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Gapped Hamiltonians & Topological Phases

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Mathematical Challenges in Quantum Mechanics
(PhD Lecture)

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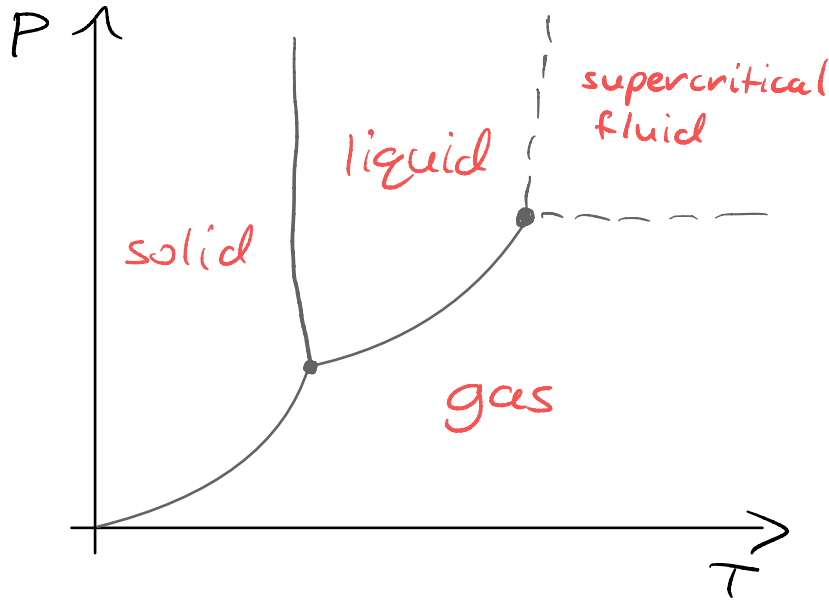
Outline

(2)

- Topological phases of matter
- Example: SSH model
- Periodic lattice Hamiltonians
& vector bundles
- Other symmetries & the 'Periodic table'
 - Talk by G. M. Graf.

Phases of matter

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→ A 'classification' of matter into various types as we vary physical parameters.

→ Phase transition between types.

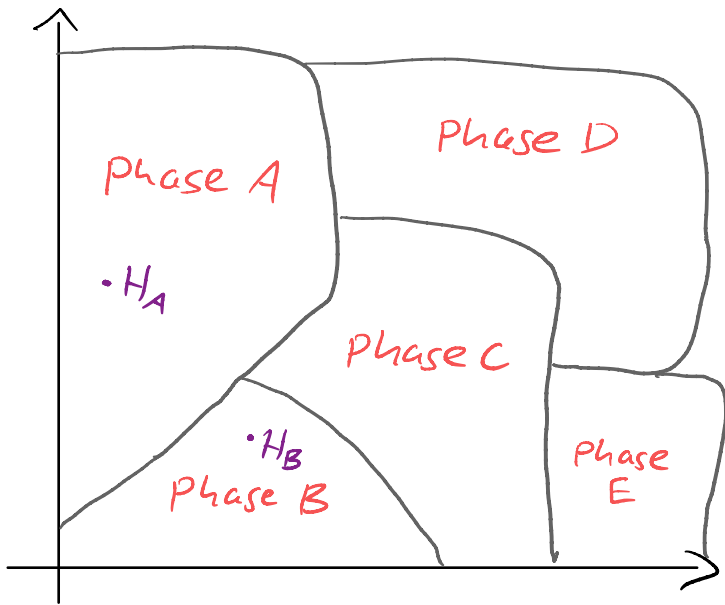
Topological phases of matter

→ Fix certain quantities
(symmetry, dimension)

→ Classify gapped
Hamiltonians w.r.t
some physical parameters.

Any path $H_A \longleftrightarrow H_B$
leaves space at some point
(gap closes, symmetry broken)

→ Topological phase transition



Why do this?

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Robustness

Phases classified by $\text{Ind}(\cdot)$
(e.g. Hall conductance)

$\text{Ind}(H)$ stable
under homotopy/
small perturbations

Justify toy models

simple system H_0
(can explicitly calculate)

$\longleftrightarrow$ realistic model H
(hard)

$$\text{If } H_0 \sim H \Rightarrow \text{Ind}(H_0) = \text{Ind}(H)$$

Setting

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Complex Hilbert space $\mathcal{H}$ and self-adjoint
Hamiltonian $H = H^*$

$$\text{e.g. } \mathcal{H} = L^2(\mathbb{R}^d, \mathbb{C}^n), \quad H = -\sum_{j=1}^d \partial_j^2 + V, \quad V \in L^\infty(\mathbb{R}^d, \mathbb{R}).$$

Often easier to work with lattice systems
and discrete models.

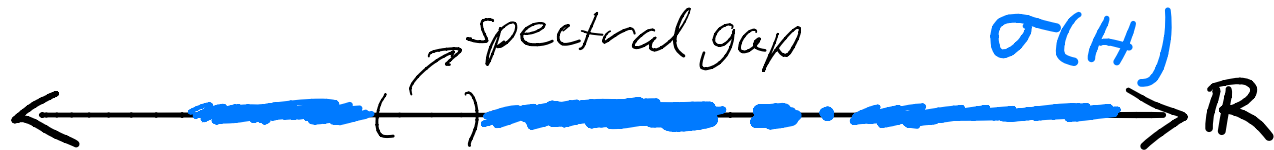
$$\text{e.g. } \mathcal{H} = \ell^2(\mathbb{Z}^2), \quad H = S_1 + S_1^* + S_2 + S_2^* + V, \quad V \in \ell^\infty(\mathbb{Z}^2, \mathbb{R})$$

$$(S_1 \psi)(m, n) = \psi(m-1, n), \quad (S_2 \psi)(m, n) = \psi(m, n-1), \quad \psi \in \ell^2(\mathbb{Z}^2)$$

Setting (cont.)

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Key assumption: $\sigma(H)$ has a gap (insulator)



Aim: Define a phase label $\text{Ind}(H)$ that:

→ Is a homotopy invariant

→ Describes a physical property
(possibly on boundary/edge)

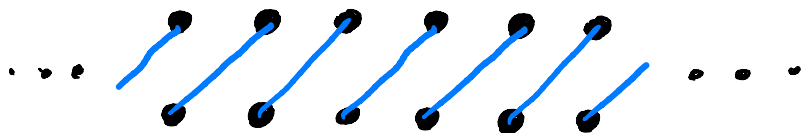
From now on, will work in concrete settings

SSH Model (Su-Schrieffer-Heeger, '80)

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$$\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2,$$

$$H = \begin{pmatrix} 0 & s - im \\ s^* + im & 0 \end{pmatrix}$$



$$m \in \mathbb{R}, \quad \text{const.}, \quad (S\psi)(n) = \psi(n-1)$$

Note: $H \sigma_3 = -\sigma_3 H$ (chiral symmetry, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$)

H periodic $\rightarrow$ diagonalised by Fourier transform

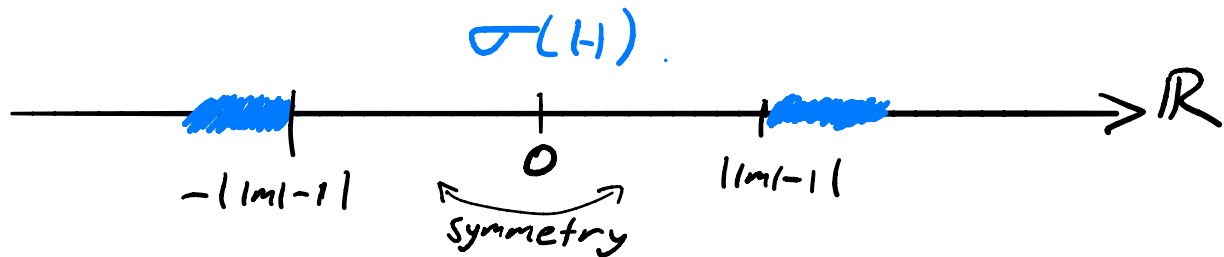
$$\mathcal{F}: \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2 \xrightarrow{\sim} L^2(\mathcal{T}) \otimes \mathbb{C}^2, \quad \mathcal{T} = \{z \in \mathbb{C} \mid |z| = 1\}$$

$$\mathcal{F} H \mathcal{F}^{-1} \sim \{H(k)\}_{k \in \mathcal{T}} \quad H(k) = \begin{pmatrix} 0 & e^{ik} - im \\ e^{-ik} + im & 0 \end{pmatrix} \in M_2(\mathbb{C}).$$

SSH (cont.) $H = \begin{pmatrix} s - im & \\ s^* + im & \end{pmatrix} \sim \left\{ \begin{pmatrix} 0 & e^{ik} - im \\ e^{-ik} + im & 0 \end{pmatrix} \right\}_{k \in \mathcal{D}}$ (9)

$\sigma(H) = \bigcup_{k \in \mathcal{D}} \sigma(H(k)), \quad \sigma(H(k)) = \left\{ \pm \sqrt{m^2 + 1 - 2m \sin(k)} \right\}$

→ gapped for $m \neq \pm 1$



Define:

$$\text{Ind} \left(H \sim \begin{pmatrix} 0 & e^{ik} - im \\ e^{-ik} + im & 0 \end{pmatrix} \right) = \text{Wind} (e^{ik} + im) = \begin{cases} -1, & m \in (-1, 1) \\ 0, & |m| > 1. \end{cases}$$

$$\text{Ind} \begin{pmatrix} s^* + im & s - im \\ s^* + im + \alpha & s - im + \alpha \end{pmatrix}, \quad \alpha \in \mathbb{R} \text{ small}$$

Physical?...

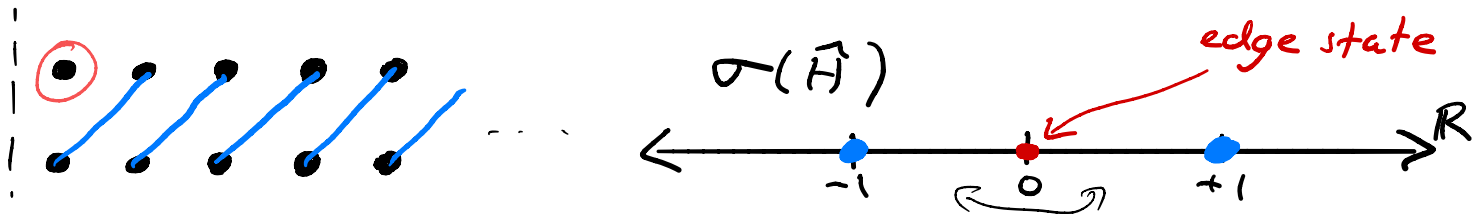
SSH - edge system

(10)

Restrict to half-space: $\Pi: \ell^2(\mathbb{Z}, \mathbb{C}^2) \rightarrow \ell^2(\mathbb{N}, \mathbb{C}^2)$ (Dirichlet BCs)

$$\hat{H} = \Pi H \Pi \simeq \begin{pmatrix} \hat{S}^* & \hat{S} \end{pmatrix} \quad (m=0 \text{ for simplicity})$$

$\hat{H}$ has 0-eigenvector $\hat{H} \begin{pmatrix} e_0 \\ 0 \end{pmatrix} = 0, \quad e_0(n) = \delta_{0,n}$



$\sigma_3 \hat{H} \sigma_3 = -\hat{H}$ (chiral symmetry) $\rightarrow$ e. value fixed at 0.

$$\begin{aligned} \# \text{ of edge states} &= \dim \text{Ker}(\hat{S}^*) = \text{Index}(\hat{S}^*) \quad (\text{Fredholm index}) \\ &= -\text{Wind}(k \mapsto \tilde{e}^{ik}) = -\text{Ind}(S^* S) \end{aligned}$$

SST - generalisations.

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Fredholm index = winding no. holds more generally

Nöther-Toeplitz Index Thm, $f \in C(\mathbb{T}, \mathbb{C} \setminus \{0\})$

$$\text{wind}(f) = -\text{Index}(\underbrace{F M_f F^{-1}}_{\text{mult. op-r on } L^2(\mathbb{T})})$$

→ Applicable to more complicated examples.

can also generalise to disordered systems.

$$(\Omega, \mathbb{P}) \ni \omega \mapsto H_\omega = \begin{pmatrix} A_\omega & A_\omega^* \end{pmatrix}, \quad A_\omega = S^* + V_\omega \text{ with } V_\omega \text{ random potential (ergodic)}$$

$$\text{Tr}_{\text{vol}}(A_\omega^{-1} i[X, A_\omega]) = -\text{Index}(\hat{A}_\omega) = \# \text{ of edge states}$$

trace per unit volume $\rightarrow \lim_{L \rightarrow \infty} \frac{1}{L^d} \text{Tr}_L(\dots)$

position operator $(X\psi)(x) = x\psi(x)$

($\mathbb{P}$ -almost surely)

Holds for 0 in a mobility gap of $\sigma(H_\omega)$ (Anderson localisation)

2D systems.

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$H = H^*$ acting on $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^n$, e.g. $H = S_1 + S_1^* + S_2 + S_2^* + V$.

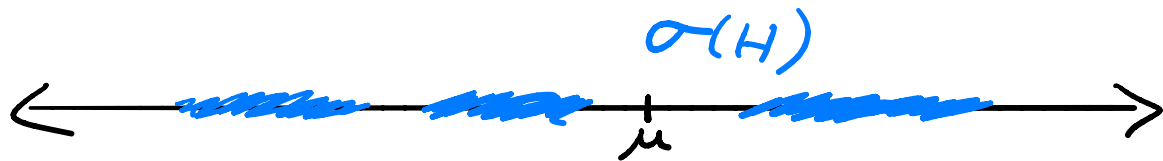
Let's assume H periodic $\rightarrow$ diagonalised by
 $\mathcal{F}: \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^n \rightarrow \ell^2(\mathbb{T}^2) \otimes \mathbb{C}^n$

$$\mathcal{F} H \mathcal{F}^{-1} \sim \{H(k)\}_{k \in \mathbb{T}^2}$$

$$H(k) \in M_n(\mathbb{C})$$

$$\sigma(H) = \bigcup_{k \in \mathbb{T}^2} \sigma(H(k))$$

Want an insulator (H gapped)

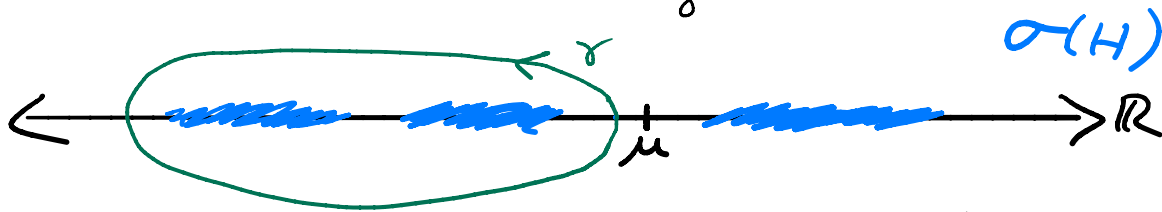


Low temperature fermions, ground state is determined by energies $\leq \mu$ (chemical potential).

Block bundle.

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Project onto $E \leq \mu$: $P_\mu = \oint_{\gamma} (z - H)^{-1} dz$, $P_\mu = P_\mu^2 = P_\mu^*$



Also have $\mathcal{F} P_\mu \mathcal{F}^{-1} \sim \{P(k)\}_{k \in \mathbb{T}^2}$ $P_\mu(k) = \oint_{\gamma} (z - H(k))^{-1} dz \in \mathcal{M}_n(\mathbb{C})$

$\mathbb{T}^2 \ni k \mapsto P_\mu(k)$ analytic for $\mu \notin \sigma(H)$

$\mathbb{T}^2 \ni k \mapsto P_\mu(k) \cdot \mathbb{C}^n$ smooth family of vector spaces

$\rightarrow$ complex vector bundle $P_\mu \rightarrow \mathbb{T}^2$

(construction works for any gapped $H = H^* \in \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^n$)
 $P_\mu \rightarrow \mathbb{T}^d$

Vector bundle basics

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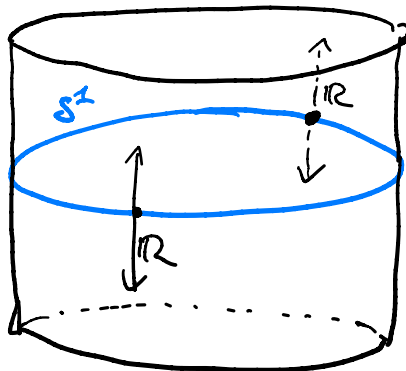
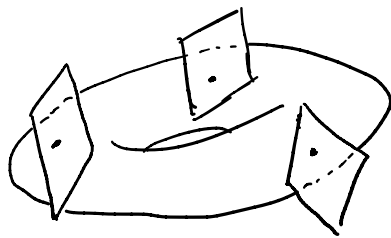
cts & surjective map $\pi: E \rightarrow M$ s.t. $\forall x \in M$,

- Fibre $E_x = \pi^{-1}(\{x\}) \simeq \mathbb{C}^m$ is a vector space

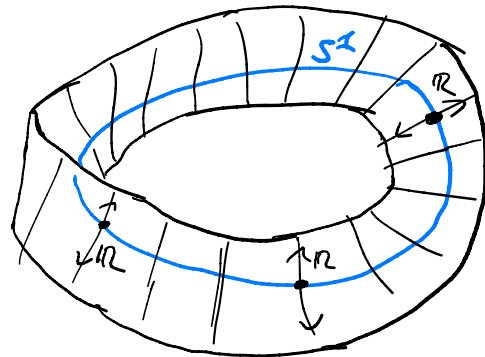
- $\exists$ open n -hood $x \in U \subset M$ & homeomorphism

$$\Phi_U: \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{C}^m \quad (\pi \circ \Phi_U^{-1}(y, v) = y \quad \forall y \in U).$$

e.g. $E = M \times \mathbb{C}^m$, $\pi(x, v) = x$ (trivial bundle).



cylinder (trivial)



Möbius bundle
(non-trivial)

2D Top. phase

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We have Bundle $P_\mu \rightarrow \mathbb{T}^2$ from Fermi projection.

How do we know if it's trivial/non-trivial?

→ Chern classes

If $Ch(P_\mu) \neq 0$ then $P \not\cong \mathbb{T}^2 \times \mathbb{C}^m$.

$$Ch(P_\mu) = \frac{1}{2\pi i} \int_{\mathbb{T}^2} Tr_{\mathbb{C}^m} (P(k) [\partial_{k_1} P(k), \partial_{k_2} P(k)]) dk \in \mathbb{Z}$$

Kubo formula: $Ch(P) \sim$ transverse conductivity

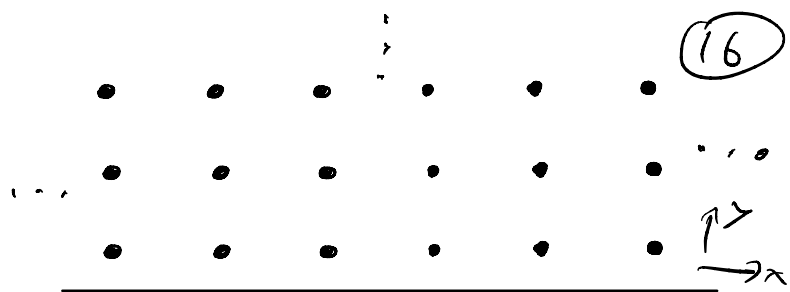
$$[TKNN] \quad \sigma_{xy} = -\frac{e^2}{h} Ch(P_\mu) \quad (T \sim 0K)$$

2D gapped $H \rightsquigarrow Ind(H) = Ch(P_\mu)$

2D Bulk-boundary

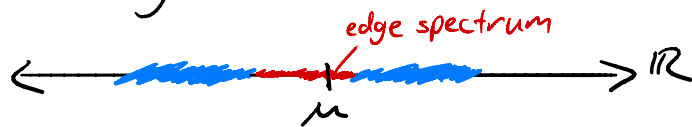
$$\Pi: \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^n \rightarrow \ell^2(\mathbb{Z} \times \mathbb{N}) \otimes \mathbb{C}^n$$

$$\hat{H} = \Pi H \Pi \quad x\text{-periodic}$$



Partial decomp. $\hat{H} \sim \{ \hat{H}(k_x) \}_{k_x \in \mathcal{D}}$, $\hat{H}(k_x) \in \mathcal{B}(\ell^2(\mathbb{N}) \otimes \mathbb{C}^n)$

May have $\mu \in \sigma(\hat{H}(k_x))$, but $\dim(\ker(\hat{H}(k_x))) < \infty \quad \forall k_x \in \mathcal{D}$

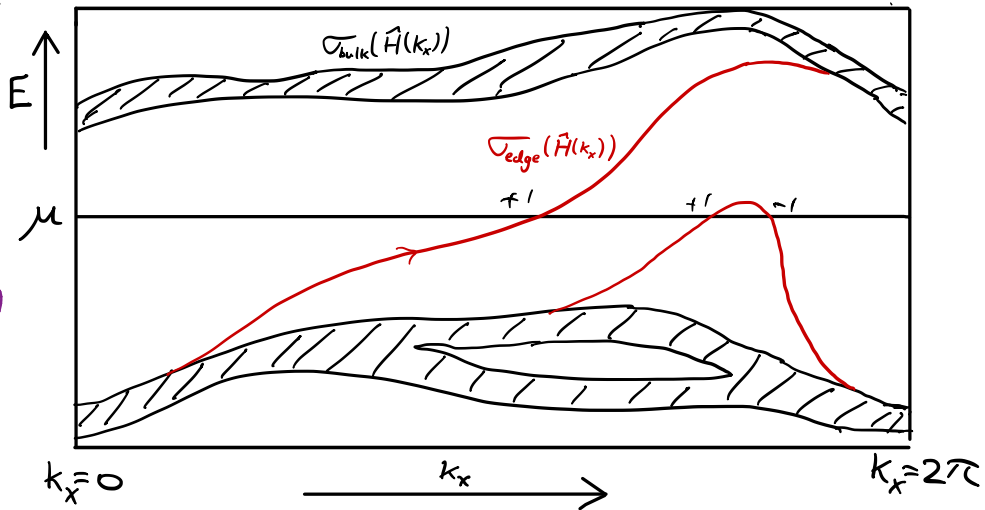


Track edge states
via spectral flow.

Thm (Hatsugai)

$$\text{Ch}(P_\mu) = -\text{sf}(\hat{H}(k_x) - \mu)$$

gapped bulk $\longleftrightarrow$ gapless edge



Free-fermionic symmetries

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Pure state $\psi \in \mathcal{H}$ invariant under $\Theta: \mathcal{H} \rightarrow \mathcal{H}$

$$|\langle \Theta \psi, \Theta \psi \rangle_x|^2 = |\langle \psi, \psi \rangle_x|^2$$

[Wigner] $\Rightarrow \Theta$ is unitary or anti-unitary.

If $H = H^*$ gapped with $\Theta H \Theta^* = \pm H$, can try & find $\text{Ind}(H, \Theta)$ to distinguish Θ -symmetric Hamiltonians.

For free fermions, often interested in

$$T H T^* = H, \quad \text{time reversal symmetry}$$

$$P H P^* = -H, \quad \text{charge conjugation (particle-hole 'symmetry')}$$

$$C H C^* = -H, \quad \text{chiral/sublattice 'symmetry'}$$

The periodic table

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TRS & PHS s.t. $T^2 = \pm \mathbb{1}$, $\Gamma^2 = \pm \mathbb{1}$.

Symmetries interact:

TRS + Chiral $\Rightarrow$ PHS $(TCH(\pi)^* = -H)$.

10 possibilities, 1-1 correspondence with:

- 1) $\mathbb{Z}_2$ -graded division algebras over $\mathbb{R}$
- 2) Morita equivalence classes of real/complex Clifford algebras.
- 3) $\mathbb{Z}_2$ -graded projective unitary/anti-unitary representations of subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2$.
- 4) 8+2 real/complex K-theory groups.

Ex: TRS vector bundles.

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Take $H = H^*$ gapped & periodic on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^n$

$$THT^* = H \quad \leadsto \quad TH(k)T^* = \overline{H(\tau(k))}, \quad \tau: \mathbb{T}^d \rightarrow \mathbb{T}^d$$

$$\tau(k) = -k.$$

Block bundle: $TP_\mu(k)T^* = \overline{P_\mu(\tau(k))}$

$\leadsto$ Real vector bundle.

$\text{Vect}_{\mathbb{R}}^{\tau}(\mathbb{T}^d) \leadsto$ isomorphism classes of Real vector bundles over $\mathbb{T}^d$.

$\text{Vect}_{\mathbb{R}}^{\tau}(\mathbb{T}^d)$ is an abelian semigroup (direct sum).

Grothendieck construction: abelian semigroup $S \rightarrow$ abelian group $\mathcal{G}(S)$

$$KR(\mathbb{T}^d, \tau) := \mathcal{G}(\text{Vect}_{\mathbb{R}}^{\tau}(\mathbb{T}^d))$$

when $\tau^2 = -\text{id} \rightarrow$ degree shift $[P_\mu] \in KR^{-4}(\mathbb{T}^d, \tau)$.

2D TRS, $T^2 = -\mathbb{1}$.

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Previously had $\text{Ind}(H) = \text{Ch}(P_n)$

If $THT^* = H$, $\text{Ch}(P_n) = 0$

$P_n \rightarrow T^2$ has 2 TRS homotopy invariants

1) $\text{Rank}(P_n) = \dim(\text{Ran}(P_n(k)))$ (not so interesting)

2) $\mathbb{Z}_2$ -valued Kane-Mele invariant

$$KR^{-4}(T^2, \tau) \cong \mathbb{Z} \oplus \mathbb{Z}_2.$$

extract $\mathbb{Z}_2$ via map $KR^{-4}(T^2, \tau) \rightarrow KR^{-4+2}(\text{pt}) \cong \mathbb{Z}_2$

can be done by 'symmetric cellular decomposition' of T^2 & long exact sequence in K-theory.

The table

"top degree"

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More generally, $\exists$ map $KR^{-n}(\mathbb{D}^d, \tau) \xrightarrow{\phi_d} KR^{-n+d}(\{pt\})$

"strong topological phase". $\text{Ind}(H) = \phi_d([P_\mu])$

δ	0	1	2	3	4	5	6	7
$KR^\delta(\{pt\})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

$$KR^{-n-8} \simeq KR^{-n}$$

Symmetry				Spatial dimension d							
CAZ	TRS	PHS	CHS	1	2	3	4	5	6	7	8
A	0	0	0		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
AI	+1	0	0				$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	+1	+1	1	$\mathbb{Z}$				$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\mathbb{Z}$		$\mathbb{Z}_2$
DIII	-1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\mathbb{Z}$	
AII	-1	0	0		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\mathbb{Z}$
CII	-1	-1	1	$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
C	0	-1	0		$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
CI	+1	-1	1			$2\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

(Taken from
[Katsura-Koma,
arXiv:1611.01928])

Concluding remarks

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→ Vector bundle picture more involved for PHS/chiral

$$\Gamma P_\mu \Gamma^* = \mathbb{1} - P_\mu$$

→ Non-periodic? Take an observable C^* -algebra $H \in A \subset B(\mathcal{H})$

$$[P_\mu] \in KO_n(A^\tau) \quad [\text{Kellendonk}]$$

Limitations

→ Physical interpretation of $\mathbb{Z}_2$ -invariants often require additional input (e.g. spin-Chern number)

→ Bulk-boundary for μ in a mobility gap still open in many cases.

→ Techniques of limited use for topological phases of gapped (many-body) ground states.

Some literature (many omissions)

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- [1] A. Alldridge, C. Max and M. R. Zirnbauer. *Bulk-boundary correspondence for disordered free-fermion topological phases*. Comm. Math. Phys., **377** (2020), no. 3, 1761–1821.
- [2] J. Bellissard, A. van Elst and H. Schulz-Baldes. *The noncommutative geometry of the quantum Hall effect*. Topology and physics. J. Math. Phys., **35** (1994), no. 10, 5373–5451.
- [3] A. Bols, J. Schenker and J. Shapiro. *Fredholm homotopies for strongly-disordered 2D insulators*. Comm. Math. Phys., **397** (2023), no. 3, 1163–1190.
- [4] D. S. Freed and G. W. Moore. *Twisted equivariant matter*. Ann. Henri Poincaré, **14** (2013), no. 8, 1927–2023.
- [5] G. M. Graf and J. Shapiro. *The bulk-edge correspondence for disordered chiral chains*. Comm. Math. Phys., **363** (2018), no. 3, 829–846.
- [6] J. Kellendonk. *On the C^* -algebraic approach to topological phases for insulators*. Ann. Henri Poincaré, **18** (2017), no. 7, 2251–2300.
- [7] R. Kennedy and M. R. Zirnbauer. *Bott periodicity for $\mathbb{Z}_2$ -symmetric ground states of gapped free-fermion systems*. Comm. Math. Phys., **342** (2016), no. 3, 909–963.
- [8] A. Kitaev. *Periodic table for topological insulators and superconductors*. In Vladimir Lebedev and Mikhail Feigel'man, editors, *Adv. Theor. Phys.: Landau Memorial Conference*, volume 1134, pages 22–30, AIP, (2009).
- [9] E. Prodan and H. Schulz-Baldes. *Bulk and Boundary Invariants for Complex Topological Insulators: From K -Theory to Physics*. Springer, Berlin (2016).
- [10] G. C. Thiang. *On the K -theoretic classification of topological phases of matter*. Ann. Henri Poincaré, **17** (2016), no. 4, 757–794.

Thank you!

ご清聴、ありがとうございました。