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Locality and spectral gaps in quantum spin systems

Angelo Lucia (Universidad Complutense de Madrid $\mapsto$ Politecnico di Milano)
MCQM PhD Lecture – 18/12/2024

Reference

"An Introduction to Quantum Spin Systems", Bruno Nachtergaele and Robert Sims, Lecture Notes

https://nextcloud.tfk.ph.tum.de/etn/wp-content/uploads/2022/09/JvN_lecture_notes_S2016_abcde-1.pdf

Quantum Spin Systems

Quantum Mechanics

Atoms, molecules, etc. are described (in a non-relativistic setting) by *quantum mechanics*.

1. States are *normalized vectors* ψ in some complex Hilbert space $\mathcal{H}$.
2. Physical quantities of interest (position, momentum, magnetization, spin, energy, etc.) are described by *self-adjoint* operators on $\mathcal{H}$ (not necessarily bounded).
3. Expectation values: $\langle A \rangle_\psi := \langle \psi | A\psi \rangle$.

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Examples

- A *free particle*: $\mathcal{H} = L_2(\mathbb{R})$. Energy is given by $\frac{1}{2m}\hat{P}^2$, where $\hat{P}$ is the *momentum* operator $\hat{P} = -i\frac{\mathrm{d}}{\mathrm{d}x}$. Position $\hat{X}$ is multiplication by x.
- A spin-1/2 particle (e.g. electron): $\mathcal{H} = \mathbb{C}^2$ (on top of positional degrees of freedom).

Quantum Spins

What is a "spin"?
Any quantum system described by a *finite dimensional* Hilbert space.

Why? (Why is such restriction justified?)

1. Restrictions on the degrees of freedom (e.g. condensed matter models)
2. Truncations of lattice QFT
3. Quantum information and quantum computation: qubits, circuits, etc.
4. Toy models

Quantum Spins

What is a “spin”?
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Why? (Why is it a useful point of view?)
Rich and complex features of quantum many-body systems can be studied and treated in a way that is mathematically accessible.

- Superfluidity, Bose-Einsten condensation
- Superconductivity
- Quantum Hall effect (integer and fractional)
- Topological order, anyons
- quantum phases of matter

Notation and definitions

Let $\mathcal{H}$ a finite dimensional Hilbert space.

- States: $\psi \in \mathcal{H}$, $\|\psi\| = 1$.
- Observables: (Hermitian) operators in $\mathcal{B}(\mathcal{H})$.
- Expectation value: $A \in \mathcal{B}(\mathcal{H}) \mapsto \langle A \rangle_\psi = \langle \psi | A\psi \rangle$.
- If $\{a_i, v_i\}_i$ are eigevalues/normalized eigenvectors of A, then

$$\langle A \rangle_\psi = \sum_i p_i a_i, \quad p_i = |\langle v_i | \psi \rangle|^2 \geq 0, \quad \sum_i p_i = 1$$

Hamiltonians and time evolution

Hamiltonian: energy operator $H \in \mathcal{B}(\mathcal{H})$.

Ground states
Eigenvectors of H with minimal energy.

Time evolution

- States: $i\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = H\psi(t)$ (Schrödinger equation)
- Observables: $\frac{\mathrm{d}}{\mathrm{d}t}A(t) = i[H, A]$ (Heisenberg equation)
- $\langle A\rangle_{\psi(t)} = \langle A(t)\rangle_{\psi}$.
- Solution to Heisenberg equation

$$A(t) = U_t^* A U_t, \quad U_t = \exp(-itH)$$

Composite systems

Question:
How to describe the joint system given by two different quantum spins, described by $\mathcal{H}_1$ and $\mathcal{H}_2$?

Answer:
Use the (Hilbert) tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$!

$\mathcal{H}_1 \otimes \mathcal{H}_2$ has a natural Hilbert space structure:

$$\langle \psi_1 \otimes \psi_2 | \phi_1 \otimes \phi_2 \rangle = \langle \psi_1 | \phi_1 \rangle \cdot \langle \psi_2 | \phi_2 \rangle$$

(and extend by linearity).

How to study a collection of an infinite number of quantum spins?

Problem

Interesting physical phenomena requires an infinite collection of quantum spins to be described.

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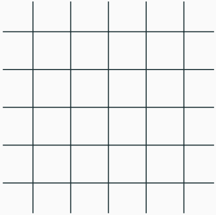
when I is infinite?

Yes, but it is not nice (usually not separable, etc.). Instead, we will take a C^*-algebraic approach.

Quantum spin system on a lattice

Lattice models

- Γ infinite graph (for example $\Gamma = \mathbb{Z}^D$)



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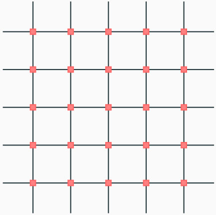

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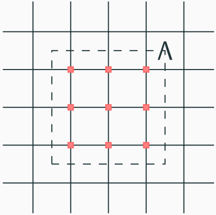

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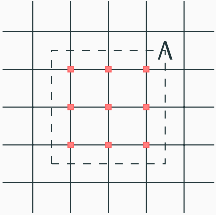

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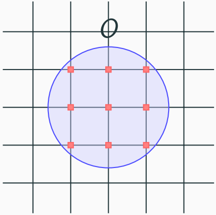

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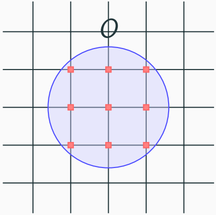

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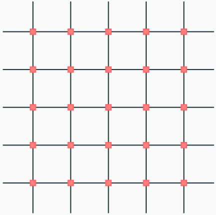
- the minimal Λ such that $O \in \mathcal{A}_\Lambda$ is denoted the *support* of O and we write $\operatorname{supp} O$.
- Let $\mathcal{A}_\Gamma^{\mathrm{loc}} = \bigcup_{\Lambda\nearrow\Gamma} \mathcal{A}_\Lambda$. $\mathcal{A}_\Gamma = \overline{A_\Gamma^{\mathrm{loc}}}^{\|\cdot\|}$ is the C^*-algebra of quasi-local observables (the *thermodynamic limit*).



How to define a Hamiltonian? Time evolution? Ground states?

Can we recover a Hilbert space?

Local interactions

Local interactions

Interactions in physics are usually *local*: they decay with distance.

Interaction
A mapping $\Phi(X)$ for each $X \subset \Gamma$ finite, such that $\operatorname{supp} \Phi(X) = X$, and $\Phi(X)$ is Hermitian.

Decay rate:

- *finite-range:* $\Phi(X) = 0$ if $\operatorname{diam} X > R$ for some $R > 0$.
- *exponential:* $\|\Phi(X)\| \leq \exp(-\alpha \operatorname{diam} X)$ for some $\alpha > 0$.
- *polynomial:* $\|\Phi(X)\| \leq \alpha^{-\operatorname{diam} X}$ for some $\alpha > 1$.

As finite-range and exponential behave similarly, we will only consider these cases.

Local Hamiltonian

With a given interaction Φ, we can define a Hamiltonian H_Λ for each finite volume

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X)$$

with a corresponding automorphism of $\mathcal{A}_\Lambda$

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}, \quad A \in \mathcal{A}_\Lambda$$

Examples

$\Gamma = \mathbb{Z}$, $\mathcal{H}_u = \mathbb{C}^2$, and

1D Ising model

$$H_{[-n,n]} = -\sum_{i=-n}^{n-1} J_{i,i+1}\, \sigma_i^Z \otimes \sigma_{i+1}^Z - \sum_{i=-n}^{n} \mu_i\, \sigma_i^X$$

XXZ model

$$H_{[-n,n]} = -\sum_{i=-n}^{n-1} \sigma_i^X \otimes \sigma_{i+1}^X + \sigma_i^Y \otimes \sigma_{i+1}^Y + \Delta\, \sigma_i^Z \otimes \sigma_{i+1}^Z$$

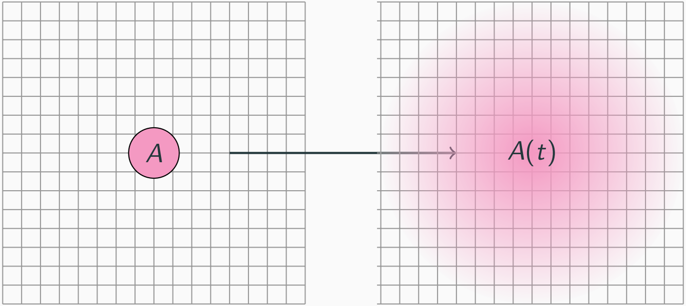
$$\sigma^X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

Evolution generatared by local interactions preserves locality

(in a sense)

Locality estimates

Finite group velocity



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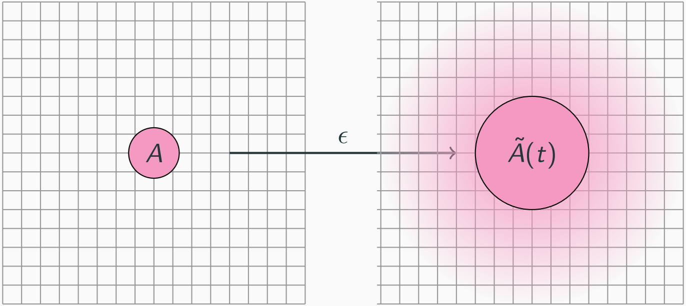

For any $t > 0$, the support of $\tau_t^\Lambda(A)$ can be the whole Λ.

But "most of the norm" of $\tau_t^\Lambda(A)$ is concentrated in a region of size linear in t!

Lieb-Robinson bounds

Theorem (Lieb, Robinson 1972)
Let $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, $X, Y \subset \Lambda$, $X \cap Y \neq \emptyset$. Then

$$\left\| \left[\tau_t^\Lambda(A), B \right] \right\| \leq C \|A\| \|B\| \min(|X|, |Y|) e^{\mu(vt - d(X,Y))}$$

for positive constants C, μ, v.

v is called Lieb-Robinson velocity or finite group velocity.

Idea: since this is true for any operator $B \in \mathcal{A}_Y$, when $t \ll \frac{1}{v} d(X, Y)$, $\tau_t^\Lambda(A)$ is supported *outside* of B (up to small errors).

The proof is in a sense "combinatorial" (counting possible "chains of interactions" between X and Y).

Infinite volume dynamics

A corollary of Lieb-Robinson bounds is that we can localize the evolution.

Lemma
For $A \in \mathcal{A}_X$, *with* $X \subset X(r) \subset \Lambda$, $r > 0$, *where* $X(r) = \{x \colon d(x, X) \leq r\}$

$$\left\| \tau_t^{\Lambda}(A) - \tau_t^{X(r)}(A) \right\| \leq C \|A\| |X| e^{\mu(vt - r)}$$

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Theorem
If we take an increasing and absorbing sequence of volumes $\Lambda_n \nearrow \Gamma$, then for each fixed finite $X \subset \Gamma$, $\tau_t^{\Lambda_n}(A)$ is a Cauchy sequence for every $A \in \mathcal{A}_X$:

$$\left\| \tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A) \right\| \to 0, \quad \text{as } n, m \to \infty$$

so we can define

$$\tau_t^{\Gamma}(A) := \lim_n \tau_t^{\Lambda_n}(A), \quad \forall A \in \mathcal{A}_{\Gamma}^{loc}$$

τ_t^{Γ} is a strongly continous group of $*$-automorphisms of $\mathcal{A}_{\Gamma}$.

Ground states and GNS representation

Let us now consider a sequence of ground states $\{\psi_n\}_n$ of H_{Λ_n}, such that

$$\omega(A) := \lim_n \langle A \rangle_{\psi_n} = \lim_n \langle \psi_n | A \psi_n \rangle \,, \quad A \in \mathcal{A}_\Gamma^{\mathrm{loc}}$$

is well defined.

Theorem
With the assumption above, there exits a Hilbert space $\mathcal{H}_\omega$, a vector $\Omega_\omega \in \mathcal{H}_\omega$, a $*$-representation π_ω of $\mathcal{A}_\Gamma$ on $\mathcal{H}_\omega$, and a densely defined self-adjoint operator H_ω on $\mathcal{H}_\omega$, such that

1. $\omega(A) = \langle \Omega_\omega | \pi_\omega(A) \Omega_\omega \rangle = \langle \pi_\omega(A) \rangle_{\Omega_\omega}$
2. $H_\omega \geq 0$ *and* $\Omega_\omega \in \ker H_\omega$
3. $\pi_\omega(\tau_t^\Gamma(A)) = U_t^* \pi_\omega(A) U_t$ *for* $U_t = \exp(-itH_\omega)$

In other words, we recover the "usual" quantum mechanic framework, but only up to a representation of $\mathcal{A}_\Gamma$ (which depends on ω).

This is a consequence of the *GNS representation* for $(\mathcal{A}_\Gamma, \omega)$.

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The *spectral gap* $\mu_\Lambda > 0$ of H_Λ if the difference between its two smallest (distinct) eigenvalues.

The *spectral gap* μ_ω of H_ω is defined as

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Lemma

$$\mu_\omega \geq \liminf_n \mu_{\Lambda_n}$$

Phase classification

Quantum phases of matter

Phases of matter are equivalence classes of systems with "similar" physical properties: we can smoothly deform one into the other.

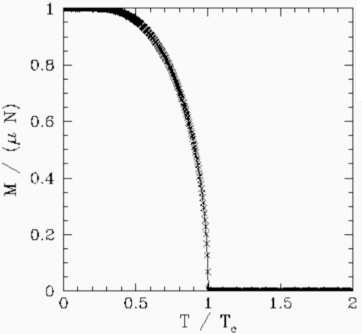

Figure 1: Example of a phase transition. Picture author: Richard Fitzpatrick 2006

Interactions with parameters

Consider a family $\Phi(s)$ of interactions, depending smoothly on a parameter $s \in [0, 1]$.

Let us assume for simplicity that each $H_\Lambda^{(s)}$ has a unique ground state, whose limit is ω_s.

Question:
Under which assumptions ω_s "depends smoothly" on $s \in [0, 1]$?

Automorphic flow

Theorem (Bachmann, Michalakis, Natchergaele, Sims. 2012)
Suppose that the Hamiltonians are uniformly gapped in s:

$$\inf_{s\in[0,1]} \mu_{\omega_s} \geq \gamma > 0.$$

Then there exists a continous family of $*$-automorphisms α_s (the spectral flow) such that

$$\omega_s = \omega_0 \circ \alpha_s$$

Moreover, α_s satisfies a Lieb-Robinsons bound.

Idea: α_s is the evolution generated by a time-dipendent interaction

$$K_\Lambda(s) = \int_{-\infty}^{\infty} \mathrm{d}t\, W_\gamma(t) e^{itH_\Lambda^{(s)}} \left(\frac{\mathrm{d}}{\mathrm{d}s} H_\Lambda^{(s)} \right) e^{-itH_\Lambda^{(s)}}$$

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The hard problem:
Prove that a given spin model has a spectral gap

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Thank you for your attention! Questions are welcome!