y\in\mathbb{Z}\setminus\mathbb{N}} Ce^{-\mu|x-y|}$$

$$= C\sum_{\substack{x>0\\ y\leq 0}} e^{-\mu x} e^{\mu y} < \infty.$$

<u>Claim</u>: If F is Fredholm and Q is a proj. s.t. $[Q,F]\in$ cpt. then QF is Fredholm.

<u>Proof</u>: <u>FACT</u>: (ATKINSON's THM.) F is Fredholm iff $\exists$ G (parametrix) : $\mathbb{1}-FG$, $\mathbb{1}-GF$ are both cpt.

<u>Claim</u>: QG is the param. of QF.

<u>Proof</u>: $\mathbb{1}-(QF)(QG) =$

$$= \mathbb{1} - QFQGQ - Q^\perp$$

$$= Q - QFQGQ$$

$$= Q(\mathbb{1} - FQG)Q$$

$$= Q\underbrace{(\mathbb{1} - FG)}_{\text{cpt.}}Q + Q\underbrace{[F,Q]}_{\text{cpt.}}GQ$$

But $S \in \mathcal{A}$ is inv. $\subseteq \mathcal{F} \Rightarrow \Lambda S \in \mathcal{F}$ ✓.

6.2 Continuity of $\mathcal{N}$ and the Logarithmic Property

Assume $S, \tilde{S} \in \mathcal{A}$. W.T.S. $\exists\ \varepsilon > 0$: if $\|S - \tilde{S}\| < \varepsilon$ then $\mathcal{N}(S) = \mathcal{N}(\tilde{S})$.

$$\|\Lambda S - \Lambda \tilde{S}\| \equiv \|\Lambda S \Lambda + \Lambda^\perp - \Lambda \tilde{S} \Lambda - \Lambda^\perp\| = \|\Lambda (S - \tilde{S}) \Lambda\| \overset{\|\Lambda\| = 1.}{\leqslant} \|S - \tilde{S}\|.$$

Now follows via cont. of index (Dieudonne).

Now, if $S, \tilde{S} \in \mathcal{A}$, $S\tilde{S} \in \mathcal{A}$ too since:

1. $\|S\tilde{S}\| \geqslant \|S\| \|\tilde{S}^{-1}\|^{-1} > 0 \rightsquigarrow$ inv.
2. Product of locals is local
   $([\Lambda, S\tilde{S}] = [\Lambda, S]\tilde{S} + S[\Lambda, \tilde{S}])$

and
$$\mathcal{N}(S) + \mathcal{N}(\tilde{S}) = \mathrm{index}(\Lambda S) + \mathrm{index}(\Lambda \tilde{S}) \underset{\text{of index}}{\overset{\text{log. prop.}}{=}} \mathrm{index}((\Lambda S)(\Lambda \tilde{S}))$$

$$(\Lambda S)(\Lambda \tilde{S}) = \Lambda S \Lambda \tilde{S} \Lambda + \Lambda^\perp = \boxed{\Lambda(S\tilde{S})} + \Lambda \underbrace{[S, \Lambda]}_{\text{cpt.}} \tilde{S} \Lambda$$

But index is stable on adding cpt. op.

Rm.: Taking any integer power of R yields surjectivity.

6.3 The PCC Theorem

Let $S, \tilde{S} \in \mathcal{A}$ be given : $\mathcal{N}(S) = \mathcal{N}(\tilde{S})$.
W.T.S. that $\exists$ cont. path $\gamma : [0,1] \to \mathcal{A}$
s.t. $\gamma(0) = S$, $\gamma(1) = \tilde{S}$.
Using the log. prop. equiv. to $\mathcal{N}(S) = 0$ then
$\exists$ path $S \rightsquigarrow \mathbb{1}$.

Recall KUIPER's THM.: The inv. op. on $\mathcal{B}(\mathcal{H})$ are path-connected. Can't this work here, since S is invertible ???

<u>NO</u>: locality will be violated.
Indeed, Kuiper works via:

$$A = |A| e^{iH} \qquad (\text{polar decomp. } \exists\, H = H^*)$$

Then $$\gamma(t) = \underbrace{\left((1-t)|A| + t\mathbb{1}\right)}_{\text{always inv.}} \underbrace{\exp(i(1-t)H)}_{\text{always unitary}}$$

If A is local & inv., so will $|A|$ be.

However, in general $H = -i \log(A)$ will NOT be local unless log is analytic f^n (only when $\sigma(A|A|^{-1}) \neq \mathbb{S}^1$).

These statements rely on the Combes-Thomas estimate: If f is analytic and S is local then $\quad f(S) \equiv \frac{i}{2\pi} \oint \frac{f(z)}{(S - z\mathbb{1})} dz \quad$ is

local too.

Need an alternative route.

① Cl.: S is local. So $\forall\ \varepsilon > 0\ \exists\ k \in \mathbb{N}$:

$$\| \underbrace{S - \sum_{j=-k}^{k} S_j R^j}_{=: S^{(k)}} \| < \varepsilon$$

where R^j is the right shift op. to the j^{th} power and S_j is a diagonal op. w/ el.

$$(S_j)_{xx} := S_{x, x+j}$$

Pf.: Locality implies that the sq.

$$\{ S^{(k)} \}_k \quad \text{is} \quad \text{Cauchy}$$

in op. norm top. and its limit is S.

$\Rightarrow$ A straight line path $S \to S^{(k)}$ passes within $\mathcal{A}$ for k suff. large!

1. Due to inv. being open it can't suddenly pass through non-invertibles.
2. Turning off hoppings only makes an op. <u>more</u> local.

Moreover, by norm cont., $\mathcal{N}(S) = \mathcal{N}(S^{(k)}) = 0$.

Now by redimerization,

$$S^{(k)} \cong \tilde{A} + \tilde{B} R + \tilde{C} R^*$$ on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^{\tilde{M}}$

for some diagonal seq. $\tilde{A}, \tilde{B}, \tilde{C} : \mathbb{Z} \to \mathrm{Mat}_{\tilde{M}}(\mathbb{C})$.

Note $\tilde{M} \gg N$ (dep. on k).

Furthermore, since redim. is unitary, this process does not change $\mathcal{N}$.

A further redimerization yields

$$S^{(k)} \cong A + BR$$ on $\ell^2(\mathbb{Z}) \otimes \mathbb{C}^M$ $(M = 2\tilde{M})$

(see [Graf-S. 2018]) but now

$$\mathcal{N}(A + BR) \in [-M, 0] \cap \mathbb{Z}$$

Indeed, if $A = 0$, $B = \mathbb{1}$ get $\mathcal{N}(R) = -M$.

if $A=\mathbb{1}, B=0$, get $\mathcal{N}(\mathbb{1})=0$.

<u>Thm.:</u> If $A+BR$ is inv. and

$$p := -\mathcal{N}(A+BR) \in \{0,\dots,M\}$$

Then $A+BR$ may be path-conn. in the class of op. $\{A+BR\}_{A,B} \subseteq$ inv. to $P_p^{\perp}+P_pR$ w/ $P_p := \mathbb{1}_p \oplus 0_{M-p}$.

<u>Proof:</u> Based on a Thm. due to [Ben-Artzi – Gohberg 1991] on the spectrum of weighted-shift op.

<u>Intuitively:</u> $A+BR$ is inv.

$\Updownarrow$

A is inv. up to cpt.

$\Updownarrow$

$\mathbb{1}+\tilde{B}R$ is inv. $\Leftrightarrow$ $\tilde{B}R$ can't have spec. including -1.

But $\tilde{B}R$ is a "weighted-shift op." which has annular spectrum.

7. Epilogue

Topics not included:

0. Mobility gap (= strong disorder).
1. Bulk-Edge Corr.
2. Cont.-Disc. Corr.
3. Bulk-Edge Violation.
4. Linear resp. theory.
5. Interacting theory.
6. Beyond AZ classes.
7. Kane's Euler materials.
8. Twisted bilayer graphene.