Functional inequalities and eigenvalue bounds for Schrödinger operators

Seminar *Mathematical Challenges in Quantum Mechanics*, online

Tobias König

Goethe–Universität Frankfurt

November 09, 2022

Plan of the lecture

Part I - Inequalities for the lowest Schrödinger eigenvalue

- Motivation : stability of the hydrogen atom
- Hardy's and Sobolev's inequality and their application to the stability of hydrogen
- Duality between Sobolev's inequality and bounds on the lowest Schrödinger eigenvalue

Part II - Inequalities for sums of Schrödinger eigenvalues

- Motivation : stability of matter
- The Lieb–Thirring inequalities
- Duality : A Sobolev inequality for orthonormal functions
- The best Lieb–Thirring constants
- The Cwikel–Lieb–Rozenblum inequality : conformal invariance and best constants

Part I - Inequalities for the lowest Schrödinger eigenvalue

The stability of the hydrogen atom

Consider an electron attracted by a nucleus at fixed position $R_0 = 0 \in \mathbb{R}^3$, i.e. the hydrogen atom. In the quantum-mechanical formulation, the energy of an electron wave function $\psi \in L^2(\mathbb{R}^3)$ with $\|\psi\|_{L^2(\mathbb{R}^3)} = 1$ is

$$\mathcal{E}[\psi] = \int_{\mathbb{R}^3} |\nabla \psi|^2 \, \mathrm{d}x - \kappa \int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|} \, \mathrm{d}x.$$

where $\kappa > 0$ contains the physical constants.

- The **ground state energy** is the lowest eigenvalue

$$\lambda_1(-\Delta - \frac{\kappa}{|x|}) = \inf \{\mathcal{E}[\psi] \, : \, \|\psi\|_2 = 1\} \tag{1}$$

  of the Schrödinger operator $-\Delta - \frac{\kappa}{|x|}$ acting on $L^2(\mathbb{R}^3)$.

Stability of the hydrogen atom : *Why is $\mathcal{E}[\psi]$ bounded from below ?*

The uncertainty principle and Hardy's inequality

The **Heisenberg uncertainty principle** states that $\psi \in L^2(\mathbb{R}^3)$ and its Fourier transform $\widehat{\psi}(\xi) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-i\xi\cdot x}\psi(x)\,\mathrm{d}x$ can never be too concentrated simultaneously :

$$\left(\int_{\mathbb{R}^3} |\xi|^2 |\widehat{\psi}(\xi)|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |x|^2 |\psi(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \geq \frac{3}{2}\int_{\mathbb{R}^3} |\psi(x)|^2\,\mathrm{d}x.$$

So $\int_{\mathbb{R}^3} |\nabla\psi(x)|^2\,\mathrm{d}x = \int_{\mathbb{R}^3} |\xi|^2|\widehat{\psi}(\xi)|^2\,\mathrm{d}\xi$ must be large if ψ with $\|\psi\|_2 = 1$ is supported near the origin.

A more useful formulation of the same idea is encoded in **Hardy's inequality**.

Theorem (Hardy's inequality)

For every $\psi \in C_0^\infty(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} |\nabla\psi(x)|^2\,\mathrm{d}x \geq \frac{1}{4}\int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|^2}\,\mathrm{d}x.$$

So if $\|\nabla\psi\|_2^2$ is not too big, then ψ cannot have much 'mass' near the origin.

Stability via Hardy's inequality

We can now give a very simple proof for $\lambda_1(-\Delta - \kappa\frac{1}{|x|}) > -\infty$ using Hardy's inequality. Indeed, Hölder's inequality gives

$$\kappa \int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|}\,\mathrm{d}x \leq \kappa \left(\frac{1}{2\kappa} \int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|^2}\,\mathrm{d}x \right)^{\frac{1}{2}} \left(2\kappa \int_{\mathbb{R}^3} |\psi(x)|^2\,\mathrm{d}x \right)^{\frac{1}{2}}$$
$$\leq \frac{1}{4} \int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|^2}\,\mathrm{d}x + \kappa^2 \int_{\mathbb{R}^3} |\psi(x)|^2\,\mathrm{d}x.$$

Hence for every $\psi \in H^1(\mathbb{R}^d)$ with $\|\psi\|_2 = 1$,

$$\mathcal{E}[\psi] = \int_{\mathbb{R}^3} |\nabla\psi|^2\,\mathrm{d}x - \kappa \int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|}\,\mathrm{d}x$$
$$\geq \int_{\mathbb{R}^3} |\nabla\psi|^2\,\mathrm{d}x - \frac{1}{4} \int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|^2}\,\mathrm{d}x - \kappa^2$$
$$\geq -\kappa^2$$

by Hardy's inequality. So $\lambda_1(-\Delta - \kappa|x|^{-1}) \geq -\kappa^2$.
(The true value is $\lambda_1(-\Delta - \kappa|x|^{-1}) = -\frac{1}{4}\kappa^2$.)

Sobolev's inequality

We now introduce another fundamental functional inequality, the **Sobolev inequality**. To achieve a bit more generality, we state it for any dimension $d \geq 3$.

Theorem (Sobolev's inequality)

Let $d \geq 3$. There is $S_d > 0$ such that, for all $u \in C_0^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 \, \mathrm{d}x \geq S_d \left(\int_{\mathbb{R}^d} |u(x)|^{\frac{2d}{d-2}} \, \mathrm{d}x \right)^{\frac{d-2}{d}} . \quad (2)$$

- By scaling (i.e. considering the family $u_\lambda(x) = u(\lambda x)$), the exponent $\frac{2d}{d-2}$ is the only one for which (2) can possibly hold.
- Sobolev's inequality can also be viewed as an uncertainty principle : fix $\phi \in H^1(\mathbb{R}^d)$ and consider $\phi_\lambda(x) = \lambda^{d/2}\phi(\lambda x)$ with $\|\phi_\lambda\|_2 = 1$, which concentrates at 0 as $\lambda \to \infty$. Then $\int_{\mathbb{R}^d} |\nabla \phi_\lambda|^2 \, \mathrm{d}x \geq \|\phi_\lambda\|^2_{\frac{2d}{d-2}} = \lambda^2 \|\phi\|^2_{\frac{2d}{d-2}}$.
- Together with Hölder's inequality, (2) implies the interpolation inequality

$$\left(\int_{\mathbb{R}^d} |\nabla u|^2 \, \mathrm{d}x \right)^\theta \left(\int_{\mathbb{R}^d} |u|^2 \, \mathrm{d}x \right)^{1-\theta} \geq S_{d,q} \left(\int_{\mathbb{R}^d} |u|^q \, \mathrm{d}x \right)^{\frac{2}{q}}$$

for any $2 \leq q \leq \frac{2d}{d-2}$ and $\theta = \theta(q)$ determined by scaling.

A bound on $\lambda_1(-\Delta + V)$ for general potential V

We will see now that the Sobolev inequality can in fact give a much more universal bound on the lowest eigenvalue

$$\lambda_1(-\Delta + V) = \inf_{u \in H^1(\mathbb{R}^d)\setminus\{0\}} \frac{\int_{\mathbb{R}^d} (|\nabla u|^2 + V|u|^2)\,\mathrm{d}x}{\int_{\mathbb{R}^d} |u|^2\,\mathrm{d}x},$$

where $V \in L^p(\mathbb{R}^d)$ (for some $p \in (1, \infty)$) is an **arbitrary** potential.

Question : *Can we bound* $|\lambda_1(-\Delta + V)|$, *or a power* $|\lambda_1(-\Delta + V)|^\gamma$ *for some* $\gamma \geq 0$, *uniformly in* $\int_{\mathbb{R}^d} V_-^p\,\mathrm{d}x$ *?*

- We will always assume that all eigenvalues $\lambda_i(-\Delta + V)$ in question are negative.
- For $\gamma = 0$, we make the convention that $\lambda_1(-\Delta + V)^0 = 0$ if $-\Delta + V$ has no negative eigenvalue.

Since $V_\mu(x) = \mu^2 V(\mu x)$ yields $\lambda_1(-\Delta + V_\mu)^\gamma = \mu^{2\gamma}\lambda_1(-\Delta + V)^\gamma$, this can only hold for $p = \gamma + \frac{d}{2}$. Thus, we are asking whether the following quantity is finite :

$$L^{(1)}_{\gamma,d} := \sup_{V \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)} \frac{|\lambda_1(-\Delta + V)|^\gamma}{\int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}}\,\mathrm{d}x}$$

The duality between $S_{d,q}$ and $L^{(1)}_{\gamma,d}$

Recall the Sobolev interpolation inequality

$$\left(\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x\right)^{\theta} \left(\int_{\mathbb{R}^d} |u|^2 \,\mathrm{d}x\right)^{1-\theta} \geq S_{d,q} \left(\int_{\mathbb{R}^d} |u|^q \,\mathrm{d}x\right)^{\frac{2}{q}}. \tag{3}$$

Concerning $\lambda_1(-\Delta + V)$, we are interested in the validity of the inequality

$$|\lambda_1(-\Delta + V)|^{\gamma} \leq L^{(1)}_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}} \,\mathrm{d}x. \tag{4}$$

Theorem (Duality for $\lambda_1(-\Delta + V)$)

Let $q \in (2, \frac{2d}{d-2}]$ and $\gamma \geq 0$ be such that $\frac{q}{2}$ and $\gamma + \frac{d}{2}$ are Hölder conjugates, i.e. $\gamma + \frac{d}{2} = \frac{q}{q-2}$. Then inequalities (3) *and* (4) *are* **dual** *to each other in the sense that one implies the other and that the best constants $S_{d,q}$ and $L^{(1)}_{\gamma,d}$ are related by*

$$S_{d,q} = \theta^{\theta}(1-\theta)^{1-\theta}(L^{(1)}_{\gamma,d})^{\frac{1}{\gamma+\frac{d}{2}}}.$$

Moreover, u optimizes (3) *iff $V = -\alpha|u|^{q-2}$ optimizes* (4) *for some $\alpha > 0$.*

Proof of the duality theorem (sketch)

If (3) holds with $S_{d,q} > 0$, then for all $u \in H^1(\mathbb{R}^d)$ with $\|u\|_2 = 1$, by Hölder

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla u|^2 + V|u|^2 &\geq \int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x - \|V_-\|_{\gamma+\frac{d}{2}} \left(\int_{\mathbb{R}^d} |u|^q \,\mathrm{d}x \right)^{\frac{2}{q}} \\ &\geq \int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x - S_{d,q}^{-1} \|V_-\|_{\gamma+\frac{d}{2}} \left(\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x \right)^{\theta} \\ &\geq \inf_{T>0} T - S_{d,q}^{-1} \|V_-\|_{\gamma+\frac{d}{2}} T^\theta = -\theta^{\frac{\theta}{1-\theta}} (1-\theta) S_{d,q}^{-\frac{1}{1-\theta}} \|V_-\|_{\gamma+\frac{d}{2}}^{\frac{1}{1-\theta}}.\end{aligned}$$

Conversely, if (4) holds with constant $L^{(1)}_{\gamma,d} > 0$, then

$$\int_{\mathbb{R}^d} |\nabla u|^2 + V|u|^2 \,\mathrm{d}x \geq - \left(L^{(1)}_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}} \,\mathrm{d}x \right)^{\frac{1}{\gamma}} \|u\|_2^2$$

for all $u \in H^1(\mathbb{R}^d)$, $V \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)$. With $V = -\alpha |u|^{q-2}$, we get

$$\int_{\mathbb{R}^d} |\nabla u|^2 - \alpha |u|^q \,\mathrm{d}x \geq -\alpha^{1+\frac{d}{2\gamma}} \left(L^{(1)}_{\gamma,d} \int_{\mathbb{R}^d} |u|^q \,\mathrm{d}x \right)^{\frac{1}{\gamma}} \|u\|_2^2$$

Optimizing in $\alpha > 0$ now yields (3), with the claimed value of $S_{d,q}$. □

Part II – Inequalities for Schrödinger eigenvalue sums

Stability of matter

The quantum-mechanical Hamiltonian $H_{N,M}^{\boldsymbol{R},Z}$ of a configuration of N electrons of charge 1 and M nuclei of charge Z at fixed positions $(R_k)_{k=1}^M$ is given by its quadratic form

$$\mathcal{E}_{N,M}^{\boldsymbol{R},Z}[\psi] = \sum_{i=1}^N \int_{\mathbb{R}^{3N}} |\nabla_{x_i}\psi|^2 \,\mathrm{d}\boldsymbol{x} - \sum_{i=1}^N \sum_{k=1}^M \int_{\mathbb{R}^{3N}} \frac{Z|\psi|^2}{|x_i - R_k|}\,\mathrm{d}\boldsymbol{x}$$
$$+ \sum_{1<i<j\leq N} \int_{\mathbb{R}^{3N}} \frac{|\psi|^2}{|x_i - x_j|}\,\mathrm{d}\boldsymbol{x} + \sum_{1\leq k<l\leq M} \int_{\mathbb{R}^{3N}} \frac{Z^2|\psi|^2}{|R_k - R_l|}\,\mathrm{d}\boldsymbol{x}.$$

The electron wave function $\psi = \psi(x_1, ..., x_N)$ lives in the antisymmetric tensor product $\bigwedge_{i=1}^N L^2(\mathbb{R}^3) \subset L^2(\mathbb{R}^{3N})$ and is normalized to fulfill $\|\psi\|_{L^2(\mathbb{R}^{3N})} = 1$.

The corresponding ground state energy is

$$E_{N,M}^{\boldsymbol{R},Z} = \lambda_1(H_{N,M}^{\boldsymbol{R},Z}) = \inf\left\{\mathcal{E}_{N,M}^{\boldsymbol{R},Z}[\psi] \,:\, \psi \in H^1_{\text{antisym}}(\mathbb{R}^{3N}), \|\psi\|_{L^2(\mathbb{R}^{3N})} = 1\right\}.$$

Stability of matter : *Why is the energy per particle* $\frac{E_{N,M}^{\boldsymbol{R},Z}}{N+M}$ *bounded from below (for fixed* Z*, uniformly in* N*,* M *and* $\boldsymbol{R}$*)?*

The Pauli exclusion principle

Adding up the one-particle bound $\int_{\mathbb{R}^d}(|\nabla\psi|^2 - \kappa\psi^2|x|^{-1})\,\mathrm{d}x \geq -\kappa^2$ from Part I, with $\kappa = MZ$, we only obtain

$$\mathcal{E}_{N,M}^{\boldsymbol{R},Z}[\psi] \geq \frac{1}{M}\sum_{i=1}^{N}\sum_{k=1}^{M}\int_{\mathbb{R}^{3N}} |\nabla_{x_i}\psi(\boldsymbol{x})|^2 - MZ\int_{\mathbb{R}^{3N}}\frac{|\psi|^2}{|x_i - R_k|}$$
$$\geq -Z^2NM^2.$$

This is linear in N, but quadratic in M !

To prove a linear bound $E_{N,M}^{\boldsymbol{R},Z} \geq N + M$, we need to take into account another fundamental physical law, the **Pauli exclusion principle** :

- Since $\psi \in \bigwedge_{i=1}^{N} L^2(\mathbb{R}^3)$, the wavefunction ψ cannot simply be a product of N identical lowest one-particle eigenfunctions.
- Rather, ψ may be the antisymmetric tensor product of the N lowest eigenfunctions.

Thus the relevant quantity we need to bound should rather be $\sum_{i=1}^{N}\lambda_i(-\Delta + V)$, where $(\lambda_i(-\Delta+V))_{i=1}^{\infty}$ are the negative eigenvalues of $-\Delta + V$.

The Lieb–Thirring inequality

Theorem (Lieb–Thirring 1975)

Let $\gamma \geq \frac{1}{2}$ *if* $d = 1$, $\gamma > 0$ *if* $d = 2$ *and* $\gamma \geq 0$ *if* $d \geq 3$. *Then there is* $L_{\gamma,d} > 0$ *such that for every* $V \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)$,

$$\sum_{i=1}^{\infty} |\lambda_i(-\Delta + V)|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma+\frac{d}{2}} \, \mathrm{d}x. \tag{5}$$

- The right side of (5) is the the same as in (4) concerning a single eigenvalue!
- For $\gamma = 0$, the left side is interpreted to be the number of negative eigenvalues. This bound usually goes by the separate name of **Cwikel–Lieb–Rozenblum (CLR)** inequality.
- From (5) one can derive the inequality

$$\int_{\mathbb{R}^{3N}} |\nabla\psi|^2 \, \mathrm{d}x \geq K_3 \int_{\mathbb{R}^3} \rho_\psi^{1+\frac{2}{d}} \, \mathrm{d}x$$

  for every antisymmetric wave function ψ with $\|\psi\|_{L^2(\mathbb{R}^{3N})} = 1$, where ρ_ψ is the one-particle density of ψ. This is the crucial ingredient needed to prove stability of matter, i.e. $E_{N,M}^{\boldsymbol{R},Z} \geq c_3(Z)(N+M)$.

Duality : A Sobolev inequality for orthonormal functions

As in the first part, the LT inequality for eigenvalues of $-\Delta + V$ is dual to an inequality of Sobolev-type. Here we only discuss the case $\gamma = 1$.

Theorem (Lieb–Thirring 1975)

There is $K_d > 0$ such that for all L^2-orthonormal functions $u_1, ..., u_N \in H^1(\mathbb{R}^d)$,

$$\sum_{i=1}^N \int_{\mathbb{R}^d} |\nabla u_i|^2 \, \mathrm{d}x \geq K_d \int_{\mathbb{R}^d} \left(\sum_{i=1}^N |u_i|^2 \right)^{1+\frac{2}{d}} . \tag{6}$$

- Again, the constant K_d is independent of N. This would be impossible without the orthogonality requirement.
- The proof of the equivalence is similar to the duality proof we saw in Part I. It uses that by variational principle

$$\sum_{i=1}^N (|\nabla u_i|^2 + V|u_i|^2) \, \mathrm{d}x \geq \sum_{i=1}^N \lambda_i(-\Delta + V).$$

The constants are related by $\left((1+\frac{2}{d})K_d\right)^{1+\frac{d}{2}} = \left((1+\frac{d}{2})L_{1,d}\right)^{-(1+\frac{2}{d})}$.

The best constant in the LT inequality

Much of the current interest in LT inequalities

$$\sum_{i=1}^{\infty} |\lambda_i(-\Delta + V)|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_-^{\gamma + \frac{d}{2}} \, \mathrm{d}x.$$

is motivated by the following question, which is, in general, open.

Question : What is the optimal value of $L_{\gamma,d}$?

To approach this question, let us look at two natural lower bounds for $L_{\gamma,d}$, corresponding to one and infinitely many negative eigenvalues.

- Recall from Part I the inequality $|\lambda_1(-\Delta + V)|^\gamma \leq L_{\gamma,d}^{(1)} \int_{\mathbb{R}^d} V_-^{\gamma + \frac{d}{2}} \, \mathrm{d}x$

  By comparing with the LT inequality, we clearly get

  $$L_{\gamma,d} \geq L_{\gamma,d}^{(1)}.$$

  In the spirit of the above discussion, we call $L_{\gamma,d}^{(1)}$ the **one-particle constant**.

The conjecture by Lieb and Thirring

- The second lower bound is derived from the asymptotic formula

$$\lim_{\hbar \to 0} \hbar^d \sum_{i=1}^{\infty} |\lambda_i(-\hbar^2 \Delta + V)|^\gamma = L_{\gamma,d}^{\mathsf{cl}} \int_{\mathbb{R}^d} V_-^{\gamma + \frac{d}{2}} \, \mathrm{d}x.$$

with $L_{\gamma,d}^{\mathsf{cl}} := (4\pi)^{-\frac{d}{2}} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\frac{d}{2})}$. (Like this, the right side is equal to the classical phase space integral $\iint_{\mathbb{R}^d \times \mathbb{R}^d} (|\xi|^2 + V(x))_-^\gamma \frac{\mathrm{d}x \, \mathrm{d}\xi}{(2\pi)^d}$.) Thus

$$\lim_{\hbar \to 0} \frac{\sum_{i=1}^\infty |\lambda_i(-\Delta + \hbar^{-2} V)|^\gamma}{\int_{\mathbb{R}^d} (\hbar^{-2} V)^{\gamma + \frac{d}{2}} \, \mathrm{d}x} = L_{\gamma,d}^{\mathsf{cl}}, \qquad \text{hence} \qquad L_{\gamma,d} \geq L_{\gamma,d}^{\mathsf{cl}}.$$

The constant $L_{\gamma,d}^{\mathsf{cl}}$ is called the **semi-classical constant**.

Conjecture (Lieb–Thirring 1976)

One always has $L_{\gamma,d} = \max\{L_{\gamma,d}^{(1)}, L_{\gamma,d}^{cl}\}$.

Known results about the best constant

The following results show that Lieb's and Thirring's original conjecture is sometimes true...

Theorem

- $L_{\gamma,d} = L^{cl}_{\gamma,d}$ *if* $\gamma \geq 3/2$, $d \geq 1$. *[LT 1976, Aizenman–Lieb 1978, Laptev–Weidl 2000]*
- $L_{\frac{1}{2},1} = L^{(1)}_{\frac{1}{2},1}$ *if* $\gamma = 1/2$, $d = 1$ *[Hundertmark–Lieb–Thomas 1998]*

... and sometimes not.

Theorem

- $L_{\gamma,d} > L^{cl}_{\gamma,d}$ *if* $\gamma < 3/2$ *and* $d = 1$ *or if* $\gamma < 1$ *and* $d \geq 2$.
- $L_{\gamma,d} > L^{(1)}_{\gamma,d}$ *if* $\gamma > \max\{2 - d/2, 0\}$ *and* $1 \leq d \leq 6$, *or if* $\gamma \geq 0$ *and* $d \geq 7$.

In particular, Lieb's and Thirring's original conjecture is not true for $0 < \gamma < 1$ and $d \geq 4$. It is still believed to be true for $\gamma \geq \frac{1}{2}$, $d = 1$. This is open for $\gamma \in (\frac{1}{2}, \frac{3}{2})$.

Conformal invariance of the CLR inequality

Recall the CLR bound

$$N(-\Delta + V) \le L_{0,d} \int_{\mathbb{R}^d} V_-^{\frac{d}{2}} \,\mathrm{d}x,$$

where $N(-\Delta + V)$ denotes the number of negative eigenvalues of $-\Delta + V$.

We now reformulate this problem on the sphere $\mathbb{S}^d$ using the **stereographic projection** $\mathcal{S} : \mathbb{R}^d \to \mathbb{S}^d$ with Jacobian determinant $J_{\mathcal{S}}(x) = \left(\frac{2}{1+|x|^2}\right)^d$. For u and V on $\mathbb{R}^d$, define v and W on $\mathbb{S}^d$ by

$$V(x) = J_{\mathcal{S}}(x)^{\frac{2}{d}} W(\mathcal{S}(x)), \qquad u(x) = J_{\mathcal{S}}(x)^{\frac{d-2}{2d}} v(\mathcal{S}(x)).$$

Then the relevant integrals transform as

$$\int_{\mathbb{R}^d} |\nabla u|^2 \,\mathrm{d}x = \int_{\mathbb{S}^d} \left(|\nabla_{\mathbb{S}^d} v|^2 + \frac{d(d-2)}{4} |v|^2 \right) \mathrm{d}\omega,$$

$$\int_{\mathbb{R}^d} V|u|^2 \,\mathrm{d}x = \int_{\mathbb{S}^d} W|v|^2 \,\mathrm{d}\omega, \qquad \int_{\mathbb{R}^d} V_-^{\frac{d}{2}} \,\mathrm{d}x = \int_{\mathbb{S}^d} W_-^{\frac{d}{2}} \,\mathrm{d}\omega.$$

A conjecture for the optimal CLR constant

By the variational principle, $N(-\Delta + V)$ is equal to

$$\sup\left\{\dim V:\ V \subset H^1(\mathbb{R}^d),\quad \int_{\mathbb{R}^d} (|\nabla u|^2 + V|u|^2)\,\mathrm{d}x \le 0 \quad \forall u \in V\right\}.$$

The integral identities above yield the equivalent CLR inequality on $\mathbb{S}^d$

$$N\left(-\Delta_{\mathbb{S}^d} + \frac{d(d-2)}{4} + W\right) \le L_{0,d} \int_{\mathbb{S}^d} W_-^{\frac{d}{2}}\,\mathrm{d}\omega,$$

Taking $W = -c =$ const. as competitors, we get that

$$L_{0,d} \ge \sup_{c>0} \frac{N\left(-\Delta_{\mathbb{S}^d} + \frac{d(d-2)}{4} - c\right)}{|\mathbb{S}^d|\,c^{d/2}} =: L_{0,d}^{\mathsf{sph}}.$$

One can compute explicitly $L_{0,d}^{\mathsf{sph}} = \begin{cases} L_{0,d}^{\mathsf{sph}} = L_{0,d}^{(1)} & \text{for } d \le 6, \\ L_{0,d}^{\mathsf{sph}} > \max\{L_{0,d}^{(1)}, L_{0,d}^{\mathsf{cl}}\} & \text{for } d \ge 7. \end{cases}$

Conjecture (Glaser–Grosse–Martin 1978)

For every $d \ge 1$, one has $L_{0,d} = L_{0,d}^{sph}$.

Further reading

Textbooks :

- E. H. Lieb, M. Loss, *Analysis. Second Edition.* Graduate Studies in Mathematics 14, AMS (2001)
- E. H. Lieb, R. Seiringer, *The stability of matter in quantum mechanics*. Cambridge University Press (2009)
- R. L. Frank, A. Laptev, T. Weidl, *Schrödinger Operators : Eigenvalues and Lieb–Thirring Inequalities*. Cambridge Studies in Advanced Mathematics, Series Number 200 (2022)

Recent review articles :

- R. L. Frank, *The Lieb-Thirring inequalities : Recent results and open problems*, arXiv :2007.09326
- R. L. Frank, *Lieb-Thirring inequalities and other functional inequalities for orthonormal systems*, arXiv :2109.13660, submitted to the Proceedings of the ICM 2022
- D. Hundertmark, *Some bound state problems in quantum mechanics*, Spectral theory and mathematical physics. A festschrift in honor of Barry Simon's 60th birthday, AMS (2007)
- L. Schimmer, *The state of the Lieb–Thirring conjecture*, The physics and mathematics of Elliott Lieb. The 90th anniversary. Volume II. EMS. 253-275 (2022).

Many thanks for your attention !